Geometry for middle school competitions

Dedication

> To our esteemed colleagues, whose invaluable contributions made the realization of this book possible. A special acknowledgment goes to Michael Podaev for his exceptional support and insights.

> Dedicated to our students, who discovered the joy of geometry through the pages of this book. Your enthusiasm and engagement have truly made the journey enjoyable.

Introduction

Within the pages of this book, we embark on a journey into the captivating realm of geometry. Allow me to make one thing clear – this is not an attempt to produce yet another run-of-the-mill geometry textbook. The existing school materials adequately cover the theoretical foundations essential for solving Olympiad-level geometry problems. Instead, our focus is on bridging the gap between possessing theoretical knowledge and mastering the art of conquering Olympiad geometry challenges.

In this book, we revisit essential geometric theory, providing both a reminder and a deeper understanding of the concepts crucial for Olympiad success. Additionally, we delve into various problem-solving methodologies tailored to the unique demands of Olympiad-style questions. Reading this book assumes you've already mastered school geometry, covering grades 7-9, and perhaps explored some aspects of AP Geometry (without trigonometry; it won't be necessary here).

Traditionally, most Olympiad problems are not particularly tied to a specific grade level, making this book valuable for high school students. Moreover, some problems presented here were indeed featured in the finals of national Olympiads for higher grades. Additionally, many students tend to overcomplicate solutions. It's essential to understand how to solve problems using simple and elegant methods.

Each chapter is organized as follows: the first part of each chapter covers theory related to the topic, along with detailed solutions to several typical problems.

The second part of the chapter is a problem set specific to that topic, with each problem labeled with its source. Most Olympiad problems included in this guide are marked with a notation like «Year.Grade.Number.» For example, a problem labeled «ACM

2016.10A.5» is the fifth problem from the 10th-grade variant 10A of the ACM Olympiad in 2016. It's worth mentioning that class numbering can vary between countries. For instance, if you are studying in the USA and the Olympiad took place in Russia, feel free to add at least 1 to the grade number.

The problem number notation allows you to gauge the difficulty: generally, the higher the number, the more challenging the problem. However, this labeling doesn't always apply to some «independent» Olympiads, which can sometimes confuse genuine Olympiad participants.

The third part of the chapter consists of several problems for independent solving. Some of them are original, making their debut here. Solutions to these problems are provided at the end of the book. Don't be surprised if you encounter the same problem multiple times in different themes; it means it can be solved in various ways.

We adopt a proof-based approach to problem-solving, even though such a level of solution presentation is required only at high-level Olympiads in many countries (for example, in the USA or the UK).

Beyond the theory and solved problems, we offer an extensive collection of real Olympiad problems after each topic. Primarily, we explore problems from Russian Olympiads (a country with a strong tradition in Olympiad geometry, including the renowned Sharygin Geometrical Olympiad) and the American Mathematical Competitions (AMC). To ensure the AMC problems pose a bit more challenge, we have selected those where reconstructing the diagram is possible purely from the textual conditions. We sincerely recommend not only finding the correct answer from the given AMC options but also approaching these problems from a proof-based perspective.

Embarking on this journey signifies your readiness to elevate your geometric prowess beyond the classroom. We won't rehash familiar theories but will delve into the intricacies, nuances, and strategic thinking required to crack the code of Olympiad geometry.

Whether you are a seasoned Olympiad participant or a budding enthusiast eager to unravel the mysteries of geometric problem-solving, this book serves as your guide. Let's navigate the challenges of Olympiads together, where theory meets application, promising an enriching journey as rewarding as the destination. Welcome to a geometric adventure like no other!

Contents

Geometric Inequalities

> «Obvious» is the most dangerous word in mathematics..
>
> —Eric Temple Bell

Theory and Practice

Although problems involving the triangle inequality in its pure form are quite rare in olympiads, it is often used in the solutions of «larger» problems.

Let's start by recalling the basic properties of triangles:

1. The larger angle of a triangle is opposite the larger side.

2. The larger side of a triangle is opposite the larger angle.

3. The sum of any two sides of a triangle is greater than the third side.

The last property is often referred to as the «triangle inequality,» and it is precisely this property that is most commonly used in problem-solving.

Problem 1.1. In a triangle, the lengths of two sides are 1 and 2024 respectively. Find the length of the third side, given that it is expressed as an integer.

Solution. Denoting the side lengths as $a = 1$, $b = 2024$, and the unknown side as c, we have three inequalities:

$$1 + 2024 > c,$$

$$1 + c > 2024,$$

$$2024 + c > 1,$$

which immediately implies $2023 < c < 2025$. Therefore, $c = 2024$. □

Problem 1.2. Prove that in any triangle, the length of any side is less than half the perimeter of the triangle.

Solution. Let a, b, and c be the side lengths of the triangle.

Without loss of generality, let's prove that

$$a < \frac{a+b+c}{2}.$$

We know that $a < b + c$. Adding a to both sides of the inequality, we get $2a < a + b + c$. Dividing both sides of the inequality by the positive number 2 yields the desired result. □

Problem 1.3. Prove that in any quadrilateral, the length of any diagonal is less than half the perimeter.

Solution. Let a, b, c, and d be the side lengths of the quadrilateral (see Figure). @@@link@@@

Consider the diagonal AC and denote its length as e.

Then, by the triangle inequality, it is true that $a+b > e$ and $c+d > e$.

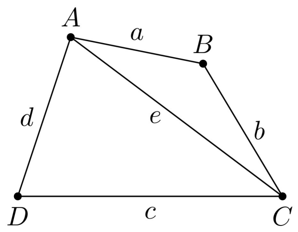

Figure 1.1: Side lengths of a quadrilateral.

Adding these inequalities (since we can do this for inequalities with the same sign), we get $a + b + c + d > 2e$, and after dividing by the positive number 2, we obtain the desired statement.

Note that in formulating the inequalities, we did not use the convexity property of the quadrilateral, so the statement holds for arbitrary quadrilaterals. □

Problem 1.4. Is it true that for any 10-line segments, there will be three that can form a triangle?

Solution. This problem also falls under the topic of «Examples and Constructions» because it is sufficient to present 10 line segments from which it is impossible to choose any three to form a triangle.

Consider the set of the following 10 line segments: $1, 2, 4, 8, 16, 32, 64, 128, 256$, and 512. Let's assume a triangle with sides $a < b < c$ does exist. Given the values of the proposed lengths, $a < \frac{c}{2}$ and $b \leqslant \frac{c}{2}$. Adding these inequalities, we get $a + b < c$, which contradicts the triangle inequality. □

Inequalities in triangles are a powerful tool, especially for proving the impossibility of certain geometric constructions.

Problem set

Problem 1.5. On the base AC of the isosceles triangle ABC, a point D is chosen, and on the extension of AC beyond vertex C, a point E is chosen such that $AD = CE$. Prove that $BD + BE > AB + BC$.

Problem 1.6. The diagonals AC and BD of the isosceles trapezoid $ABCD$ intersect at point O. It is also known that a circle can be inscribed in the trapezoid. Prove that $\angle BOC > 60^\circ$.

Problem 1.7. (OMGO — 2019.7.5) A rectangular sheet of paper was folded along the diagonal. Can the perimeter of the resulting pentagon be equal to the perimeter of the original sheet?

Problem 1.8. (MMO — 1997.9) In a triangle, one side is three times smaller than the sum of the other two sides. Prove that the angle opposite to this side is the smallest angle in the triangle.

Problem 1.9. (MMG — 2012.9) On a plane, given a square and a point P, is it possible for the distances from point P to the vertices of the square to be 1, 1, 2, and 3?

Problem 1.10. (TOT — 2016.8.9) Prove that the sum of the lengths of any two medians of an arbitrary triangle

a) is not greater than $\frac{3P}{4}$, where P is the perimeter of the triangle;

b) is not less than $\frac{3p}{4}$, where p is the semiperimeter of the triangle.

Problem 1.11. (MMO — 2002.8.5) In triangle ABC, medians AD and BE intersect at point M. Prove that if angle AMB

a) is right;

b) is acute,

then $AC + BC > 3AB$.

Problem 1.12. (ARSO — 1993.9.2) The segments AB and CD of length 1 intersect at point O, and $\angle AOC = 60°$. Prove that $AC + BD \geq 1$.

Problem 1.13. (LEO — 2017.3) Diagonals of a convex quadrilateral $ABCD$ intersect at point E. It is known that $AB = BC = CD = DE = 1$. Prove that $AD < 2$.

Problem 1.14. (LEO — 2016.3) Given an equilateral triangle ABC, point D is chosen on the extension of side AB beyond point A, point E on the extension of BC beyond point C, and point F on the extension of AC beyond point C such that $CF = AD$ and $AC + EF = DE$. Find the angle $\angle BDE$.

Problem 1.15. The perpendicular bisector of side AB of triangle ABC intersects side AC at point K, and point K divides the broken line ACB into two parts of equal length. Prove that triangle ABC is isosceles.

Problem 1.16. (TOT — 2015.8-9) Prove that in any cyclic polygon, there exist three sides that can form a triangle.

Problem 1.17. (TOT — 1985.7-8) A quadrilateral is inscribed in a rectangle (with one vertex on each side of the rectangle). Prove that the perimeter of the quadrilateral is not less than twice the diagonal of the rectangle.

Problem 1.18. (TOT — 1989.7-8) In triangle ABC, the median AM is drawn. Can the radius of the inscribed circle of triangle ABM be exactly twice the radius of the inscribed circle of triangle ACM?

Problem 1.19. (TOT — 2006.8-9) A segment of length 1 is divided into 11 segments, the length of each of which does not exceed a. For what values of a can we say that any three resulting segments can be used to form a triangle?

Problem 1.20. (TOT — 2014.8-9) Prove that among 100 red, 100 yellow, and 100 green sticks, where any three sticks of different colors can form a triangle, there must be a color for which any three sticks of that color can form a triangle.

Problem 1.21. (TOT — 2015.8-9) Given are N right-angled triangles. For each triangle, one leg was chosen, and the sum of their lengths was found. Then the sum of the lengths of the remaining legs was found, and finally, the sum of the lengths of all hypotenuses was found. It turned out that these three found numbers are the side lengths of a certain right-angled triangle. Prove that for all original triangles, there is a consistent ratio between the longer leg and the shorter leg, if

а) $N = 2$;

б) N — is any number bigger than 1.

Problem 1.22. (TOT — 1991.8-9) The sides AB, BC, CD, and DA of quadrilateral $ABCD$ are respectively equal to the sides $A'B'$, $B'C'$, $C'D'$, and $D'A'$ of quadrilateral $A'B'C'D'$. Additionally, it is known that $AB \parallel CD$ and $B'C' \parallel D'A'$. Prove that both quadrilaterals are parallelograms.

Problem 1.23. (Sharygin — 2012.8.5) Does there exists a convex quadrilateral and a point P inside it such that the sum of the distances from P to the vertices is greater than the perimeter of the quadrilateral?

Problem 1.24. (MMO — 2011.11.3) In an isosceles triangle ABC, point D is chosen on the base BC, and points E and M are chosen on the side AB such that $AM = ME$, and the segment DM is parallel to side AC. Prove that $AD + DE > AB + BE$.

Problems for further training

Problem .1. Prove that the sum of the diagonals of any convex quadrilateral is less than its perimeter but greater than half of its perimeter.

Problem .2. The angle at vertex M of triangle ACM is $60°$. Prove that

$$AM + MC \leqslant 2 \cdot AC.$$