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COMPETITIVE ARITHMETICS

Mathematical Competitions.
Levels A1-A2

1. Competitive Arithmetics

A stylized illustration of a bear sitting and holding a large abacus. The bear and the abacus are rendered in a geometric, low-poly style with various hatching and shading patterns. The bear is facing left, and the abacus is positioned in front of its chest. The entire illustration is framed by a decorative border of geometric shapes and patterns.

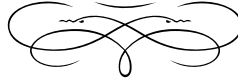
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Mathematical Competitions.
Levels A1-A2
Book 1. Competitive Arithmetics

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Dedication



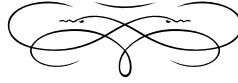
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To our esteemed colleagues, whose invaluable contributions made the realization of this book possible. A special acknowledgment goes to Michael Podaev for his exceptional support and insights.

“

Dedicated to our students, who discovered the joy of arithmetics through the pages of this book. Your enthusiasm and engagement have truly made the journey enjoyable.

Introduction



“ Once you ask me whom I love more — you or mathematics.
I will reply, «Let’s suppose, mathematics.»
And you will leave, never knowing that it was a proof by contradiction.

—Popular joke

Introduction to the series

Begin your preparation for Competition Mathematics with our carefully crafted series. These books are designed to inspire a love for problem-solving and foster critical thinking. They are ideal for both budding mathematicians and passionate enthusiasts.

Inside, you will find a wide range of challenges, puzzles, and problems. Each one is selected to enhance your mathematical abilities. Experience the challenge of solving complex equations and gain confidence by deciphering complex geometric puzzles. Every book has engaging content to stimulate your mind and expand your skills.

If you're preparing for regional competitions, national tournaments, or simply want to deepen your mathematical knowledge, this series is an invaluable resource. The books provide clear explanations, strategic insights, and numerous practice problems. They aim to build your confidence and equip you with the skills needed to tackle any mathematical challenge.

While school mathematics forms a foundation, this series goes beyond it without requiring advanced knowledge to understand the material. Our course covers a wide range of topics, reflecting the diverse nature of Olympiad problems. Solving a geometry problem may require knowledge of combinatorics, while a number theory problem might involve understanding invariants and the pigeonhole principle.

Olympiad problems are generally not restricted to specific grade levels, making these books suitable for high school students. Some of the problems included have been featured in the final stages of national math Olympiads for higher grades. The goal is to demonstrate how to solve problems using straightforward and elegant methods, avoiding unnecessary complexity.

We have categorized competition mathematics into levels similar to the international standards used for foreign language proficiency. This approach is based on the concept of the «language» of competition mathematics. Traditional grade-based divisions are often outdated, as understanding a topic might only require elementary-level math. Moreover, the topics in these books are interconnected. Without a grasp of a topic at level A1, understanding its expanded form at level A2 can be challenging.

Here's what to expect at each level:

Let's use an analogy with foreign languages:

Level A1. You understand (generally) foreign speech and can talk about family, activities, hobbies, travels, weather, and buying things. In short, the standard tourist set. Can you conjugate basic verbs and be familiar with different tenses? The question «How are you?» doesn't stump you. Congratulations! You have a good A1 level! This is enough for survival.

Similarly, in olympiad math — you can «survive» at beginner-level olympiads, understand what is required in problems, and formulate solutions. You likely won't need math knowledge beyond seventh grade to understand topics at this level. (The problem might be from an 11th-grade olympiad, but the solving method remains the same.)

At level A2, you can discuss preferences in art, cultural differences, and main social trends, etc. You form complex sentences («This is Peter, whose dad works at the bank. I've already told you about him»), can write to a friend on Facebook, describe a vacation, and understand the essence of any conversation in the language.

You can recognize and solve middle-level Olympiad problems. You will be able to avoid common mistakes and present your solutions effectively. Topics at this level typically require knowledge up to the eighth grade.

This series of books generally covers levels A1 and A2 of competition math: you will understand any problem from most competitions, formulate your solution, and even change the solution of ChatGPT to match the real competition problem. However, you are still far from being a native speaker.

What is in these Books?

This series uses a proof-based approach to problem-solving, which is usually reserved for advanced levels in countries like the USA and the UK. However, this method helps build a solid foundation in mathematics.

Each chapter is divided into four parts:

1. The first part covers the theoretical background and provides detailed solutions to typical problems.
2. The second part presents a problem set labeled by source. Olympiad problems are marked with notations like «Year.Grade/Round.Number.» For example, «ACM 2016.10A.5» is the fifth problem from the 10th-grade 10A variant of the ACM Olympiad 2016. Grade numbering may vary between countries, so adjust accordingly. Non-grade-specific Olympiads, like AIME, are marked by version (I or II) instead of grade.

You will encounter many problems from the Russian Olympiads (a country with a strong tradition in Olympiad mathematics) and various US mathematical competitions (such as AMC and AIME). We sincerely recommend not only finding the correct answer from the given AMC options but also approaching these problems from a proof-based perspective.

The problem number usually provides a sense of difficulty; generally, a higher number indicates a more challenging problem. However, this labeling doesn't always apply to some «independent» Olympiads, which can sometimes confuse genuine Olympiad participants.

3. The third part includes problems for independent solving, with some original problems introduced here.
4. Solutions are found in the fourth part.

The series consists of the following books:

1. Competitive Arithmetics
2. Ideas and Methods
3. Introduction to Discrete Mathematics
4. Introduction to Competitive Geometry
5. Competitive Number Theory
6. Competitive Geometry

This series is designed for both experienced Olympiad participants and newcomers to mathematical problem-solving. It offers a journey where theory and application meet, providing a rewarding experience. Welcome to a unique math adventure!

Introduction to this Book

«Competitive Arithmetics» is the first book of the six-part series, aimed at enhancing the mathematical capabilities of middle school students, particularly those engaged in competitive math. This book is crafted to challenge students with problems that surpass typical grade-level content, incorporating both algebraic and arithmetic elements.

Each chapter covers essential concepts such as arranging arithmetic signs, solving cryptarithms, understanding inverse operations, and addressing simultaneous equations. Even if students are already familiar with these topics, revisiting the theory is beneficial as it may refresh forgotten methods and introduce elegant new approaches. Dr. Babicheva's approach ensures that students build a strong foundation in arithmetic while developing the logical reasoning skills crucial for excelling in mathematics competitions.

List of Competitions Used in this Book

- «Математический праздник», in English mean «Mathematical Festival». We note it in the book as «MF». The official site (in Russian) is <https://olympiads.mccme.ru/matprazdnik/>
- Городская устная математическая олимпиада для 6–7 классов, mean «City Oral Mathematical Olympiad for 6–7 grades». We note it in the book as «COM». The official site (in Russian) is <https://olympiads.mccme.ru/ustn/>
- Турнир городов, mean «Tournament of Towns». We note it in the book as «TOT». The official site is <https://www.turgor.ru/en/>
- Школьный этап Всероссийской олимпиады школьников, mean «first stage of All-Russian School Olympiad». We note it in the book as «1ARSO». The official site (in Russian) is <https://vserosolimp.edsoo.ru/>
- Муниципальный этап Всероссийской олимпиады школьников, mean «second stage of All-Russian School Olympiad». We note it in the book as «2ARSO». The official site (in Russian) is <https://vserosolimp.edsoo.ru/>
- American Mathematics Competitions. We note it in the book as «AMC». The official site is <https://maa.org/math-competitions>
- American Invitational Mathematics Examination. We note it in the book as «AIME». The official site is <https://www.maa.org/math-competitions>
- Mock American Invitational Mathematics Examination. We note it in the book as «Mock AIME».
- American High School Mathematics Examination. We note it in the book as «AHSME». The official site is <https://www.maa.org/math-competitions/amc>
- American Junior High School Mathematics Examination. We note it in the book as «AJHSME». The official site is <https://www.maa.org/math-competitions/amc>
- Московская математическая олимпиада, mean «Moscow Mathematical Olympiad». We note it in the book as «ММО». The official site (in Russian) is <https://mmo.mccme.ru/>
- Региональный этап олимпиады им. Леонарда Эйлера, mean Regional stage of Leonhard Euler Math Olympiad. We note it in the book as «ER». The official site (in Russian) is <http://matol.ru/>
- Объединённая межвузовская математическая олимпиада школьников, mean United Interuniversity Mathematical Olympiad for schoolchildren. We note it in the book as «ОММО». The official site (in Russian) is <https://olympiads.mccme.ru/ommo/>
- Cyprus Mathematical Olympiad. We note it in the book as «Cyprus MO».

The official site is <https://www.cms.org.cy/en/activities/cyprus-mathematical-olympiad>

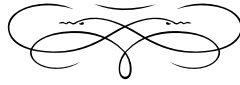
- University of Northern Colorado Math Contest. We note it in the book as «UNCO Math Contest». The official site is <https://uncmathcontest.wordpress.com/>
- iTest. We note it in the book as «iTest». The non official site is <https://artofproblemsolving.com/wiki/index.php/ITest>
- Junior Mathematical Olympiad. We note it in the book as «JMO». The official site is <https://ukmt.org.uk/junior-challenges/junior-mathematical-olympiad>
- Primary Mathematics World Contest. We note it in the book as «PMWC». The non official site is <https://www.txst.edu/mathworks/PMWC/previous-pmwc-tests.html>
- UNM-PNM Statewide High School Mathematics Contest. We note it in the book as «UNM-PNM». The official site is <http://mathcontest.unm.edu/>
- Indonesian Mathematics Olympiad. We note it in the book as «Indonesian MO». The official site is <http://tomi.or.id/>
- Турнир им. Ломоносова, mean «Lomonosov Tournament». We note it in the book as «LT». The official site (in Russian) <https://turlom.olimpiada.ru/>
- Турнир Архимеда, mean «Archimedes Tournament». We note it in the book as «AT». The official site (in Russian) is <http://www.arhimedes.org/>
- Олимпиада «Ломоносов», mean «Lomonosov Competition», competition of Moscow State University. We note it in the book as «Lomonosov». The official site (in Russian) is <https://olymp.msu.ru/>
- Курчатов, mean «Kurchatov Competition». We note it in the book as «Kurchatov». The official site (in Russian) is <https://olimpiadakurchatov.ru/>
- Московская математическая регата, mean «Moscow mathematical regatta». We note it in the book as «MMG». The official site (in Russian) is <https://olympiads.mccme.ru/regata/>
- «Покори Воробьёвы горы», mean «Conquer Vorobyovy Gory», competition of MSU. We note it in the book as «PVG». The official site (in Russian) is <https://pvg.mk.ru/>



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Arrange the arithmetic signs



“

During a math lesson, Leo is asked the following questions:

- What is 3 plus 4?
- 9.
- What about 11 plus 5?
- 22.
- You're doing such a bad job. Who's your dad?
- The waiter, he taught me.

–Popular joke

Theory and Practice

You have most likely encountered problems that involve arranging signs, numbers, and brackets as early as primary school. So, you must not be surprised that these types of problems very often appear in various mathematics olympiads and puzzle competitions. Let's dig deeper into this topic and start with a classic warm-up problem that you frequently see at mathematics workshops.

Example 1.1. Express every number from 1 to 10, using only four fours and any number of arithmetic operators or brackets.

Solution: For this problem, there are numerous ways to represent each number. For example,

$$1 = 4 \div 4 + 4 - 4 = 44 \div 44 = (4 + 4 - 4) \div 4;$$

$$2 = 4 \div 4 + 4 \div 4;$$

$$3 = (4 + 4 + 4) \div 4;$$

$$4 = 4 + (4 - 4) \div 4;$$

$$5 = (4 \cdot 4 + 4) \div 4;$$

$$6 = 4 + (4 + 4) \div 4;$$

$$7 = 44 \div 4 - 4;$$

$$8 = 4 + 4 + 4 - 4;$$

$$9 = 4 + 4 + 4 \div 4;$$

$$10 = (44 - 4) \div 4.$$

□

Sometimes, in order to solve a problem, you should first attempt to simplify it.

Example 1.2. (Lomonosov — 2016.5–8.5): Replace the asterisks in the following expression with multiplication and division signs to make the equality true.

$$1 * 3 * 3^2 * 3^4 * 3^8 * 3^{16} * 3^{32} * 3^{64} = 3^{99}.$$

Solution: Note that all numbers in this problem are powers of three. Let's recall the rules for multiplication and division of numbers with the same exponent base: $a^x \cdot a^y = a^{x+y}$, $a^x \div a^y = a^{x-y}$. Thus, this problem can be reduced to the following:

Insert addition and subtraction signs instead of asterisks in the expression

$$0 * 1 * 2 * 4 * 8 * 16 * 32 * 64 = 99.$$

So that the equality becomes true.

The number 64 can only be preceded by the «+» sign. This is because otherwise, if it were subtracted, the sum of all remaining numbers (63) would not exceed 64, making the result negative. This contradicts our equality requirement, since 99 is a positive number.

Thus, the problem is now reduced to $0 * 1 * 2 * 4 * 8 * 16 * 32 + 64 = 99$. After subtracting 64 from both sides, we get

$$0 * 1 * 2 * 4 * 8 * 16 * 32 = 35.$$

By applying the same reasoning to the number 32, we obtain

$$0 * 1 * 2 * 4 * 8 * 16 + 32 = 35, \quad 0 * 1 * 2 * 4 * 8 * 16 = 3.$$

Moving forward, there should also be a plus sign before the number 16. Otherwise, the sum of the remaining numbers (15) will never produce a positive result.

Thus,

$$0 * 1 * 2 * 4 * 8 + 16 = 3, \quad 0 * 1 * 2 * 4 * 8 = -13.$$

Using similar reasoning, we deduce that there should be a minus sign before 8. Therefore,

$$0 * 1 * 2 * 4 - 8 = -13, \quad 0 * 1 * 2 * 4 = -5.$$

Similarly, we find that there is a minus sign before 4:

$$0 * 1 * 2 - 4 = -5, \quad 0 * 1 * 2 = -1,$$

from which the solution $0 + 1 - 2 = -1$ is obvious.

Hence, the answer to this problem is

$$1 \cdot 3 \div 3^2 \div 3^4 \div 3^8 \cdot 3^{16} \cdot 3^{32} \cdot 3^{64} = 3^{99}.$$

□

Problem set

Problem 1.1. (MF – 2000.6.1): Replace the symbols «*» with «+» or «-» in the expression:

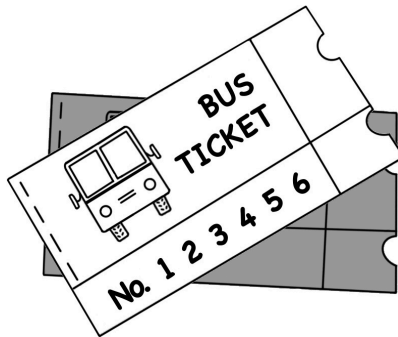
$$* 1 * 2 * 4 * 8 * 16 * 32 * 64 = 27.$$

to make the equation correct.

Problem 1.2. (MF – 1997.6.1): Leo laid out an addition problem with cards featuring digits and then swapped two cards. As you can see, the equality was violated. Which cards did Leo swap?

$$\begin{array}{r} + 314159 \\ + 291828 \\ \hline 585787 \end{array}$$

Problem 1.3. (MF – 1991.6.1;7.1): A bus ticket is considered lucky if you can place the signs of four arithmetic operations and parentheses between its digits so that the value of the resulting expression is equal to 100. Is the ticket number 123456 lucky?



Problem 1.4. (MF – 2014.7.1): Using three different arithmetic operation signs and the equality sign, obtain a correct equation from today's date: 16032014.

Problem 1.5. (COM – 2004.6.4): Leo was laying out mathematical equations with matches. He «wrote down» digits as follows:

1234567890

When Leo got distracted, Max rearranged exactly one match in each digit of one of John's correctly written equations, resulting in:

$$36 + 59 = 93$$

Restore the Original Equation.

Problem 1.6. (MF – 1995.7.4): Use parentheses to make the equation correct:

$$1 - 2 \cdot 3 + 4 + 5 \cdot 6 \cdot 7 + 8 \cdot 9 = 1995.$$

Problem 1.7. (MF – 2003.7.1): Use parentheses and arithmetic operation signs to make the equation correct:

$$\frac{1}{2} \frac{1}{6} \frac{1}{6009} = 2003.$$

Skill Assessment Problems

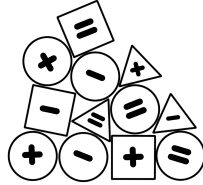
Skill Assessment Problem 1.1. Insert «+» or «−» between some of the seven consecutive nines to obtain the number 1989:

$$9 \ 9 \ 9 \ 9 \ 9 \ 9 \ 9 = 1989.$$

If there is no sign between certain digits, for example, between two nines 9 9, consider this notation as a multi-digit number.

Skill Assessment Problem 1.2. Using the digit 7 exactly four times, along with arithmetic operations and parentheses, represent all integers from 0 to 10.

Solutions to Skill Assessment Problems



Solution to Problem 1.1: The symbols can be placed as follows: $999 + 999 - 9 = 1989$. □

Solution to Problem 1.2: Represent each of the required numbers:

$$0 = 7 - 7 + 7 - 7;$$

$$1 = 77 \div 77;$$

$$2 = 7 \div 7 + 7 \div 7;$$

$$3 = (7 + 7 + 7) \div 7;$$

$$4 = 77 \div 7 - 7;$$

$$5 = 7 - (7 + 7) \div 7;$$

$$6 = (7 \cdot 7 - 7) \div 7;$$

$$7 = 7 - (7 - 7) \div 7;$$

$$8 = (7 \cdot 7 + 7) \div 7;$$

$$9 = 7 + (7 + 7) \div 7;$$

$$10 = (77 - 7) \div 7.$$

□

Are you enjoying the book so far?

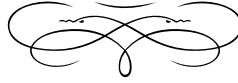
Your quick, honest review helps us immensely and only takes a minute!

Use your phone to scan the QR code and go directly to the Amazon review page.



Thank you for taking the time—it truly means the world to us!

Cryptarithms



“

Leo came home and said:
«I don't know if I can trust my Math teacher. Yesterday, he said that $10 = 6 + 4$, and today, he said that $10 = 7 + 3$.».

—Popular joke

Theory and Practice

A classic cryptarithm, also known as an alphametic, is a type of mathematical puzzle where digits in a numerical expression are replaced by letters. Identical digits are represented by the same letter, while distinct digits are represented by different letters. To solve these puzzles, you must determine which digits (ranging from 0 to 9) correspond to each letter in the equation. This process often involves deciphering the original expression, exploring all possible digit assignments, and explaining why other solutions are not feasible—unless the problem simply asks for any valid solution. In mathematics competitions, if you are unsure whether there is a need to explore all possible scenarios, it is generally advisable to solve the problem for all potential cases or seek clarification from the event organizers.

These are four general rules that are true for alphametics:

1. Identical digits are represented by identical letters, and distinct digits are represented by distinct letters.
2. The arithmetic expression formed after replacing letters with digits must be mathematically accurate.
3. Numbers cannot commence with 0; for instance, the number 0123 is not allowed.
4. Unless stated otherwise, the problems are in base 10, meaning that letters substitute for some or all of the 10 digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Let's consider an example of a typical alphametic problem. To avoid confusion between digits and letters, all letters are written in bold font.

Example 2.1. Solve the following Cryptarithm: $\mathbf{AAA} - \mathbf{AA} - \mathbf{A} = \mathbf{BB}$.

Solution: By rewriting a part of the expression in the column form, we see that

$$\begin{array}{r} \mathbf{A A A} \\ - \quad \mathbf{A A} \\ \hline \mathbf{A 0 0} \end{array}$$

Thus, the original cryptarithm can be reduced to the cryptarithm $\mathbf{A00} - \mathbf{A} = \mathbf{BB}$.

Recall that $A00 = A \cdot 100$, therefore $100 \cdot A - A = BB$. By simplifying, we get $99 \cdot A = BB$.

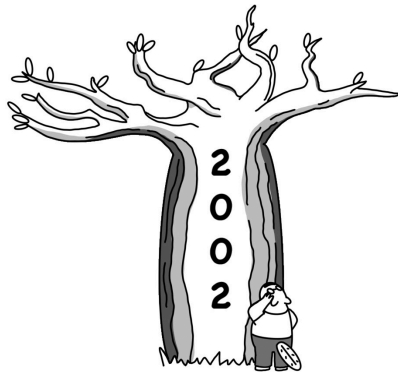
If $A \geq 2$, then $99 \cdot A$ is at least $99 \cdot 2 = 198$, it means that the right-hand side of the expression consists of at least three digits. However, the number BB has two digits, therefore $A = 1$, giving $99 \cdot 1 = 99$.

Therefore, the solution to this cryptarithm is

$$111 - 11 - 1 = 99.$$

□

Now, let's consider an example of a multiplication cryptarithm.



Example 2.2. Solve the Cryptarithm $BAO \cdot BA \cdot B = 2002$.

Solution: By examining the leading digits of the numbers, we notice that they are the same. Let's try to estimate the value of B . If $B = 1$, then $BAO \cdot BA \cdot B$ is between $100 \cdot 10 \cdot 1 = 1000$ and $199 \cdot 19 \cdot 1 = 3781$.

If B is greater than or equal to 2, then $BAO \cdot BA \cdot B$ is greater than or equal to $200 \cdot 20 \cdot 2 = 8000$, which exceeds the required value.

Thus, the cryptarithm can be represented as $1\mathbf{A}\mathbf{O} \cdot 1\mathbf{A} \cdot 1 = 2002$, meaning $1\mathbf{A}\mathbf{O} \cdot 1\mathbf{A} = 2002$.

Now, let's evaluate the second digit, i.e., \mathbf{A} . If $\mathbf{A} = 2$, then $1\mathbf{A}\mathbf{O} \cdot 1\mathbf{A}$ is between $120 \cdot 12 = 1440$ and $129 \cdot 12 = 1548$, which is too small.

If $\mathbf{A} = 3$, then $1\mathbf{A}\mathbf{O} \cdot 1\mathbf{A}$ is between $130 \cdot 13 = 1690$ and $139 \cdot 13 = 1807$, which is again too small.

If $\mathbf{A} = 4$, then $1\mathbf{A}\mathbf{O} \cdot 1\mathbf{A}$ is between $140 \cdot 14 = 1960$ and $149 \cdot 14 = 2086$, falling within the required range.

If \mathbf{A} has a value of 5 or more, then $1\mathbf{A}\mathbf{O} \cdot 1\mathbf{A}$ is no less than $150 \cdot 15 = 2250$, which exceeds the required value.

Thus, the cryptarithm can be represented as $14\mathbf{O} \cdot 14 \cdot 1 = 2002$, meaning $14\mathbf{O} = 2002 \div 14 = 143$, i.e., $\mathbf{O} = 3$.

The answer to the cryptarithm is $143 \cdot 14 \cdot 1 = 2002$. □

Example 2.3. Solve the Cryptarithm: $\mathbf{T} - \mathbf{W} - \mathbf{O} = \mathbf{T} \div \mathbf{W} : \mathbf{O} = 2$

Solution: This cryptarithm is relatively simple and can be solved through trial and error. First, we can transform the original cryptarithm into the system of two simultaneous equations, where the first equation is $\mathbf{T} - \mathbf{W} - \mathbf{O} = 2$ and the second is $\mathbf{T} : \mathbf{W} \div \mathbf{O} = 2$. From the second equation, it follows that $\mathbf{T} = 2 \cdot \mathbf{W} \cdot \mathbf{O}$. Thus, \mathbf{T} is an even digit.

Let's consider all the possible cases.

If $\mathbf{T} = 0$, then the condition $\mathbf{T} \div \mathbf{W} \div \mathbf{O} = 2$ is not satisfied.

If $\mathbf{T} = 2$, then $2 = 2 \cdot \mathbf{W} \cdot \mathbf{O}$, so $\mathbf{W} \cdot \mathbf{O} = 1$, i.e., $\mathbf{W} = \mathbf{O} = 1$. However, since different letters correspond to different digits, we encounter a contradiction.

If $\mathbf{T} = 4$, then $4 = 2 \cdot \mathbf{W} \cdot \mathbf{O}$, so $\mathbf{W} \cdot \mathbf{O} = 2$, i.e., \mathbf{W} and \mathbf{O} are 1 and 2. Substituting this into the first equation, $4 - 2 - 1 = 2$, we encounter a contradiction.

If $\mathbf{T} = 6$, then $6 = 2 \cdot \mathbf{W} \cdot \mathbf{O}$, so $\mathbf{W} \cdot \mathbf{O} = 3$, i.e., \mathbf{W} and \mathbf{O} are 1 and 3. Substituting this into the first equation, $6 - 3 - 1 = 2$, we get a mathematically accurate expression. Hence, the two solutions to the cryptarithm are $\mathbf{T} = 6$, $\mathbf{W} = 1$, $\mathbf{O} = 3$ and $\mathbf{T} = 6$, $\mathbf{W} = 3$, $\mathbf{O} = 1$.

If $\mathbf{T} = 8$, then $8 = 2 \cdot \mathbf{W} \cdot \mathbf{O}$, so $\mathbf{W} \cdot \mathbf{O} = 4$, i.e., since they are different, \mathbf{W} and \mathbf{O} are 1 and 4. Substituting this into the first equation, $8 - 4 - 1 = 2$, we encounter a contradiction.

Answer: $\mathbf{T} = 6$, $\mathbf{W} = 1$, $\mathbf{O} = 3$, or $\mathbf{T} = 6$, $\mathbf{W} = 3$, $\mathbf{O} = 1$. □

Problem Set

Problem 2.1. (AT – 2013.1): Substitute Digits for Letters to Make the Equation True:

$$22 + \mathbf{TURN} + \mathbf{IR} = 2013.$$

You are not supposed to find all solutions.

Problem 2.2. (MF – 2001.6.1): Solve the Cryptarithm:

$$\mathbf{AH} \cdot \mathbf{UH} = 2001.$$

Problem 2.3. (COM – 2011.6.1): Solve the cryptarithm: $\mathbf{LETO} + \mathbf{LES} = 2011$. Find all possible solutions.

Problem 2.4. (COM – 2006.6.1): In the following addition example, all ten digits are hidden under the asterisks, each appearing exactly once. Find at least one such example.

$$\begin{array}{r} * \\ + * * \\ * * * \\ \hline * * * * \end{array}$$

Problem 2.5. (COM – 2010.7.1): Replace Letters with Digits in the Cryptarithm

$$\mathbf{A} + \mathbf{B} = \mathbf{C} - \mathbf{B} = \mathbf{D} \cdot \mathbf{B} = \mathbf{E} - \mathbf{F} = \mathbf{G}$$

so that all the equalities become true. In this case, the same letters correspond to the same digits, and different letters should correspond to different digits. *Find all solutions to the cryptarithm.*

Problem 2.6. (COM – 2006.7.1): Find all solutions to the cryptarithm and prove that there are no other solutions:

$$\mathbf{AR}^{\mathbf{A}} = \mathbf{RAT}.$$

(Different letters represent different digits; the same letter represents the same digit.)

Problem 2.7. (MF – 2013.6.2): (We are not the poets, so the rhyme was lost in the translation)

Here is a fairly simple cryptarithm:
EH is four times greater than **OI**,
AI is four times greater than **OH**.
 Find the sum of all four.

Problem 2.8. (MF – 2000.6.3): The code of the combination lock is a two-digit number. Jean forgot the code but remembers that the sum of its digits, added to their product, equals the number itself. Write down all possible code variants, so that Jean can quickly open the lock.

Problem 2.9. (COM – 2002.6.3): Solve the Cryptarithm: **TIK + TAK = AKT**.

Problem 2.10. (MF – 2011.6.4): Find all the Solutions to the Cryptarithm

$$\mathbf{I + HE + HE + HE + HE + HE + HE + HE + HE = US.}$$

Problem 2.11. (MF – 2014.7.3): Replace the letters in the word **MATEMATIKA** with digits and addition/subtraction signs to create a numerical expression equal to 2014. (Repeated letters represent repeated digits or signs, and different letters represent different digits. It is sufficient to provide only one example.)

Problem 2.12. (COM – 2008.6.8): Each letter in the words **HI** and **TEENS** corresponds to a certain digit, such that identical digits correspond to identical letters, while different digits correspond to different letters.

It is known that $\mathbf{H \cdot I = T \cdot E \cdot E \cdot N \cdot S}$ and $\mathbf{H + I = T + E + E + N + S}$. What is $\mathbf{H \cdot I + T \cdot E \cdot E \cdot N \cdot S}$?

Problem 2.13. (AT – 2015.5): John is creating another cryptarithm. To complete the work, he wants to choose values for the letters so that the number **PANORAMA** is

divisible by the number **PANAMA** without a remainder. Will he be able to do this? (In the rebus, repeated letters should represent repeated digits, and different letters should represent different digits.)

Problem 2.14. (COM – 2016.7.6): Mrs. Owless wrote the equation

$$\mathbf{MA \cdot TE \cdot MA \cdot TI \cdot KA = 2016000}$$

and asked Esther to replace repeated letters with repeated digits and different letters with different digits to make the equation true. Does Esther have a chance to complete the task, or are there no such replacements?

Problem 2.15. (Doubly True Roman Numerals by Frank Mrazik) Solve the following cryptarithms:

- $\mathbf{I + I + I + I + I + I + I + II = IX}$
- $\mathbf{I + I + I + II + II + IV = XI}$
- $\mathbf{VI + VI = XII}$
- $\mathbf{I + II + IX = XII}$
- $\mathbf{I + IV + VII = XII}$
- $\mathbf{IV + IV + IV = XII}$
- $\mathbf{I + I + IV + VI = XII}$
- $\mathbf{I + II + IV + V = XII}$
- $\mathbf{I + III + IV + IV = XII}$
- $\mathbf{II + II + IV + IV = XII}$
- $\mathbf{II + III + III + IV = XII}$
- $\mathbf{I + I + I + III + VI = XII}$
- $\mathbf{I + I + II + II + VI = XII}$
- $\mathbf{I + I + II + III + V = XII}$
- $\mathbf{I + I + III + III + IV = XII}$
- $\mathbf{I + II + II + II + V = XII}$
- $\mathbf{I + I + I + II + II + V = XII}$
- $\mathbf{I + I + I + I + I + I + II + IV = XII}$
- $\mathbf{III + XI = XIV}$
- $\mathbf{II + VI + VI = XIV}$
- $\mathbf{III + V + VI = XIV}$
- $\mathbf{I + IV + IV + V = XIV}$

Problem 2.16. (JMO – 2021.B4): The letters **A**, **B**, and **C** stand for different non-zero digits. Find all possible solutions to the word-sum shown.

$$ABC + BCA + CAB = ABBC$$

Problem 2.17. (JMO – 2015.B6): The letters **a**, **b**, **c**, **d**, **e**, and **f** represent single digits and each letter represents a different digit. The letters satisfy the following equations:

$$a + b = d, \quad b + c = e, \quad \text{and} \quad d + e = f.$$

Find all possible values of **a**, **b**, **c**, **d**, **e**, and **f**.

Problem 2.18. (JMO – 2019.B1): In this word-sum, each letter stands for one of the digits 0-9 and represents the same digit each time it appears. Different letters stand for different digits. No number starts with 0.

$$JMO + JMO + JMO = IMO.$$

Find all possible solutions of the word-sum shown here.

Problem 2.19. (AMC – 2005.12A.8): Let **A**, **M**, and **C** be digits with

$$(100A + 10M + C)(A + M + C) = 2005.$$

What is **A**?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 2.20. (AMC – 2003.12A.5): The sum of the two 5-digit numbers **AMC10** and **AMC12** is 123422. What is **A + M + C**?

- (A) 10 (B) 11 (C) 12 (D) 13 (E) 14

Skill Assessment Problems

Skill Assessment Problem 2.1. Find all solutions to the cryptarithm:

$$\mathbf{ARKA} + \mathbf{RKA} + \mathbf{KA} + \mathbf{A} = 2014.$$

Distinct letters represent distinct digits, and repeated letters represent repeated digits.

Skill Assessment Problem 2.2. Solve the Following Cryptarithm: $\mathbf{MARRY} \cdot 5 = \mathbf{HAPPY}$.

Solutions to Skill Assessment Problems

Solution to Problem 2.1: Let's consider all possible values for the letter **A**. Since the condition implies that $\mathbf{ARKA} < 2014$, **A** can be equal to 1 or 2. The sum of the last digits of this example should be 4, so **A** can only be equal to 1.

Thus, the problem can be reduced to $1\mathbf{RK}1 + \mathbf{RK}1 + \mathbf{K}1 + 1 = 2014$. By dropping 4, dividing by 10, and removing 1 in the resulting hundreds, the problem is further simplified to $\mathbf{RK} + \mathbf{RK} + \mathbf{K} = 101$. From the last digits of the resulting equation, we deduce that $3 \cdot \mathbf{K}$ ends with 1, from which, recalling the multiplication table, we obtain $\mathbf{K} = 7$. Substituting, we get $\mathbf{R} = 4$. Thus, the given rebus has a unique solution:

$$1471 + 471 + 71 + 1 = 2014.$$

□

Solution to Problem 2.2: Firstly, we can see that $\mathbf{M} = 1$, because even $5 \cdot 20000 = 10000$ has more than 5 digits.

As $\mathbf{HAPPY} : 5$, we get that either $\mathbf{Y} = 0$ or $\mathbf{Y} = 5$. Let's consider the case $\mathbf{Y} = 0$.

In this case, $5 \cdot \mathbf{MARR} = \mathbf{HAPP}$, therefore, as $\mathbf{P} \neq 0$, we get $\mathbf{P} = 5$. Thus, $5 \cdot \mathbf{MARR} = \mathbf{HA} \cdot 100 + 55 \rightarrow \mathbf{MARR} = \mathbf{HA} \cdot 20 + 11$. As $\mathbf{HA} \cdot 20$ ends with 0, we deduce that $\mathbf{R} = 1$. However, since $\mathbf{M} = 1$, we got a contradiction.

Hence, $\mathbf{Y} = 5$. In this case, $5(\mathbf{MARR} \cdot 10 + 5) = \mathbf{HAPP} \cdot 10 + 5 \rightarrow \mathbf{MARR} \cdot 10 + 5 = \mathbf{HAPP} \cdot 2 + 1 \rightarrow \mathbf{MARR} \cdot 10 + 4 = \mathbf{HAPP} \cdot 2 \rightarrow \mathbf{MARR} \cdot 5 + 2 = \mathbf{HAPP}$.

As $\mathbf{MARR} \cdot 5$ ends with either 0 or 5, \mathbf{HAPP} ends with either 2 or 7.

If $\mathbf{P} = 2$, then $\mathbf{MARR} \cdot 5 + 2 = \mathbf{HA} \cdot 100 + 22$. Therefore, $\mathbf{MARR} \cdot 5 = \mathbf{HA} \cdot 100 + 20$. And finally, $\mathbf{MARR} = \mathbf{HA} \cdot 20 + 4$.

Thus, $\mathbf{R} = 4 \rightarrow \mathbf{MA} \cdot 100 + 44 = \mathbf{HA} \cdot 20 + 4 \rightarrow \mathbf{MA} \cdot 100 + 40 = \mathbf{HA} \cdot 20 \rightarrow \mathbf{MA} \cdot 5 + 2 = \mathbf{HA} \rightarrow \mathbf{A} = 2$ or $\mathbf{A} = 7$.

As $\mathbf{P} = 2$, $\mathbf{A} = 7$, we get $(10\mathbf{M} + 7) \cdot 5 + 2 = 10\mathbf{H} + 7$. Thus, $(10\mathbf{M} + 7) \cdot 5 = 10\mathbf{H} + 5 \rightarrow 10\mathbf{M} + 7 = 2\mathbf{H} + 1 \rightarrow 10\mathbf{M} + 6 = 2\mathbf{H} \rightarrow 5\mathbf{M} + 3 = \mathbf{H} \rightarrow \mathbf{M} = 1, \mathbf{H} = 8$.

This gives $17445 \times 5 = 87225$

If $\mathbf{P} = 7$, then $\mathbf{MARR} \cdot 5 + 2 = \mathbf{HA} \cdot 100 + 77 \rightarrow \mathbf{MARR} \cdot 5 = \mathbf{HA} \cdot 100 + 75 \rightarrow \mathbf{MARR} = \mathbf{HA} \cdot 20 + 15 \rightarrow \mathbf{R} = 5$.

However, $\mathbf{Y} = 5$, which leads to a contradiction.

Therefore, the unique solution is $17445 \times 5 = 87225$.

□

Inverse Operations



“

In a Math class at school, the teacher asks:
«How much do you have to pay in total if you owe 155 dollars to the butcher, 100 dollars to the baker, 124 dollars at the store, and 210 dollars at the grocery store? Come on, Leo, answer!»
«I don't know, in such cases, we move to another neighborhood» .

—Popular joke

Theory and Practice

The so-called Inverse Operations method is a fairly straightforward way of solving mathematical problems. In this method, you are asked to find a certain value and know the result of applying a specific set of operations to that initial value. You are probably already familiar with this topic, as it is often covered in elementary school mathematics curricula. In some cases, the solution can be obtained without introducing any equations by applying the given sequence of steps in reversed order until we find the initial value.

Let's take a look at a classic problem that involves the inverse operations method.



Example 3.1. Jean went to the market to buy groceries. At each stall, he spent half of the coins he had, plus an additional half-coin. He visited five stalls on his way and spent all of his coins. How many coins did Jean have initially?

Solution: Let's look at what Jean did in a reversed order. After finishing shopping at the fifth stall, Jean had 0 coins remaining. Before spending half a coin, he had half a coin left. Before spending half of the remaining money, he had $\frac{1}{2} \cdot 2 = 1$ coin left. Thus, after shopping at the fourth stall, he had one coin left.

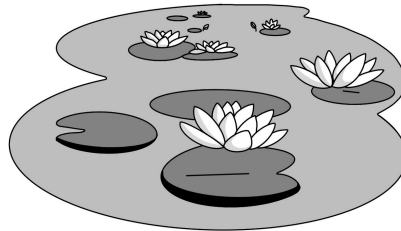
At the fourth stall, he had $1 + \frac{1}{2}$ coins left before spending half a coin. Therefore, before spending half of the remaining money, he had $(1 + \frac{1}{2}) \cdot 2 = 3$ coins left. Thus, after the third stall, he had 3 coins left.

At the third stall, he had $3 + \frac{1}{2}$ coins left before spending half a coin. Therefore, before spending half of the remaining money, he had $(3 + \frac{1}{2}) \cdot 2 = 7$ coins left. Thus, after the second stall, he had 7 coins left.

At the second stall, he had $7 + \frac{1}{2}$ coins left before spending half a coin. Therefore, before spending half of the remaining money, he had $(7 + \frac{1}{2}) \cdot 2 = 15$ coins left. Thus, after the first stall, he had 15 coins left.

Finally, at the first stall, before spending half a coin, he had $15 + \frac{1}{2}$ coins left. Hence, before spending half of the remaining money, he had $(15 + \frac{1}{2}) \cdot 2 = 31$ coins. Therefore, initially, Jean had 31 coins. \square

Let's consider one of the trickier, yet simple problem, that many students find challenging at first.



Example 3.2. The surface of Lake Beaver is covered with lotus plants. After each week, the area covered by lotus plants doubles, eventually covering the entire lake in 20 weeks. How many weeks will it take for half of the lake's area to be covered?

Solution: If you take a closer look at this problem, you will see that the solution involves only one step. Since the number of lotus plants doubles each week, in the last week, the area covered will go from half of the lake to the entire lake. Therefore, the answer to this problem is $20 - 1 = 19$ weeks. \square

Sometimes, a problem asks you to consider more than one variable.



Example 3.3. Alice and Beatrice were playing with gold coins. At the beginning of the game, Alice lost half of his coins to Beatrice, who then lost half of his remaining coins back to Alice, and finally, Alice lost half of his coins again. By the end of the game, Alice had 19 coins, and Beatrice had 47. How many coins did Alice have before the game started?

Solution: We know that the game ended when Alice had 19 coins and Beatrice had 47 coins.

Before that, Alice lost half of his coins, so he had twice as many before the last iteration. Thus, Alice had $19 \cdot 2 = 38$, and, as the total number of coins does not change in the course of the game, Beatrice had $47 - 19 = 28$ coins.

On the second-to-last iteration, Beatrice lost half of his coins, so before, he had twice as many. Thus, Beatrice had $28 \cdot 2 = 56$, and Alice had $38 - 28 = 10$ coins.

Finally, on the first iteration, Alice lost half of his coins, so before, he had twice as many. Thus, Alice had $10 \cdot 2 = 20$ coins, and Beatrice had $56 - 10 = 46$ coins. \square

Problem set

Problem 3.1. (MF – 1999.6.1): Several points were marked along a straight line. An additional point was also marked between every two adjacent points. This «compression» was repeated two more times (a total of 3 times). As a result, 113 points were marked on the straight line. How many points were initially marked?

Problem 3.2. (MF – 1996.7.2): Two pirates played with gold coins. After the game started, the first pirate lost half of his coins (gave them to the second), then the second pirate lost half of his coins to the first pirate, and finally, the first pirate lost half of his coins again. As a result, the first pirate ended up with 15 coins, and the second one had 33. How many coins did the first pirate have at the beginning of the game?

Problem 3.3. (MF – 1993.7.3): Solve the Equation:

$$1993 = 1 + 8 \div (1 + 8 \div (1 - 8 \div (1 + 4 \div (1 - 4 \div (1 - 8 \div x))))).$$

Problem 3.4. (LT – 1989.12): John had a basket full of trinkets. First, he met Anna and gave her half of his trinkets and half of a trinket. Then he met Banna and gave her half of the remaining trinkets and half of a trinket. After meeting Canna and giving her half of the remaining trinkets and half of a trinket, the basket was empty. How many trinkets did John have initially? (It was not possible to find out what trinkets were since there were none left by the end of the problem.)

Problem 3.5. (PMWC – 1997.T6): During a rebuilding project by contractors 'A,' 'B,' and 'C,' there was a shortage of tractors. The contractors lent each other tractors as needed. At first, 'A' lent 'B' and 'C' as many tractors as they each already had. A few months later, 'B' lent 'A' and 'C' as many as they each already had. Still later, 'C' lent 'A' and 'B' as many as they each already had. By then, each contractor had 24 tractors. How many tractors did each contractor originally have?

Problem 3.6. (AHSME — 1974.7): A town's population increased by 1,200 people, and then this new population decreased by 11%. The town now had 32 less people than it did before the 1,200 increase. What is the original population?

- (A) 1,200 (B) 11,200 (C) 9,968 (D) 10,000 (E) none of these

Skill Assessment Problems



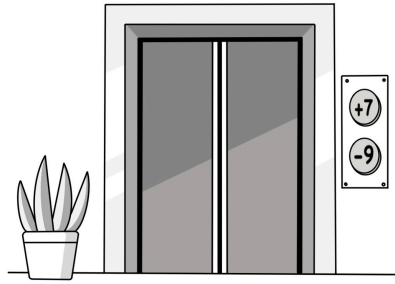
Skill Assessment Problem 3.1. Alice, Beatrice, and Clarice shared 120 walnuts. Initially, Alice gave Beatrice and Clarice as many walnuts as each of them had. Then Beatrice gave Clarice and Alice as many as each of them had. Finally, Clarice gave Alice and Beatrice as many as each of them had by that time. As a result, everyone got an equal share. How many walnuts did each of them have at the beginning?

Skill Assessment Problem 3.2. Three beavers had a total of 24 blackberry pies, with each having 3 pies less than their age. The youngest beaver was clever and suggested doing a pie exchange. First, he kept half of his pies and shared the rest equally between the other two. After that, the beaver in the middle kept half of his pies and shared the rest equally between the youngest and the oldest beavers. In the end, the oldest beaver did the same operation as the youngest suggested. After they had done this, it turned out that everyone had an equal number of pies. How old is each beaver?

Skill Assessment Problem 3.3. There are two operations that can be applied to some natural numbers:

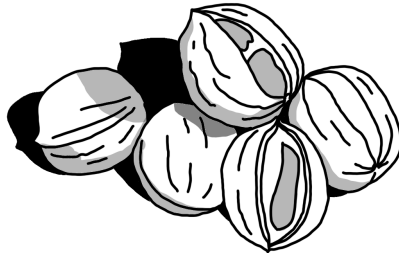
1. Multiplication by 2;
2. Rearrangement of its digits (but placing 0 in the first position is not allowed).

Is it possible to obtain the number 811 from the number 1 using these operations?



Skill Assessment Problem 3.4. Jack's house has 15 floors and a peculiar elevator with only two buttons: «+7» and «-9». Can Jack, using only the elevator, go from the 3rd floor to the 12th floor?

Solutions of Skill Assessment Problems



Solution to Problem 3.1: By the end of the exchange, everyone had an equal share, i.e., 40 each. One step earlier, Alice and Beatrice had half that amount, i.e., 20 each, and Clarice had 80.

One step earlier, Alice and Clarice had half again: Alice had 10, Sam had 40, and Beatrice had $20 + 40 + 10 = 70$ walnuts.

So, initially, Beatrice and Clarice had half of that: 35 and 20 respectively, and Alice had $10 + 35 + 20 = 65$ walnuts. \square

Solution to Problem 3.2: Since there were 24 pies in total, in the end, each beaver got $24 : 3 = 8$ pies. Before the oldest beaver shared his pies with the other two, he had $8 \cdot 2 = 16$ pies. So, he gave each of the other two $8 : 2 = 4$ pies, meaning that each of them had $8 - 4 = 4$ pies before that. Similar reasoning for the other two steps reveals that before the middle beaver shared his pies, he had 8 pies, the youngest had 2 pies, and the oldest had 14 pies. In the beginning, the oldest beaver had 13 pies, the middle beaver had 7, and the youngest had 4. Now, it's easy to calculate the age of each beaver.

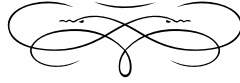
Answer: 16, 10, 7. \square

Solution to Problem 3.3: Let's assume that it is possible to obtain the number 811. Then, the last operation could be either multiplication by 2 (which is impossible, as 811 is odd) or rearrangement of digits.

By rearranging, we can obtain the numbers: 181, 811, and 118. The first two are also not divisible by 2, leaving us with only 118. Division by 2 gives 59, which is odd, just as 95. Therefore, it is impossible to obtain the number 811 from 1 using these operations. \square

Solution to Problem 3.4: Let's assume such a route exists. Then, to reach the 12th floor, one can only come from the 5th floor, and to get there, one can only come from the 14th floor, and so on. However, to reach the 7th floor, one cannot come from any other floor. Thus, it is impossible to get from the 3rd floor to the 12th floor using only the elevator. \square

Procrustes' Method



“

«Leo, let's say you have 10 dollars. You ask your father for another 10 dollars. How much money will you have in total?»

«10 dollars, Mrs. Owless.»

«That's not true, Leo. You really don't know mathematics!»

«And you, Mrs. Owless, really don't know my father!»

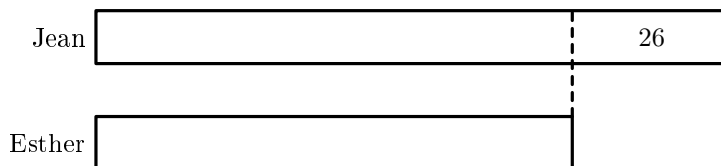
—Popular joke

Theory and Practice

If you are familiar with Greek mythology, you might recall the notorious figure Procrustes, a robber known for his gruesome practice of making his victims conform to the size of an iron bed by either stretching them or amputating their legs. Despite its brutality, Procrustes' method of «cutting off the excess or adding what is missing» can actually be quite valuable in solving mathematical problems. Known as the Procrustes' Method, this approach allows for the simplification of certain problems without the need to introduce new equations or complex systems of equations.

Example 4.1. Jean and Esther together have 100 dollars, and Jean has 26 dollars more than Esther. How much money does each of them have?

Solution: Let's represent the amounts of Jean's and Esther's money with rectangles.



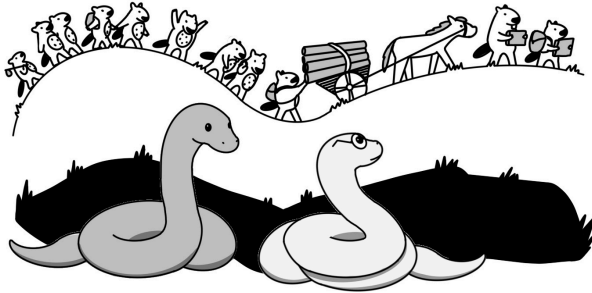
Let's cut off the «extra piece» of Jean's money (that is, the difference or 26 dollars). We are left with two identical large rectangles, totaling $100 - 26 = 74$ dollars. Therefore, one large rectangle is worth $74 \div 2 = 37$ dollars. This represents Esther's money. From that, we conclude that Jean has $37 + 26 = 63$ dollars. Thus, the answer is 63 and 37 dollars. \square

Example 4.2. The sum of four consecutive integers is 58. Find those integers.

Solution: Subtracting one from the second number gives the first number. Subtracting two from the third number gives the first number. Finally, subtracting three from the fourth number gives the first number again. In total, we subtracted $1 + 2 + 3 = 6$,

and as a result, we are left with four times the first number. Since $58 - 6 = 52$, the first number is $52 \div 4 = 13$, thus, the consecutive numbers are 13, 14, 15, and 16. \square

Example 4.3. Several beavers (walking upright, so they have 2 legs each) loaded their belongings onto a pony and set out on a long journey to conquer the Amazon. They were noticed by anacondas, who counted 36 legs and 15 heads in the caravan. How many beavers were there?

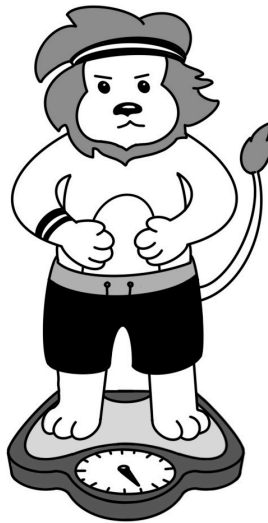


Solution: Let's consider a somewhat cruel solution to this problem. Suppose Amazonian anacondas have teeth and can bite off the legs of beavers and ponies. Initially, each beaver has 2 legs, and each pony has 4. If the anacondas bite off 2 legs from each beaver and pony, the ponies will remain two-legged, and the beavers will be legless. How many legs have the anacondas bitten off so far? They have bitten off 2 legs from each head owner, i.e., $15 \cdot 2 = 30$ legs. Therefore, there are $36 - 30 = 6$ legs left, which are divided among the ponies. Thus, there were $6 \div 2 = 3$ ponies and $15 - 3 = 12$ beavers in total. \square

Problem set

Problem 4.1. (MF – 1996.6.1): There are two coins in two wallets, but in one wallet, there are twice as many coins as in the other. How is that possible?

Problem 4.2. (COM – 2016.6.1): Anne has five friends, and each of them has pots of honey at home: Banny has 1, Canny has 2, Danny has 3, Eddy has 4, and Freddy has 5. Anne visits each friend, eats one pot of honey, and takes the rest with her. When she approached the last house, she had 10 pots of honey. Whose house could Anne have visited first?



Problem 4.3. (COM – 2008.7.1): After a morning run, the Lion loses a kilogram, and by the evening (after eating buns), his weight increases by a third. By the evening of the third day (after he started running), the Lion found that he had doubled his weight. How much did he weigh before he started running?

Problem 4.4. (COM – 2017.6.2): Both Sasha and Vanya were born on March 19th. Each of them celebrates their birthday with a cake with candles where the number of candles is equal to their age. In the year they met, Sasha had as many candles on

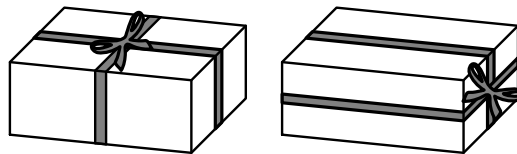
his cake as Vanya has today. It is known that the total number of candles on the four cakes of Vanya and Sasha (then and today) is 216. How old is Vanya today?

Problem 4.5. (MF – 2011.7.2): Along the path between the houses of Jean and Esther, flowers grew in a row: 15 peonies and 15 tulips were in an arbitrary order. Setting off from home to visit Jean, Esther watered all the flowers in a row. After the 10th tulip, the water ran out, and 10 flowers remained unwatered. The next day, going from home to visit Esther, Jean picked all the flowers in a row for her. After picking the 6th tulip, he decided that it was enough for a bouquet. How many flowers were left to grow along the path?

Problem 4.6. (MF – 2013.6.3;7.1): A dog and a cat simultaneously grabbed a sausage roll with their teeth from different sides. If the dog bites off his piece and runs away, the cat will get 300 g more than the dog. If the cat bites off its piece and runs away, the dog will get 500 g more than the cat. How much sausage will be left if both bite off their pieces and run away?

Problem 4.7. (COM – 2012.6.3): Cities A , B , and C are connected by direct roads, forming a triangle. It is known that the direct route from A to B is 200 km shorter than the detour through C , and the direct route from A to C is 300 km shorter than the detour through B . Find the distance between the cities B and C .

Problem 4.8. (MF – 2012.6.4;7.3): The cake is packed in a box with a square base. The height of the box is half the length of the square base's side. A ribbon of 156 cm long can be used to tie the box and make a bow on top (as shown on the left). To tie it with exactly the same bow on the side (as shown on the right), a ribbon of length 178 cm is needed. Find the dimensions of the box.



Problem 4.9. (COM – 2003.6.5;7.2): On the Island of Bad Luck, Mondays are canceled, so Tuesday immediately follows Sunday. Over the past year (from December

15, 2002, to December 14, 2003), Sundays on the island coincided with our Sundays exactly eight times. What day of the week is on the island today (February 16, 2003)?



Problem 4.10. (AT — 2017.2): Over the semester, Leo received 35 grades in Math. Just before the New Year, he retakes all his 2s and 3s: in the electronic journal, twos were changed to threes, and «old» threes were changed to fours. At the same time, the number of threes remained the same, and the average score increased by 0.4. How many twos did Leo have initially? The existing grades in the Russian school system are 2, 3, 4, and 5.

Problem 4.11. (MF — 2013.7.3): A bag of sunflower seeds was passed around the table. The first person took a 1 seed, the second took 2, the third took 3, and so on; each subsequent person took one more seed than the previous one. It is known that in the second round, the total number of seeds taken was 100 more than in the first round. How many people were sitting at the table?

Problem 4.12. (COM — 2012.6.6): The rope was folded in half, then folded in half again, and then folded in half again. After that, all layers of the rope were cut at the same place. What could be the length of the rope if it is known that any two of the resulting pieces had lengths of 9 meters and 4 meters?

Problem 4.13. (COM — 2004.6.6): Four friends participated in a Math competition. Vitya solved the greatest number of problems — eight, and Petya solved the least number of problems — five. Each problem of the competition was solved by exactly three friends. How many problems were there in the competition?

Problem 4.14. (MF – 2016.6.6): Forty children danced in a circle. Twenty-two held the hand of a boy, and thirty held the hand of a girl. How many girls were dancing?

Problem 4.15. (COM – 2016.7.5): Jean gave Esther a bouquet of red flowers and black roses. Each black rose has 4 petals and 4 stamens, and there are two leaves on the stem. Each red flower has 8 petals and 10 stamens, and there are three leaves on the stem. The number of leaves in the bouquet was 108 less than the number of petals. How many stamens were in the bouquet?

Problem 4.16. (JMO – 2022.A2): Seven consecutive odd numbers add up to 105. What is the largest of these numbers?

Problem 4.17. (JMO – 2017.B1): Some amount of money is to be divided equally between a group of children. If there were 20p more than this amount, each child would receive 70p. However, if each child were to receive 60p, then £2.10 would be left.

How many children are there in the group?

Problem 4.18. (JMO – 2019.B5): My 24-hour digital clock displays hours and minutes only.

How many displayed times in a 24-hours period contain at least one occurrence of the digit 5?

Problem 4.19. (AMC – 2017.12B.1): Kymbrea's comic book collection currently has 30 comic books in it, and she is adding to her collection at the rate of 2 comic books per month. LaShawn's collection currently has 10 comic books in it, and he is adding to his collection at the rate of 6 comic books per month. After how many months will LaShawn's collection have twice as many comic books as Kymbrea's?

- (A) 1 (B) 4 (C) 5 (D) 20 (E) 25

Problem 4.20. (AMC — 2011.12B.2): Josanna's test scores to date are 90, 80, 70, 60, and 85. Her goal is to raise her test average by at least 3 points with her next test. What is the minimum test score she would need to accomplish this goal?

- (A) 80 (B) 82 (C) 85 (D) 90 (E) 95

Problem 4.21. (iTest — 2008.22): Tony plays a game in which he takes 40 nickels out of a roll and tosses them one at a time toward his desk, where his change jar sits. He awards himself 5 points for each nickel that lands in the jar, and takes away 2 points from his score for each nickel that hits the ground. After Tony is done tossing all 40 nickels, he computes 88 as his score. Find the greatest number of nickels he could have successfully tossed into the jar.

Problem 4.22. (iTest — 2007.10): My grandparents are Arthur, Bertha, Christoph, and Dolores. The oldest of them is only 4 years older than the youngest. Each grandfather is two years older than his wife. If Bertha is younger than Dolores, what is the difference between Bertha's age and the mean of my grandparents' ages?

Problem 4.23. (Cyprus MO — 2007.5): If the remainder of the division of a with 35 is 23, then the remainder of the division of a with 7 is

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 4.24. (AMC — 2007.12B.5): The 2007 AMC 12 contests will be scored by awarding 6 points for each correct response, 0 points for each incorrect response, and 1.5 points for each problem left unanswered. After looking over the 25 problems, Sarah has decided to attempt the first 22 and leave the last 3 unanswered. How many of the first 22 problems must she solve correctly in order to score at least 100 points?

- (A) 13 (B) 14 (C) 15 (D) 16 (E) 17

Problem 4.25. (iTest — 2006.7): The sum of 17 consecutive integers is 2006. Find the second largest integer.

Problem 4.26. (AMC – 2004.12A.11): The average value of all the pennies, nickels, dimes, and quarters in Paula's purse is 20 cents. If she had one more quarter, the average value would be 21 cents. How many dimes does she have in her purse?

- (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 4.27. (AMC – 2004.12A.10): The sum of 49 consecutive integers is 7^5 . What is their median?

- (A) 7 (B) 7^2 (C) 7^3 (D) 7^4 (E) 7^5

Problem 4.28. (AMC – 2002.10A.16): Let $a + 1 = b + 2 = c + 3 = d + 4 = a + b + c + d + 5$. What is $a + b + c + d$?

- (A) -5 (B) $-10/3$ (C) $-7/3$ (D) $5/3$ (E) 5

Problem 4.29. (AHSME – 1956.6): In a group of cows and chickens, the number of legs was 14, more than twice the number of heads. The number of cows was:

- (A) 5 (B) 7 (C) 10 (D) 12 (E) 14

Skill Assessment Problems

Skill Assessment Problem 4.1. Max has coins worth 2 and 5 sol. If he takes all of his 2 sol coins, he will be short of 60 sol to buy four volumes of the encyclopedia. If he takes all of the 5 sol coins, he will be short of 60 sol to buy five volumes. If he uses all his coins, he is short of 60 sol to buy six volumes. How much sol does one volume cost?



Skill Assessment Problem 4.2. Tatiana bought croissants with three types of fillings – raspberry, blueberry, and strawberry. Croissants with raspberry filling make up half of all purchased. There are 14 fewer croissants with blueberry filling than with raspberry. And there are twice less croissants with strawberry filling than with raspberry and blueberry altogether. How many croissants of each type did Tatiana buy?

Solutions of Skill Assessment Problems

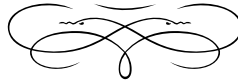
Solution to Problem 4.1: If Max takes all his coins, he will be short of $60 + 60 = 120$ solidi to buy $4 + 5 = 9$ volumes. On the other hand, he will lack 60 solidi for 6 volumes. Therefore, three volumes cost 60 solidi, and one volume costs 20. \square

Solution to Problem 4.2: Since croissants with raspberry filling are half of the total, the number of croissants with blueberry and strawberry fillings altogether is the same as with raspberry filling:

raspberry	blueberry & strawberry
raspberry	
blueberry	strawberry

At the same time, there are 14 fewer croissants with blueberry filling than with raspberry filling, so these 14 items have strawberry filling. Hence, the number of croissants with raspberry and blueberry fillings is $14 \cdot 2 = 28$. Thus, the total number of croissants is $28 + 14 = 42$. Therefore, there are $42 \div 2 = 21$ croissants with raspberry filling, 7 with blueberry filling, and 14 with strawberry filling. \square

Introduction to Simultaneous Equations



- “ — Can you really calculate faster than a computer?
- Yes.
 - What is 25 times 25?
 - Nine!
 - That’s incorrect...
 - But it’s fast!

—Popular joke

Theory and Practice

When attempting to formulate an equation to describe a problem, we often realize that expressing it with a single equation can be quite challenging. These equations together form a system of equations that accurately describes the problem at hand.

Most students encounter systems of equations while studying algebra in middle school. Although this topic might seem complex, it is more manageable than it appears. We will explore the most common types of problems involving systems of equations, and you'll discover that many similar problems can be solved using this straightforward approach.

Let's examine an example problem.

Example 5.1. When asked about the ages of her three children, the mother replied: «Pete and Kolya together are 19 years, Pete and Anna together are 14 years, and the youngest two are 7 years old together.» How old is each of the children? Explain the answer.

Solution: Let Kolya be X years old, Pete be Y years old, and Anna be Z years old. Then:

$$\begin{cases} X + Y = 19, \\ Y + Z = 14, \\ X + Z = 7. \end{cases}$$

If we sum all the equations, we get:

$$X + Y + Y + Z + X + Z = 19 + 14 + 7.$$

It means that if we take the age of each child twice and add them together, we get $19 + 14 + 7 = 40$ years. If we take the age of each child once, their total age will be $40 \div 2 = 20$ years.

Recall that when we added only X and Y , we got 19, and when we added X , Y , and Z , we got 20. The difference between these sums is Z , so $Z = 20 - 19 = 1$.

Similarly, $Y + Z = 14$, so $X = 20 - 14 = 6$; $X + Z = 7$, so $Y = 20 - 7 = 13$.

Thus, we found that Kolya is 6 years old, Pete is 13 years old, and Anna is 1 year old. \square

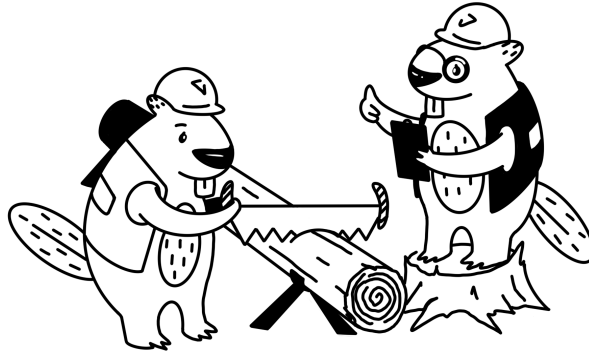
Consider another age-related problem. Here, it is essential to use another factor: time.

Example 5.2. The family consists of three people: a father, a mother, and a son. Currently, the sum of their ages is 74 years, and 10 years ago, this sum was 47 years. How old is the father now if he is 28 years older than his son?

Solution: Currently, the sum of ages is 74 years, and 10 years ago, it was 47 years. Then, in total, the father, the mother, and the son aged by $74 - 47 = 27$ years. The father and mother definitely aged by 10 years, so the son aged by 7 years in 10 years, meaning he was born 7 years ago. Therefore, the father is now $7 + 28 = 35$ years old. In this problem, we didn't even have to formulate equations; all the calculations were performed mentally. \square

Here is another problem that can be solved either by a system of equations or even mentally.

Example 5.3. There are 30 logs with lengths of 3 and 4 meters, with a total length of 100 meters. How many cuts do you need to perform in order to make all the logs have a length of 1? (Each sawing can be performed on exactly one log).



Solution: We create a system of equations. Let X be the number of logs of 3 meters and Y be the number of logs of 4 meters. Then, their total length is $X \cdot 3 + Y \cdot 4 = 100$. There is a total of 30 logs, so $X + Y = 30$. Therefore, if we take 3 times the number of logs of each type, we get that $X \cdot 3 + Y \cdot 3 = 30 \cdot 3$, $X \cdot 3 + Y \cdot 3 = 90$. By subtracting the obtained expression from the expression for the total length, we get that $Y = 10$, and, therefore, $X = 30 - 10 = 20$. Each 4-meter log needs 3 cuts, and each 3-meter log needs 2 cuts, so a total of $20 \cdot 2 + 10 \cdot 3 = 70$ cuts is needed. \square

Solution without a system of equations. One cut divides one log into 2 logs, thus adding one log. In total, there should be $100 : 1 = 100$ logs. There were 30 logs. Thus, $100 - 30 = 70$ cuts are needed.

Problem set

Problem 5.1. (TOT – 1995/96.10-11.1): One hundred people were asked the question: «Will the new president be better than the previous one?» Among them, a people think it will be better, b people think it will be the same, and c people think it will be worse. Sociologists constructed two indicators of «optimism» among those surveyed: $m = a + \frac{b}{2}$ and $n = a - c$. It turned out that $m = 40$. Find n .

Problem 5.2. (2ARSO – 2010/11.9.1): For three positive numbers, it is known that if one of them is chosen and the sum of the squares of the other two is added to it, the result is the same, regardless of the chosen number. Is it true that all numbers are equal?

Problem 5.3. (TOT – 1994/95.8-9.1): The cashier had 30 coins of 10, 15, and 20 cents, totaling 5 dollars. Prove that he had more 20-cent coins than 10-cent coins.

Problem 5.4. (MMG – 2014/15.9.1): Max added the fourth power and the square of some non-zero number and told the result to Leo. Can Leo uniquely determine Max's number?

Problem 5.5. (JMO – 2021.B2): The Smith family went to a restaurant and bought two Pizzas, three Chilies, and four Pastas. They paid £53 in total.

The Patel family went to the same restaurant and bought five Pizzas, six Chilies, and seven Pastas. They paid £107 in total.

How much more does a Pizza cost than a Pasta?

Problem 5.6. (AMC – 2023.10B.5): Maddy and Lara see a list of numbers written on a blackboard. Maddy adds 3 to each number in the list and finds that now the sum of all numbers is 45. Lara multiplies each number in the list by 3 and finds that the sum of her new numbers is also 45. How many numbers are written on the blackboard?
(A) 10 (B) 5 (C) 6 (D) 8 (E) 9

Problem 5.7. (JMO – 2018.A10): The digits in some four-digit integers are in increasing order from left to right. When we reverse the order of the digits, we obtain a four-digit integer whose digits decrease from left to right. A third four-digit integer uses exactly the same digits but in a different order. The sum of the three integers is 26352.

What is the value of the smallest of these three integers?

Problem 5.8. (JMO – 2018.B1): Polly Garter had her first child exactly when she turned 20, her second child exactly two years later, and her third child exactly two years after that.

How old was Polly when her age was equal to the sum of her three children's ages?

Problem 5.9. (AMC – 2018.10A.8): Joe has a collection of 23 coins, consisting of 5-cent coins, 10-cent coins, and 25-cent coins. He has 3 more 10-cent coins than 5-cent coins, and the total value of his collection is 320 cents. How many more 25-cent coins does Joe have than 5-cent coins?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 5.10. (UNM-PNM Statewide High School Mathematics Contest – 2014.II.3): Two people, call them A and B , are having a discussion about the ages of B 's children.

A: "What are the ages, in years only, of your four children?"

B: "The product of their ages is 72."

A: "Not enough information."

B: "The sum of their ages equals your eldest daughter's age."

A: "Still not enough information."

B: “My oldest child, who is at least a year older than her siblings, took the AMC 8 for the first time this year.”

A: “Still not enough information.”

B: “My youngest child is my only son.”

A: “Now I know their ages..”

What are their ages?

Problem 5.11. (UNM-PNM Statewide High School Mathematics Contest — 2014.II.1): Four siblings, BRYAN, BARRY, SARAH, and SHANA, have their names monogrammed on their towels. Different letters may cost different amounts to monogram. If it costs \$21 to monogram BRYAN, \$25 to monogram BARRY, and \$18 to monogram SARAH, how much does it cost to monogram SHANA?

Problem 5.12. (AMC — 2012.10A.8): The sums of three whole numbers taken in pairs are 12, 17, and 19. What is the middle number?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Problem 5.13. (UNCO Math Contest — 2010.II.1): Find a 3-digit integer less than 200 where each digit is odd and the sum of the cubes of the digits is the original number.

Problem 5.14. (UNCO Math Contest — 2008.II.2): Let $S = \{a, b, c, d\}$ be a set of four positive integers. If pairs of distinct elements of S are added, the following six sums are obtained: 5, 10, 11, 13, 14, 19. Determine the Values of a , b , c , and d . [Hint: there are two possibilities.]

Problem 5.15. (iTest — 2008.8): The Math team at Jupiter Falls Middle School meets twice a month during the Summer, and the Math team coach, Mr. Fischer, prepares

some Olympics-themed problems for his students. One of the problems Joshua and Alexis work on boils down to a system of equations: $2x + 3y + 3z = 8$,

$$3x + 2y + 3z = 808,$$

$$3x + 3y + 2z = 80808,$$

find $x + y + z$.

Problem 5.16. (iTest – 2007.2): Find $a + b$ if a and b satisfy $3a + 7b = 1977$ and $5a + b = 2007$.

- (A) 488 (B) 498

Problem 5.17. (AMC – 2007.10A.10): The Dunbar family consists of a mother, a father, and some children. The average age of the members of the family is 20, the father is 48 years old, and the average age of the mother and children is 16. How many children are in the family?

- (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Problem 5.18. (AMC – 2003.10A.13): The sum of three numbers is 20. The first is four times the sum of the other two. The second is seven times the third. What is the product of all three?

- (A) 28 (B) 40 (C) 100 (D) 400 (E) 800

Problem 5.19. (AMC – 2002.12B.19): If a, b , and c are positive real numbers such that $a(b + c) = 152$, $b(c + a) = 162$, and $c(a + b) = 170$, then abc is

- (A) 672 (B) 688 (C) 704 (D) 720 (E) 750

Problem 5.20. (AMC – 2002.10A.9): There are 3 numbers A , B , and C , such that $1001C - 2002A = 4004$, and $1001B + 3003A = 5005$. What is the average of A , B , and C ?

- (A) 1 (B) 3 (C) 6 (D) 9 (E) Not uniquely determined

Problem 5.21. (AJHSME – 1987.24): A multiple choice examination consists of 20 questions. The scoring is $+5$ for each correct answer, -2 for each incorrect answer, and 0 for each unanswered question. John's score on the examination is 48. What is the maximum number of questions he could have answered correctly?

- (A) 9 (B) 10 (C) 11 (D) 12 (E) 16

Problem 5.22. (AIME – 1986.4): Determine $3x_4 + 2x_5$ if x_1, x_2, x_3, x_4 , and x_5 satisfy the system of equations below.

$$2x_1 + x_2 + x_3 + x_4 + x_5 = 6$$

$$x_1 + 2x_2 + x_3 + x_4 + x_5 = 12$$

$$x_1 + x_2 + 2x_3 + x_4 + x_5 = 24$$

$$x_1 + x_2 + x_3 + 2x_4 + x_5 = 48$$

$$x_1 + x_2 + x_3 + x_4 + 2x_5 = 96$$

Skill Assessment Problems

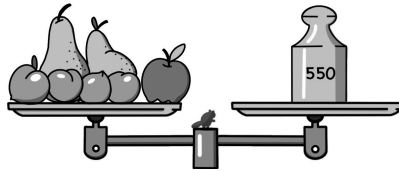
Skill Assessment Problem 5.1. It is known that 4 peaches, 2 pears, and an apple together weigh 550 grams, while a peach, 3 pears, and 4 apples together weigh 450 grams. Find the combined weight of a peach, a pear, and an apple.



Skill Assessment Problem 5.2. Russian, German, French, and English stamps were purchased. The cost of the stamps without the Russian one is 40 rubles, without the German one is 45 rubles, without the French one is 44 rubles, and without the English stamp is 27 rubles. How much does the Russian stamp cost?

Skill Assessment Problem 5.3. Doctor Aybolit distributed 2006 miraculous tablets among four sick animals. The rhinoceros received one tablet more than the crocodile, the hippopotamus received one tablet more than the rhinoceros, and the elephant received one tablet more than the hippopotamus. How many tablets did the elephant get?

Solutions of Skill Assessment Problems



Solution to Problem 5.1: Let's write a system of equations:

$$4p + 2g + a = 550,$$

$$p + 3g + 4a = 450.$$

Adding the two equations: $5p + 5g + 5a = 1000$. Dividing by 5:

$$p + g + a = 200.$$

Answer: 200 grams. □

Solution to Problem 5.2: Let's write a system of equations:

$$g + f + a = 40,$$

$$r + f + a = 45,$$

$$r + g + a = 44,$$

$$r + g + f = 27.$$

Adding the four equations: $3r + 3g + 3f + 3a = 156$. Dividing by 3:

$$r + g + f + a = 52.$$

Then the cost of the Russian stamp is: $52 - 40 = 12$ rubles.

Answer: 12 rubles. □

Solution to Problem 5.3: Let's write equations:

$$r = c + 1, \quad h = r + 1 = c + 2, \quad e = h + 1 = c + 3,$$

$$c + c + 1 + c + 2 + c + 3 = 2006.$$

From this, $c = 500$, $e = 503$.

Answer: 503.



How Much Does it Cost?



“

«One and a half kilograms. That will be 35 dollars!»

«Hmm, you know, I'm a math teacher...»

«...um, well, 30 dollars...»

«Teaching in high school!»

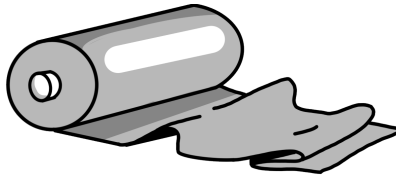
«One kilogram and 354 grams. That will be 27 dollars and 8 cents.»

—Popular joke

Theory and Practice

Problems related to purchasing items of varying costs are some of the oldest and most well-known. Since this book focuses on the elegance of arithmetic, we'll avoid using variables and instead highlight the arithmetic method, which is often clearer and more visually appealing. An example of this approach can be found in a work by Anton Chekhov.

Example 6.1. In Anton Chekhov's story «Tutor», the son of a retired general, Udodov, is taught by the gymnasium student Yegor Ziberov, who suggests solving the following problem: «A merchant bought 138 yards (2.33 ft) of black and blue cloth for 540 rubles. The question is, how many yards of each color did he buy if the blue cloth costs 5 rubles per yard and the black one costs 3 rubles per yard?»



The student could not solve it arithmetically, stating that «...it is not an arithmetic problem, but of indeterminate equations...». Hearing these words, the student's father said that «...it can be solved without algebra,» clicked on the abacus, and got the correct answer.

Solution: The solution to the problem is based on the assumption that, for example, if the merchant bought only one type of cloth — let's say, blue — then he would have paid $138 \cdot 5 = 690$ rubles. The difference of $690 - 540 = 150$ rubles can be compensated by replacing a yard of expensive cloth with a cheaper one. Thus, replacing one yard of blue cloth with one yard of black cloth reduces the difference by $5 - 3 = 2$ rubles. Therefore, a total of $150 \div 2 = 75$ yards needs to be replaced. In the end, we get 75 yards of black cloth and $138 - 75 = 63$ yards of blue cloth. \square

Let's consider another example.

Example 6.2. A pencil and a notebook cost 7 euros, and 5 such pencils and 2 notebooks together cost 23 euros. Determine the price of a notebook and a pencil.

Solution: We will present the solution in tabular form.

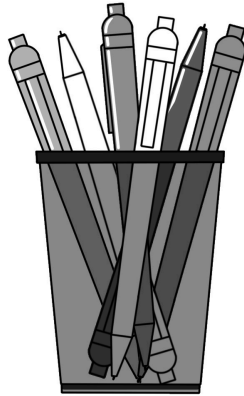
Notebooks	Pencils	Cost
1	1	7 euros
2	5	23 euros
First option		
2	2	14 euros
2	5	23 euros
$2 - 2 = 0$	$5 - 2 = 3$	$23 - 14 = 9$ euros
	1	$9 \div 3 = 3$ euros
1		$7 - 3 = 4$ euros

Notebooks	Pencils	Cost
1	1	7 euros
2	5	23 euros
Second option		
5	5	$7 \cdot 5 = 35$ euros
2	5	23 euros
$5 - 2 = 3$	$5 - 5 = 0$	$35 - 23 = 12$ euros
1		$12 \div 3 = 4$ euros
	1	$7 - 4 = 3$ euros

□

As you might have noticed, this topic is closely intertwined with the «Procrustes' Method» problems and the problems that involve the «Introduction to Simultaneous Equations.»

Let's solve a problem with a slightly more complex formulation.



Example 6.3. (1ARSO – 2018.5.3): A seller purchased a batch of pens and sold them. Some buyers bought one pen for 10 euros, while others bought 3 pens for 20 euros. It turned out that the seller made the same profit with each purchase. Find the price that the seller spent on one pen.

Solution: Let's denote the sought purchase price as x . Then, the profit from selling one pen is $10 - x$, and from selling three pens it is $20 - 3x$. According to the problem, both profits are equal: $10 - x = 20 - 3x$, which implies $x = 5$. \square

Problem Set

Problem 6.1. (MF — 2004.6.2): A kilogram of beef with bones costs 78 rubles, a kilogram of boneless beef costs 90 rubles, and a kilogram of bones costs 15 rubles. How many grams of bones are in one kilogram of beef?

Problem 6.2. (MF — 2001.6.2): Ofenya¹ bought a batch of pens at a wholesale market and now offers customers either one pen for 5 rubles or three pens for 10 rubles. Ofenya receives the same profit from each purchase. What is the wholesale price of a pen?

Problem 6.3. (MF — 1995.7.2): (It was a poem; we are not good at it)

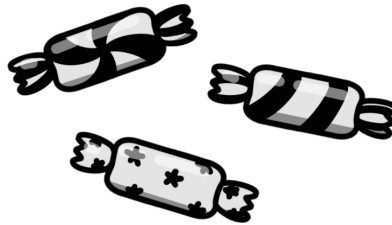
One sapphire and two topazes
are worth three times more than an emerald.
But seven sapphires and one topaz
are worth eight times more than an emerald.
We ask you to determine,
if a sapphire is more valuable than a topaz?

Problem 6.4. (MF — 1992.7.2): In January, you could buy the price for 40 screws or for 60 bolts was 1 dollar. In February, screws and bolts were sold in sets of 25 screws and 25 bolts for 1 dollar per set. To assemble a tractor, 600 screws and 600 bolts are required. In which month did the assembly of the tractor cost more, assuming that other expenses did not change?

Problem 6.5. (MF — 1993.7.5): Gulliver ended up in the land of Lilliputians with 7,000,000 rubles. He immediately bought drinking yogurt in bottles for 7 rubles each (an empty bottle cost 1 ruble at that time), spending all of his money. After drinking all the yogurt, he returned the bottles and immediately bought more yogurt for all the money. At the same time, he noticed that both the cost of yogurt and the cost of an empty bottle doubled. Then he drank all the yogurt again, returned the

¹Street vendor, peddler

bottles, and again bought yogurt for all the money, and so on. At each visit to the store, both the cost of yogurt and the cost of an empty bottle doubled. How many bottles of yogurt did Gulliver drink?



Problem 6.6. (MF — 2010.6.4): The exchange office carries out two types of operations:

- 1) Give 2 euros — get 3 dollars and a candy as a gift;
- 2) Give 5 dollars — get 3 euros and a candy as a gift.

When wealthy Leo came to the exchange office, he had only dollars. When he left, he had fewer dollars than before, no euros, but 50 candies. How much did Leo spend on such a «gift» in dollars?

Problem 6.7. (AMC — 2008.12B.4): A semipro baseball league has teams with 21 players each. League rules state that a player must be paid at least \$15,000 and that the total of all players' salaries for each team cannot exceed \$700,000. What is the maximum possible salary, in thousands of dollars, for a single player?

- (A) 270 (B) 385 (C) 400 (D) 430 (E) 700

Problem 6.8. (iTest — 2007.26): Julie runs a website where she sells university-themed clothing. On Monday, she sells thirteen Stanford sweatshirts and nine Harvard sweatshirts for a total of \$370. On Tuesday, she sells nine Stanford sweatshirts and two Harvard sweatshirts for a total of \$180. On Wednesday, she sells twelve Stanford sweatshirts and six Harvard sweatshirts. If Julie didn't change the prices of

any items all week, how much money did she take in (total number of dollars) from the sale of Stanford and Harvard sweatshirts on Wednesday?

Problem 6.9. (AMC — 2007.10A.5): The school store sells 7 pencils and 8 notebooks for \$4.15. It also sells 5 pencils and 3 notebooks for \$1.77. How much do 16 pencils and 10 notebooks cost?

- (A) \$1.76 (B) \$5.84 (C) \$6.00 (D) \$6.16 (E) \$6.32

Problem 6.10. (AMC — 2006.12A.5): Doug and Dave shared a pizza with 8 equally-sized slices. Doug wanted a plain pizza, but Dave wanted anchovies on half the pizza. The cost of a plain pizza was 8 dollars, and there was an additional cost of 2 dollars for putting anchovies on one half. Dave ate all the slices of anchovy pizza and one plain slice. Doug ate the remainder. Each paid for what he had eaten. How many more dollars did Dave pay than Doug?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 6.11. (PMWC — 1997.I6): John and Mary went to a bookshop and bought some exercise books. They had \$100 each. John could buy 7 large and 4 small ones. Mary could buy 5 large and 6 small ones and had \$5 left. How much was a small exercise book?

Problem 6.12. (PMWC — 1997.I10): Mary took 24 chickens to the market. In the morning, she sold the chickens at \$7 each, and she only sold out less than half of them. In the afternoon, she discounted the price of each chicken, but the price was still an integral number in dollars. In the afternoon, she could sell all the chickens, and she got a total of \$132 for the whole day. How many chickens were sold in the morning?

Skill Assessment Problems

Skill Assessment Problem 6.1. Beatrice bought 14 buns of two types priced at 6 solidi and 17 solidi respectively. In total, he paid 183 solidi. How many buns of each type did Beatrice buy?

Skill Assessment Problem 6.2. A problem from Magnitsky's «Arithmetic»: «Someone bought 112 sheep, old and young, for 49 rubles and 20 altyns. For an old one, he paid 15 altyns and 2 dinars, and for a young one, he paid 10 altyns. How many old and young sheep has he bought (1 altyn = 3 kopeck (100 kopeks are worth 1 ruble), 1 dinar = $\frac{1}{2}$ kopeck)?».

Skill Assessment Problem 6.3. Alice, Beatrice, and Clarice went to «Captain's Cove». Alice ordered 4 caraway cakes, a cup of coffee, and 10 doughnuts, paying 12 copper coins. Beatrice got 3 caraway cakes, a cup of coffee, and 7 doughnuts for 10 copper coins. How much did Clarice pay for a cake, a cup of coffee, and a doughnut?

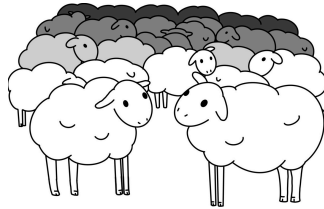
Skill Assessment Problem 6.4. In Beaverland, trade is very developed. Once upon a time, Shlomo bought a batch of pencil sharpeners wholesale and began to sell them. A family set of 7 sharpeners cost 20 logs, and a single set of 1 sharpener cost 4 logs. For how many logs did Shlomo buy each sharpener if selling a family set resulted in three times more profit than selling a single set?



Solutions to Skill Assessment Problems

Solution to Problem 6.1: If all 14 buns bought by Beatrice cost 6 solidi each, the total cost would be $14 \cdot 6 = 84$ solidi – 99 solidi less than the total expenses according to the given information. Note that replacing a cheaper bun with a more expensive one increases the total cost by 11 solidi. It is easy to calculate that we need to replace 9 such buns in total. Thus, there will be 9 buns priced at 17 solidi, and the remaining 5 will be priced at 6 solidi.

The problem can also be solved from the perspective of expensive buns: $14 \cdot 17 = 238$, $238 - 183 = 55$, $55 : 11 = 5$ – priced at 6 solidi; 9 – priced at 17 solidi. \square



Solution to Problem 6.2: For convenience, let's convert all ancient prices into ones used today:

$$49 \text{ rubles and } 20 \text{ altyns} = 49 \text{ rubles } 60 \text{ kopecks};$$

$$15 \text{ altyns and } 2 \text{ dinars} = 46 \text{ kopecks};$$

$$10 \text{ altyns} = 30 \text{ kopecks}.$$

Let's assume that only old sheep were bought:

$$112 \cdot 46 \text{ kopecks} = 51 \text{ rubles } 52 \text{ kopecks}.$$

This exceeds the required amount by 1 ruble 92 kopecks. We will need to replace a total of

$$192 : (46 - 30) = 12 \text{ rams}.$$

Thus, 12 young sheep and $112 - 12 = 100$ old sheep were bought. \square

Solution to Problem 6.3: From the problem setting, it is obvious that:

$$8\text{cakes} + 2\text{ cups} + 20\text{ doughnuts} = 12 \cdot 2 = 24\text{coins};$$

$$9\text{cakes} + 3\text{ cups} + 21\text{ doughnuts} = 10 \cdot 3 = 30\text{coins}.$$

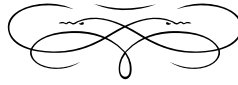
From this, subtracting the first equation from the second, we get:

$$1\text{cake} + 1\text{ cup} + 1\text{ doughnut} = 30 - 24 = 6\text{ coins}.$$

□

Solution to Problem 6.4: Denote the purchase price of the sharpeners by x logs, then the profit from selling one sharpener is $4 - x$ logs, and from selling seven, the profit is $20 - 7x$ logs. According to the problem setting, selling a family set brought him three times more profit: $20 - 7x = 3 \cdot (4 - x)$, hence $x = 2$. □

Movement



“

A biologist, a mathematician, and a physicist were asked to come up with a way to always win horse races. After a month, they were asked to share their progress.

Biologist said:

– In a month, I bred a breed of horses that have extraordinary speed and almost always win. To refine it, I need a couple more months.

Mathematician said:

– I’ve almost developed a theory that describes the probability of winning in each specific race. Now, I need approximately six more months, \$1000, and an assistant to test it in action to reduce statistical errors.

Physicist said:

– To continue the work, I need \$1000000, a well-equipped laboratory, a team of researchers, and another ten years. But I already have a theory of the victory of a liquid spherical horse in a vacuum.

—Popular joke

Theory and Practice

You've likely dealt with motion problems—whether it's an object moving in a straight line, two objects moving towards each other or finding the velocity of a stream.

Why is this topic in a Competitive Mathematics book? Despite seeming simple, motion problems are common in various competitions.

To get started, let's begin with a tricky problem involving motion on a staircase.

Example 7.1. (Regional stage of the All-Russian Physics Olympiad, 2012, 9th grade) Cheburashka and Crocodile Gena decided to compete in climbing up and down a staircase starting from the first floor of the House of Friendship. It turns out that by the time Crocodile Gena reaches the sixteenth floor, Cheburashka can run to the fourth floor and back to the first floor three times. Which floor will Cheburashka reach by the time Crocodile Gena climbs up to the sixth floor and back to the first floor? Assume that Cheburashka and Gena run up and down at constant speeds.

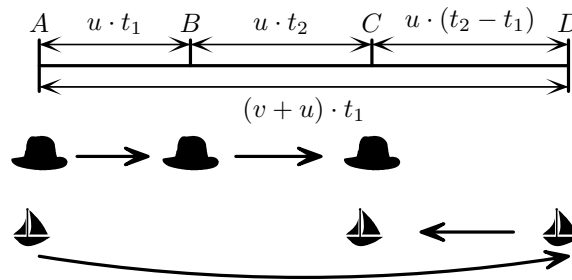
In this book, unless stated otherwise, the first floor is considered to be the ground floor in a building with no basement levels. Therefore, given this quite superficial assumption, to go from the street to the n -th floor, in reality, one needs to ascend a distance of $n - 1$ floors.

Solution: We are given that while Cheburashka runs $2 \cdot 3 \cdot (4 - 1) = 18$ floors, Crocodile Gena runs $16 - 1 = 15$ floors. Therefore, in equal intervals of time, Cheburashka runs $\frac{18}{15} = \frac{6}{5}$ times the distance run by Crocodile Gena. Hence, while Crocodile Gena was running to the sixth floor and back, that is, $2 \cdot (6 - 1) = 10$ floors in total, Cheburashka ran a total of $10 \cdot \frac{6}{5} = 12$ floors, reaching the $12 + 1 = 13$ floor. \square

Another common type of problem involves the concept of relative motion, often featuring scenarios such as motion along a river, an escalator, or a conveyor belt. For simplicity, we assume that the velocity of the stream, escalator, or belt is simply

added to the velocity of the object being considered in the problem (although, in the case of moving along the river, from a physical point of view, this is not entirely accurate; however, you will only learn about that in university when studying relativistic mechanics). So, in the context of Math competition, if it is given that the speed of a boat in still water is v and the speed of the stream is u , then if the boat moves in the same direction as the stream (that is, downstream), the boat's speed is $v + u$, and in case the boat moves in the opposite direction (upstream), the speed is $v - u$.

Example 7.2. An inattentive traveler was boating downstream, when suddenly he dropped his hat into the water. He noticed the loss only $t_1 = 1$ hour later. Upon realizing this, he immediately turned around and started paddling back, maintaining the same engine power. After what time t_2 will he be able to retrieve his hat?



Solution: In the picture, point A denotes a place where the traveler lost his hat; point B is where the hat was when he realized the loss; point D is where he was at that time; point C is where the traveler picked up his hat. Let v be the speed of the boat in still water, and u be the speed of the river.

The speed of the boat downstream is $v + u$, so $AD = (v + u)t_1$. When boating upstream, it is $v - u$, hence $CD = (v - u)t_2$. The hat moves with the speed of the river, u , so $AB = ut_1$ and $BC = ut_2$. Since $AD = AB + BC + CD$, we can write

$$(v + u)t_1 = ut_1 + ut_2 + (v - u)t_2,$$

i.e.,

$$vt_1 + ut_1 = ut_1 + ut_2 + vt_2 - ut_2,$$

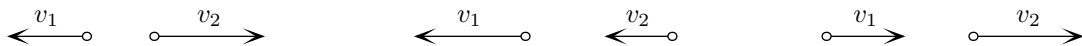
. This implies $t_1 = t_2 = 1$ hour. □

A simple answer always makes you wonder if there is an even simpler solution to the problem. It turns out that there is one, in this case, but for that, we need to consider some ideas about relative motion.

Imagine sitting on a train and looking out the window. You see another train nearby that seems still. When your train starts moving, it might feel like the other train is moving in the opposite direction at the same speed. If you're passing a moving train, it could seem like you're speeding up. But if both trains are going in the same direction, it might feel like you're barely moving.

Another example of relative motion is watching the Sun in the sky. It seems to rise in the east, travel across the sky, and set in the west. But we know this is due to the Earth rotating on its axis, not the Sun moving. This example is just for illustration; explaining such complex motions isn't the focus of this book.

Now, let's describe all of that more formally. Suppose we have two objects: the speed of the first is v_1 , and the speed of the second is v_2 . Let's introduce the concept of relative velocity, which is the velocity of the first object relative to the second or vice versa. If v_1 and v_2 are directed in different directions, we will call the relative velocity the quantity $v_1 + v_2$. If they are in the same direction, then it is $|v_2 - v_1|$. The direction of the relative velocity is determined by whether the distance between the objects increases or decreases. For example, the speed of the second object relative to the first will be directed away from it if the distance between the objects is increasing. This occurs when both speeds are directed away from each other, or the speed of the first is greater and directed away from the second object, or it is smaller and directed towards it.



In the opposite case, when the distance between the objects is decreasing, i.e., either both speeds are directed towards each other, or the speed of the first is smaller and directed away from the second object, or it is greater and directed towards it, the speed of the second object relative to the first will be directed towards it.

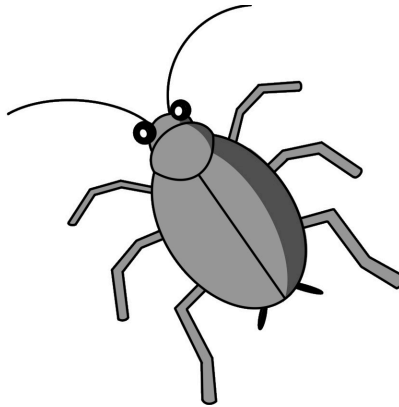


At the same time, the distance between the objects changes in accordance with their

relative speed.

These considerations become particularly clear and vivid when considering problems related to the topics mentioned earlier (movement along a river with a current, escalator, or conveyor belt). Let a boat with its own speed v sail along a river with an accompanying current, the speed of which is u . It is easy to understand that the fact that the boat's speed is simply added to the current speed actually means that the boat's speed relative to the «flow», i.e., relative to some object moving with the current, is v . What will this mean in the context of this problem? Initially, the traveler drifted away from the hat with a speed of v for a time t_1 , and then we are asked, during what time did he return to it with the same speed? Of course, now we can answer $t_2 = t_1 = 1$ hour without any calculations.

Problem set



Problem 7.1. (MF – 2005.6.1): Valentin the Cockroach claimed to be able to run at a speed of 50 m/min. No one believed him, and rightly so: Valentin got confused and thought that a meter was 60 centimeters and a minute was 100 seconds. At what speed (in «normal» m/min) does Valentin cockroach run?

Problem 7.2. (COM – 2010.6.2): Dad, Max, and Leo are walking to school. While Dad takes 3 steps, Max takes 5 steps, and Leo takes 5 steps. Max and Leo counted that together; they took 400 steps. How many steps did Dad take?

Problem 7.3. (MF – 2007.7.1): Jean and Esther live in the same building. Jean lives on the 6th floor. Leaving Jean, Esther went up instead of going down, as she needed. When she reached the last floor, Esther realized her mistake and went down to her floor. It turned out that Esther walked one and a half times more than if she had immediately gone down. How many floors are in the building?

Problem 7.4. (MF – 2007.7.1): One morning at 9:00, a pedestrian, Jean, left the village of Jeanville for the village of Estherville. At the same time, cyclist Esther set off towards him from Estherville. It is known that until the moment of the meeting, Jean managed to walk a third of the way between the villages. However, if Jean had left an hour earlier, he would have managed to walk half the way to the meeting. At what time did Jean and Esther meet?

Problem 7.5. (AT – 2014.2): (Ancient Problem) A military column is moving towards the headquarters at a speed of 6 km/h. At 9:00, the company commander sent a homing pigeon to the headquarters with a message. The pigeon delivered the message, immediately flew back, and returned to the column. At what time did the pigeon reach the headquarters if its speed is 10 km/h, and it returned at 9:45?

Problem 7.6. (AT – 2012.3): Max and Leo swim in the pool along adjacent lanes. They start simultaneously from opposite ends of the pool, «meet», and swim on. Having swum to the end of the lane, they instantly turn around, «meet» again, and so on. Max swims the lane in 5 minutes, and Leo in 7 minutes. How much time will Max catch up with Leo after the start, swimming in the same direction?

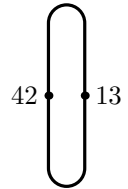
Problem 7.7. (AT – 2017.3): There is a straight road between Fox's hole and Bird's yard. The Fox is heading towards the Bird's yard, and from there, the Dog runs towards it at the same speed. The Dog, sensing the Fox at a distance of 100 m, will run after it at three times the speed. The Fox, sensing the Dog at a distance of 60 m, will run away from it at twice the speed. Can the Fox hide in the hole if the distance from the Bird's yard to the Fox's hole is 300 m?

Problem 7.8. (MF – 2008.7.3): Alice lives in a nine-story building. The elevator from her floor to the first floor takes 1 minute. Due to her small height, Alice cannot reach the button for her floor. Therefore, when going upstairs, she presses the button she can reach and then walks. The entire way up takes 1 minute 10 seconds. The elevator moves up and down at the same speed, and Alice climbs twice as slowly as the elevator. On which floor does Alice live?



Problem 7.9. (MF – 1992.6.4;7.5): Alice and Beatrice were going down an escalator. In the middle of the escalator, the hooligan Beatrice snatched Alice's hat and threw it onto the opposite escalator. The victimized Alice ran back up the escalator to then go down and retrieve the hat. The cunning Beatrice ran down the escalator to then go up and get there before Alice. Who will succeed first, if the speeds of the girls relative to the escalator are constant and do not depend on the direction of movement?

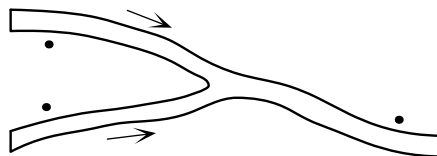
Problem 7.10. (COM – 2015.6.4): The cabins of a ski lift are numbered consecutively from 1 to 99. Jean sat in cabin No. 42 of the lift at the foot of the mountain and, at some point, noticed that he was level with the descending cabin No. 13 (see the diagram on the right). After 15 seconds, his cabin aligned with cabin No. 12. How long will it take for Jean to reach the top of the mountain?



Problem 7.11. (COM – 2010.6.5;7.3): Classmates Alice, Beatrice, and Clarice live in the same stairwell. They go to school at constant but different speeds without waiting for each other. However, if someone manages to catch up with another, then they slow down to walk together with the one they caught up with.

Once Alice went out first, Beatrice second, Clarice third, and any two of them arrived at school together. The next day, Clarice went out first, Beatrice second, and Alice third. Can all three of them come to school together?

Problem 7.12. (MF – 2012.7.4): On each of the two river branches, there is a pier one kilometer before their confluence, and another pier is located two kilometers after the confluence (see the figure below).



A boat reached from one of the piers to another (it is not known which one) in 30 minutes and from the other to the third in 18 minutes. In how many minutes can it

reach the first pier from the third? (The flow rate of the river is constant and the same in all its parts. The boat's speed when it moves without current is also constant.)

Problem 7.13. (MF — 1999.7.4): Two pedestrians set out at sunrise. Each walked at a constant speed. One walked from A to B , the other from B to A . They met at noon and, without stopping, one arrived in B at 4 PM, and the other arrived in A at 9 PM. At what time was the sunrise that day?

Problem 7.14. (COM — 2011.6.7): Martians divide the day into 13 hours. After the *Martian Hare* dropped the clock into tea, the rotation speed of the second hand changed, while the rotation speed of the other hands remained the same. It is known that every midnight, all three hands coincide. How many moments in time can there be in a day when all three hands coincide? Find all possible answers.

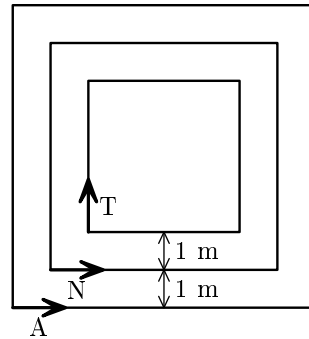


Problem 7.15. (COM — 2005.7.5): Dad swims from the bridge to the beach in 9 minutes and from the beach to the bridge in 12 minutes. The son, on the other hand, swims from the bridge to the beach in 12 minutes. How long does it take for the son to swim from the beach to the bridge?

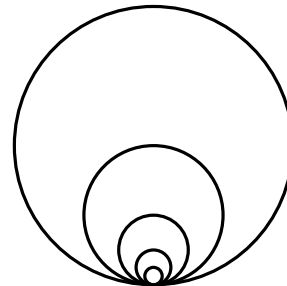
Problem 7.16. (MF — 2004.7.6): There is a highway leading from Flower Town to Sunny City with a length of 12 km. At the second kilometer of this highway, there is a railway crossing that is closed for three minutes and open for three minutes, and so on. At the fourth and sixth kilometers, there are traffic lights that burn red for two minutes and green for three minutes, and so on. Leo sets out from Flower Town to

Sunny City at the moment when the railway crossing has just closed, and both traffic lights have just switched to red. In what minimum time (in minutes) can he reach Sunny City without violating the rules if his electric car travels on the highway at a constant speed (Leo does not know how to brake or accelerate)?

Problem 7.17. (MF – 2013.7.5): Three square tracks with a common center are spaced one meter apart. Three ants start simultaneously from the bottom left corners of the tracks and run at the same speed: A and N counterclockwise, and T clockwise. When A reached the bottom right corner of the big track, the other two, having not yet completed a full circle, were on the right sides of their tracks, and all three were on the same straight line. Find the sides of the squares.



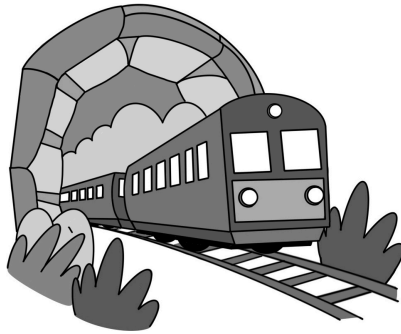
Problem 7.18. (COM – 2006.7.8): The labyrinth consists of five circles (shown in the figure at the right). The lengths of the circles are 10, 20, 40, 80, and 160 meters. A person begins to walk through the labyrinth at a constant speed, bypassing all its circles clockwise in the order of their increasing lengths.



After passing the largest circle, he moves to the smallest one and starts all over again. After some time, another person begins to walk through the labyrinth, who walks at the same speed and according to the same plan as the first one but bypasses the circles counterclockwise. Prove that these two people will definitely meet.

Problem 7.19. (COM – 2009.7.9): Ponchik was snacking in a roadside cafe when a bus passed by him. Three doughnuts after the bus, a motorcycle passed Ponchik, and another three doughnuts later, a car passed. Past Syropchik, who was snacking at another cafe on the same road, they passed in a different order: first, the bus; after three doughnuts, the car, and another three doughnuts later, the motorcycle. It is known that Ponchik and Syropchik always eat doughnuts at the same constant speed. Find the speed of the bus if the speed of the car is 60 km/h, and the speed of the motorcycle is 30 km/h.

Problem 7.20. (JMO — 2023.B2) Laura went for a training ride on her bike. She covered the first 10% of the total distance in 20% of the total time of the ride. What was the ratio of her average speed over the first 10% of the distance to her average speed over the remaining 90% of the distance?



Problem 7.21. (JMO — 2019.B3): It takes one minute for a train traveling at a constant speed to pass completely through a tunnel that is 120 meters long. The same train traveling at the same constant speed takes 20 seconds from the instant its front enters the tunnel to its being completely inside the tunnel.

How long is the train?

Problem 7.22. (AMC — 2023.10B.4): A cart rolls down a hill, traveling 5 inches the first second and accelerating so that during each successive second-time interval, it travels 7 inches more than during the previous 1-second interval. The cart takes 30 seconds to reach the bottom of the hill. How far, in inches, does it travel?

(A) 215 (B) 360 (C) 2992 (D) 3195 (E) 3242

Problem 7.23. (AMC — 2023.10B.11): Emily sees a ship traveling at a constant speed along a straight section of a river. She walks parallel to the riverbank at a uniform rate, faster than the ship. She counts 210 equal steps walking from the back of the ship to the front. Walking in the opposite direction, she counts 42 steps of the same size from the front of the ship to the back. In terms of Emily's equal steps, what is the length of the ship? (A) 70 (B) 84 (C) 98 (D) 105 (E) 126

Problem 7.24. (AMC – 2023.10B.6): Chantal and Jean start hiking from a trailhead toward a fire tower. Jean is wearing a heavy backpack and walks slower. Chantal starts walking at 4 miles per hour. Halfway to the tower, the trail becomes really steep, and Chantal slows down to 2 miles per hour. After reaching the tower, she immediately turns around and descends the steep part of the trail at 3 miles per hour. She meets Jean at the halfway point. What was Jean’s average speed, in miles per hour, until they met?

- (A) $\frac{12}{13}$ (B) 1 (C) $\frac{13}{12}$ (D) $\frac{24}{13}$ (E) 2

Problem 7.25. (AMC – 2017.10B.7): Samia set off on her bicycle to visit her friend, traveling at an average speed of 17 kilometers per hour. When she had gone half the distance to her friend’s house, a tire went flat, and she walked the rest of the way at 5 kilometers per hour. In all, it took her 44 minutes to reach her friend’s house. In kilometers rounded to the nearest tenth, how far did Samia walk?

- (A) 2.0 (B) 2.2 (C) 2.8 (D) 3.4 (E) 4.4

Problem 7.26. (AMC – 2014.10A.6): Suppose that a cows give b gallons of milk in c days. At this rate, how many gallons of milk will d cows give in e days?

- (A) $\frac{bde}{ac}$ (B) $\frac{ac}{bde}$ (C) $\frac{abde}{c}$ (D) $\frac{bcde}{a}$ (E) $\frac{abc}{de}$

Problem 7.27. (AMC – 2014.10A.15): David drives from his home to the airport to catch a flight. He drives 35 miles in the first hour but realizes that he will be 1 hour late if he continues at this speed. He increases his speed by 15 miles per hour for the rest of the way to the airport and arrives 30 minutes early. How many miles is the airport from his home?

- (A) 140 (B) 175 (C) 210 (D) 245 (E) 280

Problem 7.28. (UNCO Math Contest – 2015.II.4): Tarantulas A , B , and C start together at the same time and race straight along a 100 footpath, each running at a

constant speed the whole distance. When A reaches the end, B still has 10 feet more to run. When B reaches the end, C has 20 feet more to run. How many more feet does Tarantula C have to run when Tarantula A reaches the end?

Problem 7.29. (AMC – 2012.10A.19): Paula, the painter, and her two helpers each paint at constant but different rates. They always start at 8:00 AM, and all three always take the same amount of time to eat lunch. On Monday, the three of them painted 50% of a house, quitting at 4:00 PM. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at 2:12 PM. On Wednesday, Paula worked by herself and finished the house by working until 7:12 P.M. How long, in minutes, was each day's lunch break?

- (A) 30 (B) 36 (C) 42 (D) 48 (E) 60

Problem 7.30. (AMC – 2012.10A.1): Cagney can frost a cupcake every 20 seconds, and Lacey can frost a cupcake every 30 seconds. Working together, how many cupcakes can they frost in 5 minutes?

- (A) 10 (B) 15 (C) 20 (D) 25 (E) 30

Problem 7.31. (AMC – 2008.12B.10): Bricklayer Brenda would take 9 hours to build a chimney alone, and Bricklayer Brandon would take 10 hours to build it alone. When they work together, they talk a lot, and their combined output is decreased by 10 bricks per hour. Working together, they build the chimney in 5 hours. How many bricks are in the chimney?

- (A) 500 (B) 900 (C) 950 (D) 1000 (E) 1900

Problem 7.32. (AMC – 2008.12A.7): While Steve and LeRoy are fishing 1 mile from shore, their boat springs a leak, and water comes in at a constant rate of 10 gallons per minute. The boat will sink if it takes in more than 30 gallons of water. Steve starts rowing towards the shore at a constant rate of 4 miles per hour while LeRoy bails water out of the boat. What is the slowest rate, in gallons per minute, at which LeRoy can bail if they are to reach the shore without sinking?

- (A) 2 (B) 4 (C) 6 (D) 8 (E) 10

Problem 7.33. (AMC — 2008.10A.15): Yesterday, Han drove 1 hour longer than Ian at an average speed 5 miles per hour faster than Ian. Jan drove 2 hours longer than Ian at an average speed of 10 miles per hour faster than Ian. Han drove 70 miles more than Ian. How many more miles did Jan drive than Ian?

- (A) 120 (B) 130 (C) 140 (D) 150 (E) 160

Problem 7.34. (iTest — 2007.8): Joe is right in the middle of a train tunnel, and he realizes that a train is coming. The train travels at a speed of 50 miles per hour, and Joe can run at a speed of 10 miles per hour. Joe hears the train whistle when the train is a half mile from the point where it will enter the tunnel. At that point in time, Joe can run toward the train and just exit the tunnel as the train meets him. Instead, Joe runs away from the train when he hears the whistle. How many seconds does he have to spare (before the train is upon him) when he gets to the tunnel entrance?

- (A) 7.2 (B) 14.4 (C) 36 (D) 10 (E) 12 (F) 2.4 (G) 25.2

Problem 7.35. (AMC — 2007.12A.9): Yan is somewhere between his home and the stadium. To get to the stadium he can walk directly to the stadium, or he can walk home and then ride his bicycle to the stadium. He rides 7 times as fast as he walks, and both choices require the same amount of time. What is the ratio of Yan's distance from his home to his distance from the stadium?

- (A) $\frac{2}{3}$ (B) $\frac{3}{4}$ (C) $\frac{4}{5}$ (D) $\frac{5}{6}$ (E) $\frac{7}{8}$

Problem 7.36. (AMC — 2007.12A.4): Kate rode her bicycle for 30 minutes at a speed of 16 mph, then walked for 90 minutes at a speed of 4 mph. What was her overall average speed in miles per hour?

- (A) 7 (B) 9 (C) 10 (D) 12 (E) 14

Problem 7.37. (AMC – 2006.10A.15): Odell and Kershaw run for 30 minutes on a circular track. Odell runs clockwise at $250m/min$ and uses the inner lane with a radius of 50 meters. Kershaw runs counterclockwise at $300m/min$ and uses the outer lane with a radius of 60 meters, starting on the same radial line as Odell. How many times after the start do they pass each other?

- (A) 29 (B) 42 (C) 45 (D) 47 (E) 50

Problem 7.38. (AMC – 2005.12A.6): Josh and Mike live 13 miles apart. Yesterday, Josh started to ride his bicycle toward Mike's house. A little later, Mike started to ride his bicycle toward Josh's house. When they met, Josh had ridden for twice the length of time as Mike and at four-fifths of Mike's rate. How many miles had Mike ridden when they met?

- (A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Problem 7.39. (Indonesia MO – 2004.2): A trough, if filled with cold water tap, will be full in 14 minutes. To empty the full trough by opening the hole on the base of the trough, the water will be all out in 21 minutes. If the cold water tap and the hot water tap are opened simultaneously with the opening of the hole, the trough will be full in 12.6 minutes. Then, how long does it take to full the trough when only the hot water tap is opened and the hole is closed?

Problem 7.40. (AMC – 2004.12A.15): Brenda and Sally run in opposite directions on a circular track, starting at diametrically opposite points. They first meet after Brenda has run 100 meters. They next meet after Sally has run 150 meters past their first meeting point. Each girl runs at a constant speed. What is the length of the track in meters?

- (A) 250 (B) 300 (C) 350 (D) 400 (E) 500

Problem 7.41. (AMC – 2003.12A.4): It takes Anna 30 minutes to walk uphill 1 km from her home to school, but it takes her only 10 minutes to walk from school to her home along the same route. What is her average speed, in km/hr, for the round trip?

- (A) 3 (B) 3.125 (C) 3.5 (D) 4 (E) 4.5

Problem 7.42. (AMC — 2002.12B.17): Andy's lawn has twice as much area as Beth's lawn and three times as much area as Carlos' lawn. Carlos' lawn mower cuts half as fast as Beth's mower and one-third as fast as Andy's mower. If they all start to mow their lawns at the same time, who will finish first?

- (A) Andy (B) Beth (C) Carlos (D)
Andy and Carlos tie for first. (E) All three tie.

Problem 7.43. (AMC — 2002.12A.11): Mr. Earl E. Bird gets up every day at 8:00 AM to go to work. If he drives at an average speed of 40 miles per hour, he will be late by 3 minutes. If he drives at an average speed of 60 miles per hour, he will be early by 3 minutes. How many miles per hour does Mr. Bird need to drive to get to work exactly on time?

- (A) 45 (B) 48 (C) 50 (D) 55 (E) 58

Problem 7.44. (PMWC — 1997.T4): In one morning, a ferry traveled from Hong Kong to Kowloon, and another ferry traveled from Kowloon to Hong Kong at a different speed. They started at the same time and met for the first time at 8 : 20. The two ferries then sailed to their destinations, stopped for 15 minutes, and returned. The two ferries met again at 9 : 11. Suppose the two ferries traveled at a uniform speed throughout the whole journey; what time did the two ferries start their journey?

Problem 7.45. (AHSME — 1991.11): Jack and Jill run 10 km. They start at the same point, run 5 km up a hill, and return to the starting point by the same route. Jack has a 10 minute head start and runs at the rate of 15 km/hr uphill and 20 km/hr downhill. Jill runs 16 km/hr uphill and 22 km/hr downhill. How far from the top of the hill are they when they pass each other going in opposite directions (in km)?

- (A) $\frac{5}{4}$ (B) $\frac{35}{27}$ (C) $\frac{27}{20}$ (D) $\frac{7}{3}$ (E) $\frac{28}{49}$

Problem 7.46. (AHSME — 1967.27): Two candles of the same length are made of different materials so that one burns out completely at a uniform rate in 3 hours and the other in 4 hours. At what time P.M. should the candles be lighted so that, at 4 P.M., one stub is twice the length of the other?

- (A) 1:24 (B) 1:28 (C) 1:36 (D) 1:40 (E) 1:48

Skill Assessment Problems

Skill Assessment Problem 7.1. Two trains are moving towards each other with speeds of $70\frac{\text{km}}{\text{h}}$ for the first one and $80\frac{\text{km}}{\text{h}}$ for the second one. A passenger on the second train measured the time it took for the first train to pass him and got 12 seconds. Find the length of the first train (in meters).



Skill Assessment Problem 7.2. A kayaker was paddling against the current of the Thames and lost his water bottle under London Bridge. He noticed the loss only after 10 minutes, after which he turned the canoe and caught up with the bottle at Tower Bridge. Find the speed of the river current if the distance between the two bridges is 1 km.

Skill Assessment Problem 7.3. Max was running at a constant speed up and down a moving escalator, and Leo watched him, counting the number of steps Max took. While Max ran down, Leo counted 30 steps, and on the way up, he counted 70 steps. How many steps would Leo count if Max ran on a stationary escalator?

Solutions of Skill Assessment Problems

Solution to Problem 7.1: The closing speed of the trains is $70 + 80 = 150 \frac{\text{km}}{\text{h}}$. Convert 12 seconds to hours:

$$12 \text{ s} = \frac{12}{3600} \text{ h} = \frac{1}{300} \text{ h}.$$

The length of the train is obtained by multiplying the speed by the time:

$$150 \cdot \frac{1}{300} = \frac{1}{2} \text{ km, or } 500 \text{ m}.$$

□

Solution to Problem 7.2: The speed relative to the river does not depend on the direction of movement, so the canoeist will catch up with the flask in 10 minutes as well. Thus, the flask, moving with the speed of the river current, will cover the distance between the bridges in 20 minutes. Therefore, the speed of the current is $3 \frac{\text{km}}{\text{h}}$.

□

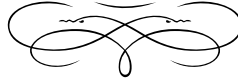
Solution to Problem 7.3: The number of steps that Max manages to run is directly proportional to the time of movement. Let V_e be the speed of the escalator and V_r be Max's speed. Then:

$$\begin{aligned} \frac{S}{V_r + V_e} &= 30; \\ \frac{S}{V_r - V_e} &= 70; \\ 4V_r = 10V_e &\Rightarrow V_e = 0.4V_r; \\ S &= 42V_r. \end{aligned}$$

Answer: 42 steps.

□

Shares and Proportions



“

Two mathematicians are flying in an airplane. One says to the other:

– Lately, I’ve been really afraid of flying because I calculated that the probability of there being a bomb on board is even higher than the probability of dying in a car crash.

The second mathematician responds:

– Yes, I did the same calculations and came to the same conclusion. But I went further. The probability of there being two bombs on the plane is negligible, so now I always carry one with me.

—Popular joke

Theory and Practice

Tasks involving shares and proportions have been around for centuries. One of the most famous examples is the ancient Roman problem of inheritance, where the division of assets among heirs had to be calculated fairly. This type of problem has stood the test of time and continues to be relevant today.

Example 8.1. The owner, upon dying, left a pregnant wife a will stating that the inheritance should be divided between the child and the wife depending on the gender of the firstborn: if the wife gives birth to a daughter, $\frac{2}{3}$ goes to the wife, $\frac{1}{3}$ to the daughter. If a son is born, the wife is to receive $\frac{1}{3}$, and the son $\frac{2}{3}$. The wife gave birth to twins: a son and a daughter. How should the inheritance be divided?

Solution: In the 2nd century BCE, the Roman Jurist Salvius Julianus proposed dividing the property into seven parts: giving four parts to the son, two to the mother, and one to the daughter. According to the will, the son would receive twice as much as the mother, who would receive twice as much as the daughter. \square



Example 8.2. (2ARSO – 2015.5.2): Wolf, Hedgehog, Fox, and Beaver shared an orange. Hedgehog got twice more segments than Fox, Fox got five times less than Beaver, and Beaver got 8 segments more than Fox. Find how many segments were in the orange, given that Wolf only got the peel.

Solution: Let Fox get x orange segments; then Hedgehog got $2x$. Beaver will get, on the one hand, $x + 8$, and on the other hand, $5x$. Therefore: $x + 8 = 5x$, from

which $x = 2$. Then, in total, there were: 2 (Fox's share) + 4 (Hedgehog's share) + 10 (Beaver's share) = 16 segments in the orange. \square

Sometimes, it is helpful to provide not only a strictly analytical and formal solution but also a visual interpretation. Let's demonstrate this with the following example.

Example 8.3. Once upon a time, in his travels, Columbus came across an island where $\frac{2}{3}$ of all men were married and $\frac{3}{5}$ of all women were married. Surprisingly, the morality of the islanders regarding marriage turned out to be quite in line with European standards of that time. What proportion of the island's population is married?

Solution: Let $2t$ people be married in total. Let's calculate the number of men and women:

$$t \div \frac{2}{3} = \frac{3}{2}t, \quad t \div \frac{3}{5} = \frac{5}{3}t.$$

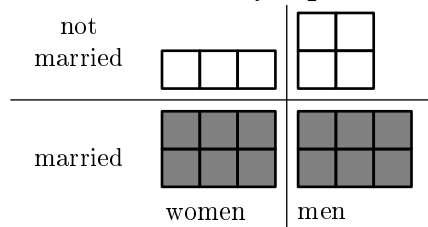
The total population on the island is:

$$\left(\frac{3}{2} + \frac{5}{3}\right)t = \frac{19}{6}t,$$

of which $2t$ are married. Thus, the proportion of the population that is married is

$$2t \div \frac{19}{6}t = \frac{12}{19} \text{ of the islanders.}$$

The situation described above can be visually represented as follows:

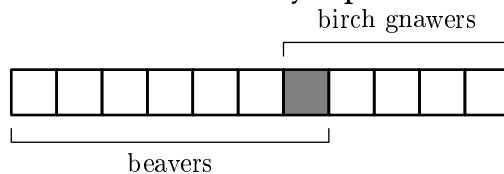


Example 8.4. It is known that among beavers, every seventh one is a birch gnawer, and among birch gnawers, every fifth one is a beaver. Which group is larger: beavers or birch gnawers?



Solution: It is evident that the entire set of presented characters can be divided into three types: «pure beavers», «pure birch gnawers», and those who are both a beaver and a birch gnawer. Let's denote the amount of beavers-birch gnawers as a , all beavers as b , and all birch gnawers as p . Then, on the one hand, $a = b/7$, and on the other hand, $a = p/5$. Hence, $b/7 = p/5$, or $5b = 7p$, i.e., there are $\frac{7}{5}$ times more beavers than birch gnawers.

The situation described above can be visually represented as follows:



□

Problem Set

Problem 8.1. (MF — 2014.6.1;7.1) Children went to the forest for mushrooms. If Alice gives half of her mushrooms to Beatrice, everyone will have an equal number of mushrooms. However, if Alice gives all her mushrooms to Clarice instead, Clarice will have as many mushrooms as all the others combined. How many children went for mushrooms?

Problem 8.2. (MF — 2011.6.1): «And it's too early for you to see this», said Baba Yaga to her 33 students and commanded, «Close your eyes!». All boys closed their right eye, and a third of the girls did, too. All girls closed their left eye, and a third of the boys did, too. How many students still saw what it was too early to see?

Problem 8.3. (MF — 2010.6.1): The loaf of sausage has thin cross-sectional rings drawn on it. If you cut along the red rings, you get 5 pieces; if you along the yellow ones, you get 7 pieces; and if you along the green ones, you get 11 pieces. How many pieces of sausage will you get if you cut along the rings of all three colors?

Problem 8.4. (MF — 2007.6.1): The same film started airing on two television channels simultaneously. On the first channel, the film was divided into parts, each lasting 20 minutes, with 2-minute commercial breaks between them. On the second channel, the film was divided into parts, each lasting 10 minutes, with 1-minute commercial breaks between them. On which channel will the film finish earlier?

Problem 8.5. (MF — 1995.6.1): After Esther ate half of the peaches from the jar, the level of the compote decreased by one-third. What fraction (of the initial level) will the level of the compote decrease if she eats half of the remaining peaches?

Problem 8.6. (MF — 2008.6.1): Alice, Beatrice, Clarice, and Dorice emptied a barrel of honey. Beatrice ate half of what Alice ate, Rabbit ate half of what Alice did not eat, and Dorice got only a tenth of the barrel. What fraction of the barrel did Clarice get?

Problem 8.7. (MF – 2006.7.1): Alice and Beatrice divided a cake between themselves. Beatrice grumbled that he got too little. Then Alice gave him a third of his share. After that, Beatrice's amount of cake tripled. What fraction of the cake did Pooh have initially, and what fraction did Beatrice have?

Problem 8.8. (MF – 1992.6.2): At the New-Vasyukov Currency Exchange, 11 tugriks can be exchanged to 14 dinars, 22 rupees to 21 dinars, 10 rupees to 3 talers, and 5 crowns to 2 talers. How many tugriks can be exchanged for 13 crowns?

Problem 8.9. (MF – 2016.6.2;7.2): In a small town, there is only one tram line. It is circular, and trams run in both directions. There are stops at the Circus, Park, and Zoo on the ring. The tram journey from the Park to the Zoo via the Circus is three times longer than without going through the Circus. The journey from the Circus to the Zoo via the Park is twice as short as without going through the Park. Which route from the Park to the Circus is shorter, via the Zoo or without going through the Zoo, and by how many times?

Problem 8.10. (MF – 2015.7.2): In an aquarium, there are three types of fish: gold, silver, and red. If the cat eats all the goldfish, there will be one less fish than $\frac{2}{3}$ of the original number. If the cat eats all the redfish, there will be four more fish than $\frac{2}{3}$ of the original number. Which type of fish, gold or silver, is more numerous, and by how much?

Problem 8.11. (MF – 2009.7.2): In each of the two vegetable gardens, Grandpa planted the same number of turnips. If the Granddaughter comes into the garden, she pulls out exactly $\frac{1}{3}$ of the turnips that are there at that moment. If Beetle comes in, she pulls out $\frac{1}{7}$ of the turnips, and if Mouse comes in, she pulls out only $\frac{1}{12}$ of the turnips. By the end of the week, there were 7 turnips left in the first garden and 4 in the second. Did Beetle come into the second garden?

Problem 8.12. (MF – 2007.6.3): The wolf, along with three piglets, wrote the detective story «Three Little Pigs-2», and then, together with Little Red Riding Hood and her grandmother, wrote the cookbook «Little Red Riding Hood-2». The publishing house paid a fee for both books to the piglet Huff-Puff. He took his share and handed

the remaining 2100 gold coins to the wolf. The fee for each book is divided equally among its authors. How much money should the wolf take for himself?

Problem 8.13. (MF – 2001.6.4): Arrange 6 distinct numbers in a circle so that each of them is equal to the product of two neighboring ones.

Problem 8.14. (COM – 2010.7.4): Waking up every morning at 8:30, the stoker fills the stove with coal to the brim. At the same time, he puts in exactly 5 kg of coal. Every evening, going to bed (he goes to bed at the same time), he again fills the stove with coal to the brim and puts in exactly 7 kg of coal. At what time does the stoker go to bed?

Problem 8.15. (MF – 1997.7.5): The test has 5 answer options for each question. An A-grade student answers all questions correctly. When a student with a lower grade manages to cheat, he answers correctly, and otherwise, he guesses (that is, among the uncheated questions, he answers correctly to $1/5$ of them). In total, the student with a lower grade answered half of the questions correctly. What fraction of the answers did he manage to cheat?

Problem 8.16. (MF – 2012.7.5): Max wrote a true statement:

«In this phrase, $1/3$ of all digits are the digit 3,
and $1/2$ of all digits are the digit 1».

And Leo wrote the phrase:

«In this phrase, $1/\dots$ of all digits are the digit *,
the fractions of digits * and * are the same and
equal to $1/\dots$, and the fraction of all other digits is $1/\dots$ ».

Insert three different digits instead of asterisks and three different numbers instead of dots so that the statement becomes true.

Problem 8.17. (AMC – 2023.10B.11): Two jars each contain the same number of marbles, and every marble is either blue or green. In Jar 1, the ratio of blue to green

marbles is $9 : 1$, and the ratio of blue to green marbles in Jar 2 is $8 : 1$. There are 95 green marbles in all. How many more blue marbles are in Jar 1 than in Jar 2?
(A) 5 (B) 10 (C) 25 (D) 45 (E) 50

Problem 8.18. (AMC — 2023.10B.9): The knights in a certain kingdom come in two colors. $\frac{2}{7}$ of them are red, and the rest are blue. Furthermore, $\frac{1}{6}$ of the knights are magical, and the fraction of red knights who are magical is 2 times the fraction of blue knights who are magical. What fraction of red knights are magical?

(A) $\frac{2}{9}$ (B) $\frac{3}{13}$ (C) $\frac{7}{27}$ (D) $\frac{2}{7}$ (E) $\frac{1}{3}$

Problem 8.19. (AMC — 2023.10B.8): Barb-the baker, has developed a new temperature scale for her bakery called the Breadus scale, which is a linear function of the Fahrenheit scale. Bread rises at 110 degrees Fahrenheit, which is 0 degrees on the Breadus scale. Bread is baked at 350 degrees Fahrenheit, which is 100 degrees on the Breadus scale. Bread is done when its internal temperature is 200 degrees Fahrenheit. What is this, in degrees, on the Breadus scale?

(A) 33 (B) 34.5 (C) 36 (D) 37.5 (E) 39

Problem 8.20. (AMC — 2015.10A.15): Consider the set of all fractions $\frac{x}{y}$, where x and y are relatively prime positive integers. How many of these fractions have the property that if both numerator and denominator are increased by 1, the value of the fraction is increased by 10%?

(A) 0 (B) 1 (C) 2 (D) 3 (E) infinitely many

Problem 8.21. (AMC — 2013.12A.16): A , B , C are three piles of rocks. The mean weight of the rocks in A is 40 pounds, the mean weight of the rocks in B is 50 pounds, the mean weight of the rocks in the combined piles A and B is 43 pounds, and the mean weight of the rocks in the combined piles A and C is 44 pounds. What is the greatest possible integer value for the mean in pounds of the rocks in the combined piles B and C ?

(A) 55 (B) 56 (C) 57 (D) 58 (E) 59

Problem 8.22. (Christmas Invitational Mathematics Examination — 2020.II.3): There are blue jelly beans and green jelly beans in a jar. Then, 15% of the blue jelly beans are removed, and 40% of the green jelly beans are removed. If afterward, the total number of jelly beans is 80% of the original number of jelly beans, then determine the percent of the remaining jelly beans that are blue.

Problem 8.23. (iTest — 2008.45): In order to save money on gas and consume less fuel, Hannah has a special battery installed in the family van. Before the installation, the van averaged 18 miles per gallon of gas. After the conversion, the van got 24 miles per gallon of gas.

Michael notes, «The amount of money we will save on gas over any time period is equal to the amount we would save if we were able to convert the van to go from 24 miles per gallon to m miles per gallon. It is also the same that we would save if we were able to convert the van to go from m miles per gallon to n miles per gallon.»

Assuming Michael is correct, compute $m + n$. In this problem, assume that gas mileage is constant over all speeds and terrain and that the van gets used the same amount regardless of its present state of conversion.

Problem 8.24. (UNCO Math Contest — 2007.II.2): In Grants Pass, Oregon $\frac{4}{5}$ of the men are married to $\frac{3}{7}$ of the women. What fraction of the adult population is married? Give a possible generalization.

Problem 8.25. (AMC — 2002.12A.10): Sarah places four ounces of coffee into an eight-ounce cup and four ounces of cream into a second cup of the same size. She then pours half the coffee from the first cup to the second and, after stirring thoroughly, pours half the liquid in the second cup back to the first. What fraction of the liquid in the first cup is now cream?

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{3}{8}$ (D) $\frac{2}{5}$ (E) $\frac{1}{2}$

Problem 8.26. (AMC – 2000.10.8): At Olympic High School, $\frac{2}{5}$ of the freshmen and $\frac{4}{5}$ of the sophomores took the AMC-10. Given that the number of freshmen and sophomore contestants was the same, which of the following must be true?

- (A) There are five times as many sophomores as freshmen.
- (B) There are twice as many sophomores as freshmen.
- (C) There are as many freshmen as sophomores.
- (D) There are twice as many freshmen as sophomores.
- (E) There are five times as many freshmen as sophomores.

Problem 8.27. (PMWC – 1997.I7): 40% of girls and 50% of boys in a class got an 'A'. If there are only 12 students in the class who got 'A's and the ratio of boys and girls in the class is 4 : 5, how many students are there in the class?

Problem 8.28. (AHSME – 1993.9): Country A has $c\%$ of the world's population and $d\%$ of the world's wealth. Country B has $e\%$ of the world's population and $f\%$ of its wealth. Assume that the citizens of A share the wealth of A equally, and assume that those of B share the wealth of B equally. Find the ratio of the wealth of a citizen of A to the wealth of a citizen of B .

- (A) $\frac{cd}{ef}$ (B) $\frac{ce}{ef}$ (C) $\frac{cf}{de}$ (D) $\frac{de}{cf}$ (E) $\frac{df}{ce}$

Problem 8.29. (AHSME – 1991.16): One hundred students at Century High School participated in the AHSME last year, and their mean score was 100. The number of non-seniors taking the AHSME was 50% more than the number of seniors, and the mean score of the seniors was 50% higher than that of the non-seniors. What was the mean score of the seniors?

- (A) 100 (B) 112.5 (C) 120 (D) 125 (E) 150

Skill Assessment Problems

Skill Assessment Problem 8.1. Jean suggested buying Iris candies. Practical Esther replied: «Let's buy lollipops instead. We can buy half more of them, and we'll have to pay only a quarter more for them.» In how many times are lollipops cheaper than Iris candies?

Skill Assessment Problem 8.2. The company conducted a survey of employees to find out which social networks they use: MySpace or Facebook. Some employees said they use MySpace, some use Facebook, and some said they use both social networks, and 40 employees said they don't use any social networks. Among all those who use social networks, 75% use MySpace, and 65% use both networks. The proportion of employees who use Facebook to the total number of all employees is $\frac{5}{6}$. How many employees work in the company?

Solutions to Skill Assessment Problems

Solution to Problem 8.1: Let S be the amount of money that Jean planned to spend on iris candies, and i be the amount of iris candies that can be purchased for this amount. Then the cost of one iris candy is S/i . Esther wants to spend a quarter more on lollipops — $\frac{5}{4}S$, and the planned amount of them is half as much — $\frac{3}{2}i$. Then the cost of one lollipop is $\frac{5}{4}S \div \frac{3}{2}i = \frac{5S}{6i}$, i.e., lollipops are $6/5$ times cheaper. \square

Solution to Problem 8.2: Since among all users of social networks, 75% use MySpace, it turns out that only 25% use Facebook. In addition, 65% use both networks, so in total, 90% use Facebook. These 90% make up $\frac{5}{6}$ of the company's employees, so 100% make up

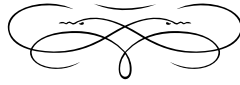
$$\frac{10}{9} \cdot \frac{5}{6} = \frac{50}{54}.$$

of all employees. Therefore, those who do not use them make up

$$1 - \frac{50}{54} = \frac{4}{54},$$

and there are 40 of them. Consequently, the total number of employees is 540. \square

Percents



“

The math teacher glanced over the student's notebook and was shocked by the complex calculations:

«One of us has gone crazy, Leo!»

The next day, Leo put an envelope on the table.

«What's in it?» the teacher asked.

«A certificate stating that I am not insane.»

—Popular joke

Theory and Practice

To understand many terms, including mathematical ones, it is useful to consider the literal translation of their mother concepts. For instance, the word «percent» comes from the Latin «pro centum», which literally translates to «for a hundred» or «from a hundred». It is convenient for us to interpret «percent» as one-hundredth or a hundredth part. The idea of expressing parts of a whole constantly in the same fractions, driven by practical considerations, originated in ancient Babylon and India, and was further developed in Rome, where «percentages» referred to the money that a debtor paid to a lender for each hundred.

Nowadays, we encounter «percentage calculations» every day. For example, when logging into your favorite online store, you might see an announcement: «The price of the book has increased by 15%». If the initial price was a dollars, such an increase means raising the price by $\frac{15}{100} = 0.15$ (15 percent is 15 hundredths), or $a + 0.15a = 1.15a$.

We can summarize it with two simple rules:

1. To increase the number a by $x\%$:

$$a \cdot \left(1 + \frac{x}{100}\right).$$

2. To decrease the number a by $x\%$:

$$a \cdot \left(1 - \frac{x}{100}\right).$$

Now, let's try to imagine what happens if the price a for a product increases two days in a row — by 15% and 27%, respectively? The resulting price will be

$$a \cdot 1.15 \cdot 1.27.$$

The concept of so-called «compound» percentages refers to cases where successive percentages are calculated based on the already increased numbers.

The basic rule for calculating «compound» percentages states that if the price a for a product increases on each of n days by $x\%$, the resulting price will be given by the formula

$$a \cdot \left(1 + \frac{x}{100}\right)^n.$$

Understanding how percentages work is crucial. One of the authors of this book encountered a graduate of a hairdressing school, who was obliged to take the state exam in mathematics. When asked, «Imagine that there are stores of the same network nearby, and the same dress is sold in them (with the same initial price), but in one, the discount on it from the initial price is 10%, and in the other, it is 90%. Where would you go?» The girl answered, «Of course, where it's 10%, it's cheaper!» The authors sincerely hope that you will never make such mistakes...

Example 9.1. Two stores sell the same product at the same price. In the first one, the price of the product is first increased by 10%, and then by 25% of the newly increased price. In the second store, the price of the same product is first increased by 25%, and then by 10% of the new price. In which store is the product more expensive?

Solution: Let's calculate the price increase coefficient: in the first store, it is $1.1 \cdot 1.25$; in the second store, it is $1.25 \cdot 1.1$. Therefore, the price of the product will be the same. □



Example 9.2. A bank offers its customers the opportunity to open a savings account with a 10% annual profit. One day, Mrs. Owless came to the bank and asked the bank

employee to open a savings account for six months at 5% first and then for another six months also at 5% — «after all, in the end, I will get the same $5 + 5 = 10\%$ per annum», Mrs. Owless assured the employee. Help the confused bank employee check if Mrs. Owless is telling the truth and calculate the correct percentage for six months.

Solution: Let's assume the sought percentage is $x\%$. Then, after six months, the price increase coefficient will be:

$$\left(1 + \frac{x}{100}\right),$$

and another six months later:

$$\left(1 + \frac{x}{100}\right)^2.$$

According to the conditions, after a year, the client should receive 10%, i.e.:

$$\left(1 + \frac{x}{100}\right)^2 = 1.1,$$

$$\left(1 + \frac{x}{100}\right) = \sqrt{1.1} \approx 1.0488,$$

$$x \approx 4.88\%.$$

Thus, the correct percentage would be less than 5%, and we see that Mrs. Owless is clever and wants to get a higher percentage than the bank offers. \square

The notion of extracting the square root is possibly new to some of you. In essence, it is the inverse operation of squaring a number. So, in order to calculate the square root of a number a , you have to answer the question, «What number must be squared to obtain a ?» Of course, as in the problem above, a calculator can be used to get the answer.

There is a well-known story about the will of the famous American statesman Benjamin Franklin. It was published in the «Collection of Various Writings by Benjamin Franklin». Here is an excerpt from it.

Benjamin Franklin's Will. «The said sum of one thousand Pounds Sterling, if accepted by the Inhabitants of the Town of Boston, shall be managed under the direction of the Select Men, who are to let out the same upon Interest at 5 percent per Annum to young married artificers ¹.

If this plan is executed and succeeds as projected without interruption for one hundred years, the Sum will then be 131,000, of which I would have the Managers of the Donation to the Town of Boston, then lay out at their discretion 100,000 in Public Works which may be judged of most general utility to the Inhabitants such as Fortifications, Bridges, Aqueducts, Public Building, Baths, Pavements, or whatever may make a living in the town more convenient to its people and render it more agreeable to strangers, resorting thither for health or a temporary residence. The remaining 31,000 Pounds, I would have continued to be let out on interest in the manner above directed for another 100 years. At the end of this second Term, if no unfortunate accident has prevented the operation, the sum will be 4,061,000 Pounds Sterling, of which I leave 1,061,000 Pounds to the Disposition of the Inhabitants of the Town of Boston and 3,000,000 to the disposition of the Government of the State, not presuming to carry my views farther. » B. Franklin was experienced in mathematical calculations, as evidenced by the non-trivial nature of the problem above.

To increase 1000 pounds at 5% per annum for 100 years, considering compound interest, it is necessary to calculate:

$$1000 \cdot 1.05^{100}.$$

The problem of quick approximate exponentiation to the hundredth power (rather than a long and torturous multiplication of all the numbers) is solved with slightly more complex methods — for example, by taking logarithms, which will be explained in more detail in the following volumes of this book.

In this example, it is clearly seen that compound interest can grow quite rapidly. For example, a growth of 1% per day for a year gives an increase of ... more than 37 (!) times, while a decrease of 1% per day over the course of a year results in 0.02 of the original amount. As a consequence, companies offering so-called microloans, unfortunately, keep thriving — the majority of financially illiterate people believe that one percent per day for a loan is very little.

¹In America at that time, there were no credit institutions yet.

Example 9.3. In Beaverland, there was a poor harvest of blackberries, and as a result, all pies with blackberries became 25% more expensive. By what percentage can one buy fewer pies for the same amount of money?

Solution: The increase by 25% means that the new price of a pie is equal to the old price multiplied by $\frac{5}{4}$. Hence, for the same amount of money, you can buy $1 \div \frac{5}{4} = \frac{4}{5}$ of the old number of pies, that is, 20% less. \square

Example 9.4. Max and Leo divided among themselves the proceeds from the sale of pies with blackberries to their friends. Max thought that if she takes 40% more coins, Leo's share will decrease by 60%. But how would Leo's share change if Max took 50% more coins for herself?

Solution: Since 40% of Merry's money is equal to 60% of Leo's money, Max has 1.5 times more coins. Therefore, an increase in Max's share by $n\%$ reduces Leo's share by $1.5n\%$. Thus, the answer is that it would decrease by 75%. \square



Example 9.5. Beavers and squirrels decided to go on a hike. Initially, the squirrels comprised 25% of the total number of animals. However, one squirrel did not come, and one beaver came instead, so the number of squirrels became only 20% of the total number. How many squirrels and how many beavers went on a hike?

Solution: Initially, there were a quarter of squirrels among the participants, and when one of them did not come, there was a fifth of the same number, as a beaver came instead. Then, if there were a total of x participants, initially, there were $x/4$ squirrels, and later there were $x/5$. Since the difference is one squirrel, you can get the equation $x/4 - x/5 = 1$, from which $x = 20$. Thus, in the final composition of the participants in the hike, there were $20/5 = 4$ squirrels and $20 - 4 = 16$ beavers. \square

Problem Set

Problem 9.1. (COM — 2015.7.1): Three pirates divided a bag of coins. The first took $\frac{3}{7}$ of all coins, the second took 51% of the remaining coins, and after that, the third had 8 coins less than the second. How many coins were in the bag?

Problem 9.2. (MF — 1994.7.1): Over two years, a factory reduced the volume of its production by 51%. During each year, the production volume decreased by the same percentage. By how much did the production volume decrease?



Problem 9.3. (MF — 1996.6.2): Alice, Beatrice, and Clarice were hunting mushrooms. Beatrice got 20% more mushrooms than Alice but 20% less than Vasya. By what percentage did Clarice get more mushrooms than Alice?

Problem 9.4. (COM — 2004.6.2): One number increased by 2%, and another increased by 3%. Could the sum increase by 5%? (Numbers are considered positive.)

Problem 9.5. (MF — 2009.6.3): In a park, there were linden and maple trees. Maples accounted for 60% of all trees. In spring, more linden trees were planted, after which maples accounted for 20%. In autumn, more maples were planted, and the share of maples became 60% again. By what factor did the number of trees in the park increase during the year?

Problem 9.6. (AT – 2015.2): For breakfast, Max and Leo ate candies. Leo ate all his candies, while Max ate only 20% of his candies. It is known that together, they ate 80% of all the candies they had before breakfast. Who had more candies before breakfast, and by what factor?

Problem 9.7. (AT – 2011.2): Jean has 10% more «D» grades than Lisa. Jean did extra work and got 10% of his «D»s replaced by better grades, while Lisa only improved 1% of hers. Who has more «D»s to improve?

Problem 9.8. (MF – 2001.7.2): Coming to the shooting competition, a player contributes 100 rubles to the cash register. After each successful shot, his money increases by 10%, and after each miss, it decreases by 10%. Could the player have 80.19 rubles after several shots?

Problem 9.9. (MF – 1998.7.2): In the Banana Republic, parliamentary elections were held, and all residents participated. All those who voted for the «Tangerine» party love tangerines. Among those who voted for other parties, 90% do not like tangerines. What percentage of votes did the «Tangerine» party receive in the elections if exactly 46% of the residents love tangerines?

Problem 9.10. (MF – 1991.7.3): At the beginning of the year, screws, nails, and bolts were sold at the same price of 1 ruble per 1 kg. On February 27, the Supreme Soviet of the USSR passed a law to increase the price of screws by 50% and decrease the price of nails by 50%. On February 28, the Supreme Soviet of the RSFSR passed a law to decrease the price of screws by 50% and increase the price of nails by 50%. Which product will be the most expensive, and which will be the cheapest in March?

Problem 9.11. (COM – 2014.7.4): At the beginning of the year, there were 25 students in the 7th grade. After seven newcomers came, the percentage of «straight A» students increased by 10 (if it was $a\%$ at the beginning of the year, now it is $(a + 10)\%$). How many «straight A» students are there now in the class?

Problem 9.12. (COM – 2008.7.6): The thick edition of the newspaper costs 30 rubles, while the thin one is cheaper. Pensioners have a discount of the same percentage for

all newspapers, so they buy the thin edition of the same newspaper for 15 rubles. It is known that in any case, the newspaper costs a whole number of rubles. How much does the thin newspaper cost without a discount, and how much does the thick newspaper cost for pensioners?

Problem 9.13. (COM – 2010.6.8): Alice buried two ingots on the Field of Wonders – one golden and one silver. In good weather, the golden ingot increases by 30%, and the silver one – by 20%. And on bad weather days, the golden ingot decreases by 30%, and the silver one – by 20%. After a week, it turned out that one of the ingots had increased, and the other had decreased. How many days was the weather good?

Problem 9.14. (COM – 2014.6.8): Max deposited a certain amount in rubles in a bank at 20% per annum. Leo took another amount in rubles, converted it into dollars, and deposited it in a bank at 10% per annum. In a year, the price of one dollar in rubles increased by 9.5%. When Leo converted his deposit back into rubles, it turned out that Max and Leo made the same profit over the year. Who initially had more money – Max or Leo?

Problem 9.15. (AMC – 2023.10B.3): In an after-school program for juniors and seniors, there is a debate team with an equal number of students from each class. Among the 28 students in the program, 25% of the juniors and 10% of the seniors are on the debate team. How many juniors are there in the program?

- (A) 5 (B) 6 (C) 8 (D) 11 (E) 20

Problem 9.16. (AMC – 2023.10B.2): Carlos went to a sports store to buy running shoes. Running shoes were on sale, with prices reduced by 20% on every pair of shoes. Carlos also knew that he had to pay a 7.5% sales tax on the discounted price. He had 43 dollars. What is the original (before discount) price of the most expensive shoes he could afford to buy?

- (A) \$46 (B) \$50 (C) \$48 (D) \$47 (E) \$49

Problem 9.17. (JMO – 2022.A3): In a class, 55% of students scored at least 55% on

a test. 65% of students scored at most 65% on the same test. What percentage of students scored between 55% and 65% (inclusive) on the test?

Problem 9.18. (JMO – 2022.B1): The sum of two numbers is 90.

40% of the first number is 15 more than 30% of the second number.

Find the two numbers.

Problem 9.19. (AMC – 2019.8.22): A store increased the original price of a shirt by a certain percentage and then lowered the new price by the same amount. Given that the resulting price was 84% of the original price, by what percent was the price increased and decreased ?

(A) 16 (B) 20 (C) 28 (D) 36 (E) 40

Problem 9.20. (AMC – 2017.10A.14): Every week, Roger pays for a movie ticket and a soda out of his allowance. Last week, Roger's allowance was A dollars. The cost of his movie ticket was 20% of the difference between A and the cost of his soda, while the cost of his soda was 5% of the difference between A and the cost of his movie ticket. To the nearest whole percent, what fraction of A did Roger pay for his movie ticket and soda?

(A) 9% (B) 19% (C) 22% (D) 23% (E) 25%

Problem 9.21. (AMC – 2014.10A.11): A customer who intends to purchase an appliance has three coupons, only one of which may be used:

Coupon 1: 10% off the listed price if the listed price is at least \$50

Coupon 2: \$20 off the listed price if the listed price is at least \$100

Coupon 3: 18% off the amount by which the listed price exceeds \$100

For which of the following listed prices will coupon 1 offer a greater price reduction than either coupon 2 or coupon 3?

- (A) \$179.95 (B) \$199.95 (C) \$219.95 (D) \$239.95 (E) \$259.95

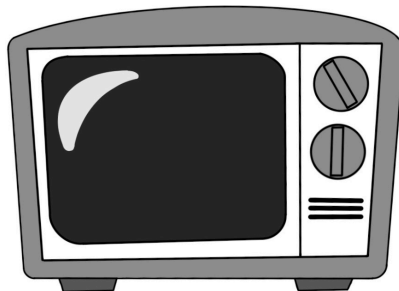
Problem 9.22. (AMC — 2008.12A.6): Heather compares the price of a new computer at two different stores. Store *A* offers 15% off the sticker price followed by a \$ 90 rebate, and store *B* offers 25% off the same sticker price with no rebate. Heather saves \$ 15 by buying the computer at store *A* instead of store *B*. What is the sticker price of the computer in dollars?

- (A) 750 (B) 900 (C) 1000 (D) 1050 (E) 1500

Problem 9.23. (AIME — 2008.I.1): Of the students attending a school party, 60% of the students are girls, and 40% of the students like to dance. After these students are joined by 20 more boy students, all of whom like to dance, the party is now 58% girls. How many students now like to dance at the party?

Problem 9.24. (Cyprus MO — 2007.24): Costas sold two televisions for €198 each. From the sale of the first one, he made a profit of 10% on its value, and from the sale of the second one, he had a loss of 10% on its value. After the sale of the two televisions, Costas had in total

- (A) profit 4 (B) neither profit nor loss
(C) loss 8 (D) profit 8 (E) loss 4



Problem 9.25. (AMC — 2007.12A.5): Last year, Mr. Jon Q. Public received an inheritance. He paid 20% in federal taxes on the inheritance and paid 10% of what he had left in state taxes. He paid a total of \$10500 for both taxes. How many dollars was his inheritance?

- (A) 30000 (B) 32500 (C) 35000 (D) 37500 (E) 40000

Problem 9.26. (AMC — 2006.12A.7): Mary is 20% older than Sally, and Sally is 40% younger than Danielle. The sum of their ages is 23.2 years. How old will Mary be on her next birthday?

- (A) 7 (B) 8 (C) 9 (D) 10 (E) 11

Problem 9.27. (AMC — 2005.12B.2): A positive number x has the property that $x\%$ of x is 4. What is x ?

- (A) 2 (B) 4 (C) 10 (D) 20 (E) 40

Problem 9.28. (AMC — 2004.12A.9): A company sells peanut butter in cylindrical jars. Marketing research suggests that using wider jars will increase sales. If the diameter of the jars is increased by 25% without altering the volume, by what percent must the height be decreased?

- (A) 10 (B) 25 (C) 36 (D) 50 (E) 60

Problem 9.29. (AHSME — 1993.5): Last year, a bicycle cost \$160, and a cycling helmet cost \$40. This year, the cost of the bicycle increased by 5%, and the cost of the helmet increased by 10%. The percent increase in the combined cost of the bicycle and the helmet is:

- (A) 6% (B) 7% (C) 7.5% (D) 8% (E) 15%

Problem 9.30. (AHSME — 1964.9): A jobber buys an article at \$24 less $12\frac{1}{2}\%$. He

then wishes to sell the article at a gain of $33\frac{1}{3}\%$ of his cost after allowing a 20% discount on his marked price. At what price, in dollars, should the article be marked?

- (A) 25.20 (B) 30.00 (C) 33.60 (D) 40.00 (E) none of these

Problem 9.31. (AHSME — 1950.23): A man buys a house for \$10,000 and rents it. He puts $12\frac{1}{2}\%$ of each month's rent aside for repairs and upkeep; pays \$325 a year taxes and realizes $5\frac{1}{2}\%$ on his investment. The monthly rent (in dollars) is:

- (A) 64.82 (B) 83.33 (C) 72.08 (D) 45.83 (E) 177.08

Skill Assessment Problems



Skill Assessment Problem 9.1. A bookseller sold a book, discounting 5% of the original price, and as a result, made a profit of 14%. How much profit (in percentage) would the merchant have made without a discount?

Skill Assessment Problem 9.2. In the first month, the price of a product increased by n cents, and in the second month, it increased by 300 cents. During the first month, the price increased by 300%, and during the second month, it increased by $n\%$. What is the new price of the product?

Solutions of Skill Assessment Problems

Solution to Problem 9.1: Let the initial price of the book be a . After the discount, it will be $0.95a$, which is $100\% + 14\% = 114\%$ of the price at which the merchant himself bought the book and which was

$$0.95a \div 1.14 = \frac{5}{6}a.$$

Calculate the income that the merchant planned to receive (in percent):

$$\left(a \div \frac{5}{6}a\right) \cdot 100\% = 120\%.$$

The merchant planned to make a profit of $120\% - 100\% = 20\%$. □

Solution to Problem 9.2: Let the initial price be x , then after the first month, the price increased to $x + n$, and after the second month, it increased to $x + n + 300$ cents. For the first month, the price increased by 300% , i.e., $n = 3x$. It is also known that for the second month, the price increased by $n\%$, i.e.,

$$(x + n) \cdot \frac{n}{100}.$$

Since the price increased by 300 cents, then

$$(x + n) \cdot \frac{n}{100} = 300.$$

Knowing that $n = 3x$, we get: $10000 = 4x^2$, which means $x = 50$. After the second month, the price became $x + n + 300 = 4x + 300 = 200 + 300 = 500$ cents. □

Blends and Concentrations



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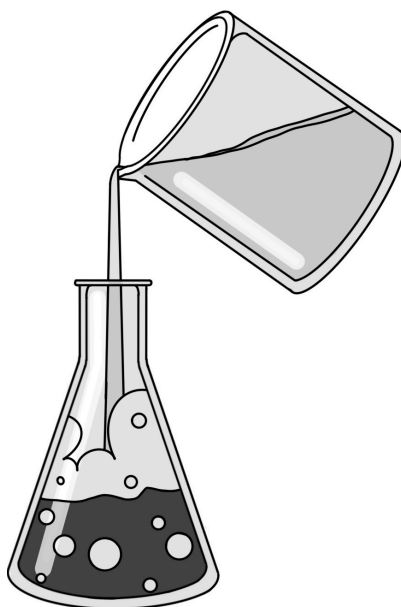
- Do you love math more than you love me?
- Darling! How could you even think that!
- Well, then prove it!
- Alright. Let R be the set of all beloved objects.

–Popular joke

Theory and Practice

The concentration of a substance in some solution is defined as the ratio of the mass or volume of that substance to the mass or volume of the entire solution, respectively. Such problems can be encountered in both olympiads and exams.

Example 10.1. In a 10% acid solution with a mass of 3 kg, 1.8 liter of pure water was added. What is the concentration of the new solution (in percentage)?



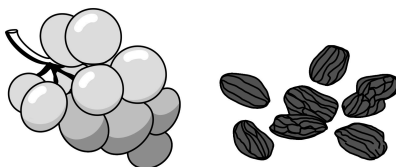
Solution: The problem can be reduced to finding the ratio of the mass of acid to the mass of the solution. Obviously, the amount of acid in the solution will be: $3 \cdot 0.1 = 0.3$ liter (there is no acid in water), and the total mass of the solution will be $3 + 1.8 = 4.8$ kg. Hence:

$$\frac{0.3}{4.8} = 0.0625 = 6.25\%.$$

□

Let's consider another well-known example.

Example 10.2. Raisins contain 5% moisture. They are made from grapes containing 90% moisture. How many kilograms of grapes are needed to obtain 20 kg of raisins?



Solution: In the process of making raisins, the mass of moisture in the grapes decreases, while the mass of the pulp remains unchanged. In raisins, the pulp constitutes 95%, thus, $19/20$ parts. In 20 kg of raisins, there should be: $19/20 \cdot 20 = 19$ kg, but these 19 kg of pulp will constitute $1/10$ part of the grapes. Thus, a total of $19 \cdot 10 = 190$ kg of grapes is needed. \square

Example 10.3. There are two glasses: one with milk and the other with water. A spoonful was transferred from the first to the second, mixed, and then a spoonful of the mixture was poured back.

- a) What is greater: the amount of water in the glass with milk or the amount of milk in the glass with water?
- b) The same question, but if the described procedure is repeated 100 times.

Solution: Let the volume of each of the glasses be a , which means that there will be a water and a milk in total. Suppose that, after some number of operations where the volumes of the solutions in the glasses are equal, the amount of water in the glass with milk is x . Then, the amount of water in the glass with water is $a - x$, and the amount of milk in the glass with it is $a - (a - x) = x$. Thus, the mentioned quantities are the same both after one operation and after 100 operations, as well as after any number of arbitrary operations without spillage, where the volume of liquid in the glasses remains equal. \square



Example 10.4. At a party on April Fools' Day, John and Ivan made themselves a cocktail. John mixed apple jam with cherry, and Ivan mixed apricot with grape. It is known that in apple jam, the sugar content is higher than in apricot, and in cherry, it is higher than in grape. Can we conclude that John is drinking a sweeter cocktail?

Solution: No, from the given information, we cannot conclude that. For example, let the sugar content in apple jam be 100%, in apricot – 90%, in cherry – 10%, and in grape – 0%. Let John take 1 kg of apple and 9 kg of cherry, and Ivan – 9 kg of apricot and 1 kg of grape. Then, the sugar content in John's mixture will be

$$\frac{1 \cdot 1 + 0,1 \cdot 9}{10} = 0,19,$$

and in Ivan's mixture

$$\frac{0,9 \cdot 9 + 0 \cdot 1}{10} = 0,81.$$

□

Lemma. The concentration of the active substance in a mixture of 2 solutions will be no less than the smallest of the concentrations of the original solutions and no greater than the largest of the concentrations of the original solutions. Equality is achieved only when the concentrations of the original solutions are equal.

Proof. Let the mass of the mixed substances be a and b , and the concentrations be x and y . Then, the concentration of the active substance in the final solution will be

$$\frac{ax + by}{a + b}.$$

Let $x < y$, and we will show that this concentration is no less than x :

$$\frac{ax + by}{a + b} - x = \frac{ax + by - ax - bx}{a + b} = \frac{b(y - x)}{a + b} \geq 0,$$

and equality is achieved only in the case of equality of x and y . The upper bound can be proven in a similar way.

Problem Set

Problem 10.1. (COM — 2005.6.5): The snack bar worker makes a milk and cherry cocktail by mixing milk and cherry juice in a blender. Milk costs 20 rubles per liter, and cherry juice costs 30 rubles per liter. It is known that the cost of milk poured into the blender is equal to the cost of juice poured into the blender. How much does a liter of milk and cherry cocktail cost?

Problem 10.2. (MMO — 1994.8.1): The coop receives apple and grape juices in identical containers and produces an apple-grape drink in identical cans. One container of apple juice can contain exactly 6 cans of the drink, and one container of grape juice can contain exactly 10 cans. When the recipe for the drink was changed, one container of apple juice became enough for exactly 5 cans of the drink. How many cans of the drink will one container of grape juice be enough for now? (The drink is not diluted with water.)

Problem 10.3. (ER — 2011.8.3): There are three-liter jars and a 100 ml measuring cup. The first jar is empty, the second jar contains 700 ml of sweet tea, and the third contains 800 ml of sweet tea. In the second jar, 50 g of sugar is dissolved, and in the third jar, 60 g of sugar. It is allowed to pour a full measuring cup of tea from any jar into any other jar. By repeating this action several times, is it possible to achieve the scenario when the first jar is empty, and the amount of sugar in the second and the third jars is equal?

Problem 10.4. (AMC — 2014.10A.5): On an algebra quiz, 10% of the students scored 70 points, 35% scored 80 points, 30% scored 90 points, and the rest scored 100 points. What is the difference between the mean and median scores of the student's scores on this quiz?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 10.5. (iTest — 2008.5): Jerry recently returned from a trip to South America, where he helped two old factories reduce pollution output by installing more modern

scrubber equipment. Factory *A* previously filtered 80% of pollutants, and Factory *B* previously filtered 72% of pollutants. After installing the new scrubber system, both factories now filter 99.5% of pollutants.

Jerry explains the level of pollution reduction to Michael: «Factory *A* is the much larger factory. It's four times as large as Factory *B*. Without any filters at all, it would pollute four times as much as Factory *B*. Even with the better pollution filtration system, Factory *A* was polluting nearly three times as much as Factory *B*.»

Assuming the factories are the same in every way except size and previous percentage of pollution filtered, find $a + b$ where a/b is the ratio in lowest terms of volume of pollutants unfiltered from both factories *after* installation of the new scrubber system to the volume of pollutants unfiltered from both factories *before* installation of the new scrubber system.

Problem 10.6. (Cyprus MO – 2006.1) A dairy industry, in a quantity of milk with 4% fat, adds a quantity of milk with 1% fat and produces 1200kg of milk with 2% fat. The quantity of milk with 1% fat, that was added is (in kg)

- (A) 1000 (B) 600 (C) 800 (D) 120 (E) 480

Problem 10.7. (AMC – 2006.12B.11): Joe and JoAnn each bought 12 ounces of coffee in a 16-ounce cup. Joe drank 2 ounces of his coffee and then added 2 ounces of cream. JoAnn added 2 ounces of cream, stirred the coffee well, and then drank 2 ounces. What is the resulting ratio of the amount of cream in Joe's coffee to that in JoAnn's coffee?

- (A) $\frac{6}{7}$ (B) $\frac{13}{14}$ (C) 1 (D) $\frac{14}{13}$ (E) $\frac{7}{6}$

Problem 10.8. (AMC – 2006.10B.9): Francesca uses 100 grams of lemon juice, 100 grams of sugar, and 400 grams of water to make lemonade. There are 25 calories in 100 grams of lemon juice and 386 calories in 100 grams of sugar. Water contains no calories. How many calories are in 200 grams of her lemonade?

- (A) 129 (B) 137 (C) 174 (D) 233 (E) 411

Problem 10.9. (PMWC – 1997.I9): A chemist mixed an acid of 48% concentration with the same acid of 80% concentration and then added 2 liters of distilled water to the mixed acid. As a result, he got 10 liters of the acid of 40% concentration. How many milliliters of the acid of 48% concentration that the chemist had used? (1 litre = 1000 millilitres)

Problem 10.10. (AHSME – 1973.33): When one ounce of water is added to a mixture of acid and water, the new mixture is 20% acid. When one ounce of acid is added to the new mixture, the result is $33\frac{1}{3}\%$ acid. The percentage of acid in the original mixture is

- (A) 22% (B) 24% (C) 25% (D) 30% (E) $33\frac{1}{3}\%$

Problem 10.11. (AHSME – 1953.9): The number of ounces of water needed to reduce 9 ounces of shaving lotion containing 50 % alcohol to a lotion containing 30 % alcohol is:

- (A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Skill Assessment Problems

Skill Assessment Problem 10.1. A group of friends gathered in a cafe, and each ordered one cup of coffee with cream. Jean drank a quarter of all the cream and a sixth of all the coffee if all their friends mixed their drinks together. How many friends were there in total? (The volume of the cups is the same for everyone, unlike the strength of the coffee.)



Skill Assessment Problem 10.2. There are two cups; one contains a certain amount of coffee, and the other contains the same amount of cream. It is allowed to pour some amount of liquid from one cup to another (in this case, the liquids are evenly mixed). Is it possible, using such operations, to have a scenario when the first cup contains a mixture in which the concentration of cream is greater than in the second cup?

Skill Assessment Problem 10.3. Two alloys weighing 6 and 12 kg, with different lead percentages, were cut, taking a piece of the same weight from each. Then, the piece that was cut of the first alloy was melted with the remainder of the second piece. The same was done with the piece that was cut of the second piece and the remainder of the first piece, with the remainder of the other piece. After that, the percentage of lead in those two new alloys became equal. Find the weights of each of the cut pieces.

Solutions of Skill Assessment Problems

Solution to Problem 10.1: If only 4 cups of coffee were drunk, then they contained all the cream, and $2/3$ of all the coffee — i.e., less than required. Therefore, there were more than four people in the company. Similarly, with 6 cups, all the coffee would have been taken, but there would not have been enough cream, so there would have been fewer than 6 people. Therefore, it follows that there were 5 friends who drank coffee. \square

Solution to Problem 10.2: For simplicity, let's assume that the amount of liquid poured into each cup was 100 g. We will prove that the percentage of cream in the first cup will never exceed 50%, and in the second one, it will never be less than 50%.

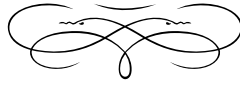
Initially, this statement is true. Suppose it is true at some point: in the first cup, there is not less coffee than cream, and in the second one, there is not less cream than coffee. Let's say we poured some amount of liquid from the first cup into the second. Then, the percentage of cream in the first cup remained the same, i.e., there was still no less coffee than cream in it. But in the second cup, together with the first one, there are 100 g of coffee and 100 g of cream. Therefore, in the second cup, after pouring, there is no less cream than coffee. So, the statement remains true during the pouring. \square

Solution to Problem 10.3: Let x be the lead concentration in the 6 kg piece and y be the lead concentration in the 12 kg piece, and a be the mass of the cut piece. Then, the condition of equal lead concentration after the specified actions looks like this:

$$\frac{(6 - a)x + ay}{6} = \frac{(12 - a)y + ax}{12},$$

from which $2(6 - a)x + 2ay = (12 - a)y + ax$, and we get $12(x - y) = 3a(x - y)$, so $a = 4$, i.e., it is necessary to cut 4 kg from each piece. \square

Progressions and Patterns



“

Driving on North American roads follows its own patterns. Here are some that I have noticed.

- The lane you're driving on is the slowest.
- If you change lanes to speed up, your new lane becomes the slowest, and the old one speeds up.
- If you have to travel through a major city during a long journey, it will be during rush hour.
- Snow, rain, and traffic jams are directly proportional to how much you are in a hurry.
- The size and visibility of a road sign are inversely proportional to its informativeness and usefulness.

—Popular joke

Theory and Practice

Sequence is a crucial concept in mathematics. For now, we define a sequence as a set of elements, where each element can be assigned a unique position. It's important to distinguish a *sequence* from a *set*. For instance, consider the sequence 4, 2, 8, 5, 7, 1 (perhaps someone recognizes the digits from the period of the fraction $\frac{3}{7}$). This sequence contains the elements of the set 1, 2, 4, 5, 7, 8. We could also represent this set as 1, 4, 8, 5, 7, 2. However, a sequence has a strict order.

In mathematical problems, sequences often follow some logic. Classic problems in this area include «continue the series» or «find the pattern» questions. What are some of the most well-known sequences? Here are a few, presented without algebraic details:

Definition 1. *Arithmetic progression* — a numerical sequence in which each successive element is the sum of the previous element and some constant number (*step* or *difference* of the progression).

Definition 2. *Geometric progression* — a numerical sequence in which each successive element is the previous element multiplied by some constant number (*common ratio* of the progression).

Definition 3. *Recursive sequence* — a sequence in which each successive element can be computed from the previous ones.

One of the most well-known recursive sequences is *Fibonacci numbers*, which are defined as follows:

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n.$$

For clarity, let's list the first few terms of this sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$$

Think about whether arithmetic and geometric progressions are recursive sequences. And how about the sequence 4, 2, 8, 5, 7, 1?

There is a legend about the greatest mathematician, Carl Friedrich Gauss (who was called the «Prince of Mathematicians» even during his lifetime), that is widely known. Young Gauss was in a classroom with older children and constantly distracted the teacher with questions. The teacher decided to keep him occupied for a long time and gave him the task of finding the sum of all natural numbers from 1 to 100, hoping that Gauss would leave him in peace. However, Gauss provided the answer in just a minute. He reasoned as follows: let the sum of the elements be denoted by S . The sum of the elements of the same sequence, written in reverse order, is the same. Let's write the sequences on top of each other and add them:

$$\begin{array}{rcccccc} 1 & + & 2 & + & \dots & + & 99 & + & 100 & = & S, \\ 100 & + & 99 & + & \dots & + & 2 & + & 1 & = & S, \\ \hline 101 & + & 101 & + & \dots & + & 101 & + & 101 & = & 2S, \end{array}$$

100 times

thus $2S = 101 \cdot 100$ and $S = 5050$.

Alternatively, the sum of the first and the last elements of the series is equal to the sum of the second and the penultimate elements. We can generalize it by noticing that $a_i + a_{100-i} = 101$ for all elements a_i of the sequence. Knowing the number of unique pairs, calculating the sum of the entire series is straightforward.

This idea forms the basis for finding the sum of any arithmetic progression.

In turn, the work of Archimedes laid the foundation for computing the sum of an infinite decreasing geometric progression (the positive geometric progression is called «decreasing» if the common ratio is less than 1). Let's apply his method to find the following sum:

$$\begin{aligned} S &= \frac{1}{4} + \frac{1}{4^2} + \dots \Big| \cdot 4, \\ 4S &= 1 + \left(\frac{1}{4} + \frac{1}{4^2} + \dots \right), \\ 4S &= 1 + S, \\ S &= \frac{1}{3}. \end{aligned}$$

Now that you are familiar with some common methods of solving problems that involve sequences, let's try a few problems where we need to find some patterns.

Example 11.1. (OMMO – 2017.11.1):

$$\frac{12 + 15}{18} + \frac{21 + 24}{27} + \dots + \frac{48 + 51}{54}.$$

Solution: This problem was given in one of the regional competitions for high school seniors, and the participants' greatest difficulty was not simplifying the resulting fraction but understanding what was hidden behind the ellipsis. Note that if we connect the numbers from smaller to larger, we will see that every 2 consecutive number differs by 3, and the arrows form a repeating geometric pattern.

$$\frac{12 + 15}{18} + \frac{21 + 24}{27} + \dots + \frac{48 + 51}{54}$$

Now, we can write down the given expression:

$$\begin{aligned} & \frac{12 + 15}{18} + \frac{21 + 24}{27} + \frac{30 + 33}{36} + \frac{39 + 42}{45} + \frac{48 + 51}{54} = \\ & = \frac{4 + 5}{6} + \frac{7 + 8}{9} + \frac{10 + 11}{12} + \frac{13 + 14}{15} + \frac{16 + 17}{18} = \\ & = \frac{(6 - 2) + (6 - 1)}{6} + \frac{(9 - 2) + (9 - 1)}{9} + \dots + \frac{(18 - 2) + (18 - 1)}{18} = \\ & = 2 - \frac{1}{2} + 2 - \frac{1}{3} + \dots + 2 - \frac{1}{6} = 2 \cdot 5 - \left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{6} \right) = \\ & = 10 - \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6} \right) - \left(\frac{1}{4} + \frac{1}{5} \right) = \frac{171}{20} = 8\frac{11}{20}. \end{aligned}$$

□

The example above shows how to tackle pattern recognition in sequence problems. But you might be wondering how to find the solution without detailing every step.

Example 11.2. What is the sum of the following:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{2017 \cdot 2018}.$$

Solution: Notice that:

$$\frac{1}{x \cdot (x + 1)} = \frac{1}{x} - \frac{1}{x + 1}.$$

Then

$$\begin{aligned} & \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{2017 \cdot 2018} = \\ & = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2017} - \frac{1}{2018}\right) = \\ & = 1 - \frac{1}{2018} = \frac{2017}{2018}. \end{aligned}$$

□



Example 11.3. When Alice went to school, Mrs. Owless, rich after winning the lottery, promised to give him 1 coin at the end of the first year, 2 coins at the end of the second, 4 coins at the end of the third, and so on. By the end of school, Alice got a total of 65535 coins. How many years did Alice finish?

Solution: First, let's solve a more general problem of finding the sum of the first n terms of a geometric progression $1, 2, 4, \dots, 2^n$:

$$S_n = 1 + 2 + 4 + \dots + 2^n,$$

$$2S_n = 2 + 4 + \dots + 2^n + 2^{n+1}.$$

By subtracting the first equation from the second, we get:

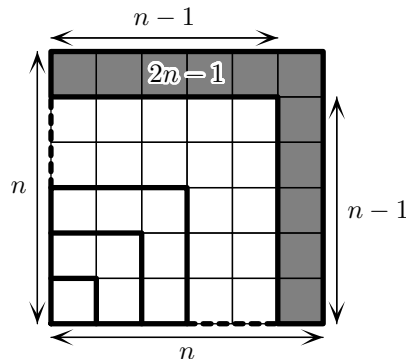
$$S_n = 2^{n+1} - 1.$$

By trial and error, we can find the appropriate n . Considering that $65536 = 2^{15+1}$, Alice finished 15 years of school. \square

Sometimes, when trying to find the sum of a sequence, geometric considerations can be useful.

Example 11.4. What is the sum of all odd numbers from 1 to $(2n - 1)$?

Solution: Let's prove that this sum is equal to n^2 . The diagram illustrates the solution: the i -th triangle contains $2i - 1$ cells, and their total area is n^2 .



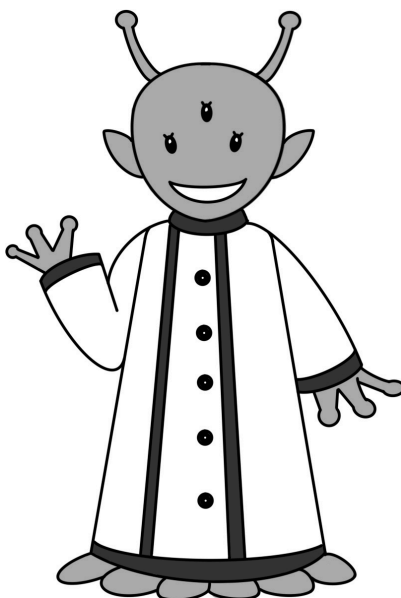
\square

Problem set

Problem 11.1. (2ARSO – 2011/2012.11.1) An infinite increasing arithmetic progression has the property that the product of any two different terms is also a term of this progression. Prove that all its terms are integers.

Problem 11.2. (LT – 1982.04): a) There are six distinct natural numbers that sum up to 22. Find these numbers and prove that there are no other solutions.

b) The same question but for 100 numbers that sum up to 5051.



Problem 11.3. (MF – 1993.6.1): When an alien from the star Tau Ceti arrived on Earth on Monday, he exclaimed: «A!». On Tuesday, he exclaimed: «AU!», on Wednesday – «AUUA!», and on Thursday – «AUUAUAAU!». What will he exclaim on Saturday?

Problem 11.4. (LT – 1999.02): On a grid paper, a figure is drawn such that each row contains an increasing number of cells: in the top row, there is one cell; in the

second row from the top, there are three cells; in the next row, there are 5 cells, and so on, with a total of n rows. Prove that the total number of cells is a square of some number.

Problem 11.5. (JMO – 2021.B5): In Sally’s sequence, each term from the third onwards is the sum of the previous two terms. Additionally, every term is a positive integer. Sally’s eighth term is 400.

Find the minimum value of the third term in Sally’s sequence.

Problem 11.6. (AMC – 2023.10B.6): Let $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$. How many terms in the sequence $L_1, L_2, L_3, \dots, L_{2023}$ are even?

- (A) 673 (B) 1011 (C) 675 (D) 1010 (E) 674

Problem 11.7. (AMC – 2023.10B.10): Maureen is keeping track of the mean score that she got on her quizzes this semester. If Maureen scores 11 on the next quiz, her mean will increase by 1. If she scores 11 on each of the next three quizzes, her mean will go up by 2. What is the mean of her quiz scores currently?

- (A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Problem 11.8. (AMC – 2023.10B.14): Suppose that S is a subset of $\{1, 2, 3, \dots, 25\}$ such that the sum of any two (not necessarily distinct) elements of S is never an element of S . What is the maximum number of elements S may contain?

- (A) 12 (B) 13 (C) 14 (D) 15 (E) 16

Problem 11.9. (AIME – 2023.II.1): The number of apples growing on each of the six apple trees forms an arithmetic sequence where the greatest number of apples growing on any of the six trees is double the least number of apples growing on any of the six trees. The total number of apples growing on all six trees is 990. Find the greatest number of apples growing on any of the six trees.

Problem 11.10. (AMC – 2018.10B.20): A function f is defined recursively by $f(1) = f(2) = 1$ and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \geq 3$. What is $f(2018)$?

- (A) 2016 (B) 2017 (C) 2018 (D) 2019 (E) 2020

Problem 11.11. (AIME – 2015.I.1): The expressions $A = 1 \times 2 + 3 \times 4 + 5 \times 6 + \cdots + 37 \times 38 + 39$ and $B = 1 + 2 \times 3 + 4 \times 5 + \cdots + 36 \times 37 + 38 \times 39$ are obtained by writing multiplication and addition operators in an alternating pattern between successive integers. Find the positive difference between integers A and B .

Problem 11.12. (AMC – 2014.10A.10): Five positive consecutive integers starting with a have an average of b . What is the average of 5 consecutive integers that start with b ?

- (A) $a + 3$ (B) $a + 4$ (C) $a + 5$ (D) $a + 6$ (E) $a + 7$

Problem 11.13. (UNM-PNM Statewide High School Mathematics Contest – 2018.II.1): Let $x \neq y$ be two real numbers. Let x, a_1, a_2, a_3, y and b_1, x, b_2, b_3, y, b_4 be two arithmetic sequences.

Calculate $\frac{b_4 - b_3}{a_2 - a_1}$.

Problem 11.14. (UNCO Math Contest – 2014.II.4): On the first slate, the Queen's jurors write the number 1. On the second slate, they write the numbers 2 and 3. On the third slate, the jurors write 4, 5, 6, and so on, writing n integers on the n th slate.

- (a) What is the largest number they write on the 20th slate?
- (b) What is the sum of the numbers written on the 20th slate?
- (c) What is the sum of the numbers written on the n th slate?

Problem 11.15. (UNCO Math Contest – 2014.II.2): Define the Cheshire Cat function $\boxed{:)}$ by

$$\begin{aligned}\boxed{:)}(x) &= -x && \text{if } x \text{ is even and} \\ \boxed{:)}(x) &= x && \text{if } x \text{ is odd}\end{aligned}$$

Find the sum $\boxed{:)}(1) + \boxed{:)}(2) + \boxed{:)}(3) + \boxed{:)}(4) + \dots + \boxed{:)}(289)$

Problem 11.16. (AIME – 2012.II.2): Two geometric sequences a_1, a_2, a_3, \dots and b_1, b_2, b_3, \dots have the same common ratio, with $a_1 = 27$, $b_1 = 99$, and $a_{15} = b_{11}$. Find a_9 .

Problem 11.17. (AMC – 2010.12A.2): A ferry boat shuttles tourists to an island every hour, starting at 10 AM until its last trip, which starts at 3 PM. One day, the boat captain notes that on the 10 AM trip, there were 100 tourists on the ferry boat and that on each successive trip, the number of tourists was 1 less than on the previous trip. How many tourists did the ferry take to the island that day?

- (A) 585 (B) 594 (C) 672 (D) 679 (E) 694

Problem 11.18. (AMC – 2010.12A.10): The first four terms of an arithmetic sequence are p , 9, $3p - q$, and $3p + q$. What is the 2010th term of this sequence?

- (A) 8041 (B) 8043 (C) 8045 (D) 8047 (E) 8049

Problem 11.19. (iTest – 2008.42): Joshua's physics teacher, Dr. Lisi, lives next door to the Kubiks and is a long-time friend of the family. An unusual fellow, Dr. Lisi spends as much time surfing and raising chickens as he does trying to map out a *Theory of Everything*. Dr. Lisi often poses problems to the Kubik children to challenge them to think a little deeper about math and science. One day, while discussing

sequences with Joshua, Dr. Lisi writes out the first 2008 terms of an arithmetic progression that begins $-1776, -1765, -1754, \dots$. Joshua then computes the (positive) difference between the 1980th term in the sequence, and the 1977th term in the sequence. What number does Joshua compute?

Problem 11.20. (iTest – 2008.43): Alexis notices Joshua working with Dr. Lisi and decides to join in on the fun. Dr. Lisi challenges her to compute the sum of all 2008 terms in the sequence. Alexis thinks about the problem and remembers a story one of her teachers at school taught her about how a young Karl Gauss quickly computed the sum $1 + 2 + 3 + \dots + 98 + 99 + 100$ in elementary school. Using Gauss's method, Alexis correctly finds the sum of the 2008 terms in Dr. Lisi's sequence. What is this sum?

Note: Dr. Lisi's sequence is $-1776, -1765, -1754 \dots$

Problem 11.21. (iTest – 2008.31): The n^{th} term of a sequence is $a_n = (-1)^n(4n + 3)$. Compute the sum $a_1 + a_2 + a_3 + \dots + a_{2008}$.

Problem 11.22. (AMC – 2008.12A.4): Which of the following is equal to the product

$$\frac{8}{4} \cdot \frac{12}{8} \cdot \frac{16}{12} \cdot \dots \cdot \frac{4n+4}{4n} \cdot \dots \cdot \frac{2008}{2004}?$$

(A) 251 (B) 502 (C) 1004 (D) 2008 (E) 4016

Problem 11.23. (UNCO Math Contest – 2007.II.1): Express the following sum as a whole number:

$$1 + 2 - 3 + 4 + 5 - 6 + 7 + 8 - 9 + 10 + 11 - 12 + \dots + 2005 + 2006 - 2007.$$

Problem 11.24. (iTest – 2007.5): Compute the sum of all twenty-one terms of the geometric series

$$1 + 2 + 4 + 8 + \dots + 1048576.$$

(A) 2097149 (B) 2097151 (C) 2097153 (D) 2097157 (E) 2097161

Problem 11.25. (AMC – 2006.10B.18): Let a_1, a_2, \dots be a sequence for which $a_1 = 2$, $a_2 = 3$, and $a_n = \frac{a_{n-1}}{a_{n-2}}$ for each positive integer $n \geq 3$. What is a_{2006} ?

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{3}{2}$ (D) 2 (E) 3

Problem 11.26. (PMWC – 2005.T6):

$$\begin{aligned} 1 + 2 &= 3 \\ 4 + 5 + 6 &= 7 + 8 \\ 9 + 10 + 11 + 12 &= 13 + 14 + 15 \\ &\vdots \end{aligned}$$

If this pattern is continued, find the last number in the 80th row (e.g., the last number of the third row is 15).

Problem 11.27. (AIME – 2005.I.2): For each positive integer k , let S_k denote the increasing arithmetic sequence of integers whose first term is 1 and whose common difference is k . For example, S_3 is the sequence 1, 4, 7, 10, \dots . For how many values of k does S_k contain the term 2005?

Problem 11.28. (AMC – 2004.12A.17): Let f be a function with the following properties:

(i) $f(1) = 1$, and

(ii) $f(2n) = n \cdot f(n)$ for any positive integer n .

What is the value of $f(2^{100})$?

- (A) 1 (B) 2^{99} (C) 2^{100} (D) 2^{4950} (E) 2^{9999}

Problem 11.29. (AMC – 2004.10A.7): A grocer stacks oranges in a pyramid-like stack whose rectangular base is 5 oranges by 8 oranges. Each orange above the first level rests in a pocket formed by four oranges below. The stack is completed by a single row of oranges. How many oranges are in the stack?

- (A) 96 (B) 98 (C) 100 (D) 101 (E) 134

Problem 11.30. (AMC – 2002.12B.8): Suppose July of year N has five Mondays. Which of the following must occur five times in the August of year N ? (Note: Both months have 31 days.)

- (A) Monday (B) Tuesday
(C) Wednesday (D) Thursday (E) Friday

Problem 11.31. (AMC – 2002.12A.15): The mean, median, unique mode, and range of a collection of eight integers are all equal to 8. The largest integer that can be an element of this collection is

- (A) 11 (B) 12 (C) 13 (D) 14 (E) 15

Problem 11.32. (AHSME – 1963.20): Two men at points R and S , 76 miles apart, set out at the same time to walk towards each other. The man at R walks uniformly at the rate of $4\frac{1}{2}$ miles per hour; the man at S walks at the constant rate of $3\frac{1}{4}$ miles per hour for the first hour, at $3\frac{3}{4}$ miles per hour for the second hour, and so on, in arithmetic progression. If the men meet x miles nearer R than S in an integral number of hours, then x is:

- (A) 10 (B) 8 (C) 6 (D) 4 (E) 2

Skill Assessment Problems

Skill Assessment Problem 11.1. Find the sum of the digits of all natural numbers from 1 to 10^9 .

Skill Assessment Problem 11.2. Find the sum: $1 + 2024 + 2024^2 + \dots + 2024^n$.

Skill Assessment Problem 11.3. Find the sum:

$$\frac{1}{2} + \frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \dots + \frac{1}{2022 \cdot 2024}.$$

Solutions of Skill Assessment Problems

Solution to Problem 11.1: In this problem, we can form 500 million pairs: (0 and 999,999,999), (1 and 999,999,998), and so on, and in each pair, the sum of the digits is 81. The only number that is left is 1,000,000,000. Therefore, the total sum of the digits is $500,000,000 \cdot 81 + 1 = 40,500,000,001$.

Answer: 40,500,000,001. □

Solution to Problem 11.2: Let S be the desired sum:

$$\begin{aligned} S &= 1 + 2024 + 2024^2 + \dots + 2024^n, \\ S &= 1 + 2024(1 + 2024 + 2024^2 + \dots + 2024^{n-1}), \\ S &= 1 + 2024(S - 2024^n), \\ 2023S &= 2024^{n+1} - 1, \\ S &= \frac{2024^{n+1} - 1}{2023}. \end{aligned}$$

In fact, by solving this problem, we learned how to find the sum of a geometric progression, deriving the formula that is covered in high school math courses. □

Solution to Problem 11.3: Similarly to the example from the theoretical part of this chapter, we can see that

$$\frac{2}{x \cdot (x + 2)} = \frac{1}{x} - \frac{1}{x + 2}.$$

Then:

$$\begin{aligned} &\frac{1}{2} + \frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \dots + \frac{1}{2022 \cdot 2024} = \\ &= \frac{1}{2} \left(1 + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \dots + \left(\frac{1}{2022} - \frac{1}{2024} \right) \right) = \\ &= \frac{1}{2} \left(\frac{3}{2} - \frac{1}{2024} \right). \end{aligned}$$

□

Inequalities in Word Problems



“

The university rector reviewed the estimate brought to him by the dean of the faculty of physics and, sighing, said:

– Why do physicists always demand such expensive equipment? For instance, mathematicians only ask for money for paper, pencils, and erasers.

And, after some thought, he added:

– Philosophers are even better. They don't need erasers at all.

—Popular joke

Theory and Practice

This chapter focuses on problems where conditions are represented through systems of equations and inequalities. Some of these inequalities can be solved using simple logical reasoning without needing special knowledge.

Example 12.1. At the Russian-French meeting, only these two nationalities were present. The French attendees had more money overall than the Russians, and the women had more money than the men. Does this mean there is definitely a French-woman at the meeting?

Solution: Consider what would happen if there were no French women at the meeting. Then, all the women at the meeting are Russians. The total money held by women exceeds that of men. This means that the money held by Russian women is greater than the combined total of Russian and French men. Since this total is at least as much as the money held by all French men—and since French men’s money is part of the total for all French attendees—Russian women’s money surpasses the total amount held by all French individuals. Thus, the amount of money for Russian women exceeds the amount of money for all the French, the amount of money for which, according to the conditions, exceeds the amount of money for Russians. Such a situation would be possible only if Russian men had a negative amount of money, but in math olympiads, such cases are usually impossible. Thus, there will definitely be Frenchwomen at the meeting. \square

Inequalities can also arise in problems related to motion, for example.

Example 12.2. A plane (with its own speed of 500 km/h) flew from Crocotown to Beavertown and back in windy weather with a wind speed of 100 km/h. The next day, it repeated the same flight in calm weather. In both cases, the engines of the plane developed the same power. On which day did it spend more time on the flight?

Solution: Let the distance from Crocotown to Beavertown be X kilometers. Then in windy weather, the plane flew one way in $\frac{X}{400}$ hours, and back in $\frac{X}{600}$ hours, which

together amounts to

$$\left(\frac{1}{400} + \frac{1}{600}\right) \cdot X = \frac{1}{240} \cdot X \text{ hours.}$$

In calm weather, one way, the plane takes $\frac{X}{500}$ hours, which amounts to $\frac{1}{250} \cdot X$ hours round trip.

Since $\frac{1}{240} > \frac{1}{250}$, the plane spent more time on the flight on the first day. \square

Example 12.3. Out of a hundred students at the Faculty of Physics and Mathematics, 80 received excellent (5) and good (4) grades on the first exam (in general physics), 72 on the second exam (in mathematical analysis), and 60 on the third exam (in analytical geometry). What is the minimum number of students who could receive excellent and good grades (i.e., those who will receive a scholarship in the next semester) on all three exams?

Solution: Note that the number of students who received grades not higher than 3 is $100 - 80 = 20$ for the first exam, $100 - 72 = 28$ for the second exam, and $100 - 60 = 40$ for the third exam. Thus, the total number of students who received at least one grade not higher than 3 does not exceed $20 + 28 + 40 = 88$, which means that the number of students with grades not lower than 4 is guaranteed to be at least 12. Let the students from 21 to 100 have passed the first exam well; those from 1 to 20 and from 49 to 100 have passed the second exam well, and those from 1 to 48 and from 89 to 100 have passed the third exam well. It is easy to see that, in this case, their number will be 12. \square

Example 12.4. Near the dam, linden and oak trees are planted, and their total number is more than 14. If the number of linden trees is doubled, and the number of oak trees is increased by 18, then there will be more oak trees than linden trees. However, if the number of oak trees is doubled without changing the number of linden trees, then there will still be more linden trees than oak trees. How many linden and oak trees are growing near the dam?

Solution: Let l and o denote the number of linden and oak trees, respectively. From the given information:

$$\begin{aligned}l + o &> 14, \\o + 18 &> 2l, \\l &> 2o.\end{aligned}$$

If a and b are integers, then $a > b \Leftrightarrow a \geq b + 1$. Rewrite the conditions taking this into account:

$$\begin{aligned}l + o &\geq 15, \\o + 18 &\geq 2l + 1, \\l &\geq 2o + 1.\end{aligned}$$

Adding all these inequalities, we get: $18 \geq 17$. Thus, either all of them become equalities, or 2 of them become equalities, and in the third one, the left side exceeds the right side by 1. The first case is not possible because then $l = 15 - o$, $15 - o = 2o + 1$, $o = \frac{14}{3}$. It is easy to understand that it is necessary for the first inequality not to become an equality. Then $l = 11$, $o = 5$. \square

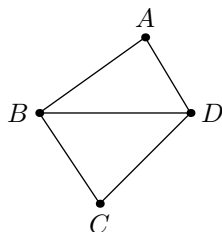
Problem set

Problem 12.1. (COM – 2003.6.1): The cashier sold all the tickets in the first row of the cinema, and by mistake, two tickets were sold for one of the seats. The sum of the seat numbers on all these tickets is 857. Which seat was sold twice?

Problem 12.2. (COM – 2017.6.1;7.1): In the Country of Fools, there are coins in denominations of 1, 2, 3, ..., 19, 20 solidi (there are no other coins). Alice had one coin. He bought ice cream and received one coin in change. Again, he bought the same ice cream and received the change in three coins of different denominations. Alice wanted to buy a third ice cream of the same kind, but he didn't have enough money. How much does the ice cream cost?

Problem 12.3. (COM – 2016.7.2): In Beatrice's room, there are clocks on each wall, and they all show the wrong time: the first clock is wrong by 2 minutes, the second one by 3 minutes, the third one by 4 minutes, and the fourth one by 5 minutes. One day, Beatrice, going outside, decided to find out the exact time and saw the following clock readings: 14:54, 14:57, 15:02, and 15:03. Help Beatrice determine the exact time.

Problem 12.4. (COM – 2012.7.2): On the map, 4 villages are marked: A , B , C , and D , connected by paths (see the figure). The directory states that on the routes $A - B - C$ and $B - C - D$, there are 10 potholes each; on the route $A - B - D$, there are 22 potholes, and on the route $A - D - B$, there are 45 potholes. Tourists want to get from A to D with as few potholes on their way as possible. Which route should they take?



Problem 12.5. (MF – 2006.6.3): Max invited Leo to visit, saying that he lives in the 10th entrance, apartment №333, but he forgot to mention the floor. Approaching the

house, Leo found that the house had nine floors. On which floor should he go up? (The number of apartments on each floor is the same, and the apartment numbers in the house start from one.)

Problem 12.6. (MF – 2001.6.3): Alice and Beatrice bought the same box of tea bags. It is known that one tea bag is enough for two or three cups of tea. Alice had enough tea for 41 cups, and Beatrice had enough for 58. How many tea bags were in the box?

Problem 12.7. (MF – 1997.6.3;7.4): In a basket, there are 30 mushrooms – Russulas and Boletes. It is known that among any 12 mushrooms, there is at least one Russula, and among any 20 mushrooms, there is at least one Bolete. How many Russulas and Boletes are there in the basket?

Problem 12.8. (COM – 2011.6.3): A moment is considered magical when the number of minutes on the digital clock coincides with the number of hours. To brew a magical potion, it needs to be placed on the fire and removed from the fire at magical moments. And to make it tasty, it needs to be brewed for one and a half to two hours. How much time does the tasty magical potion take to brew?

Problem 12.9. (COM – 2006.6.3): Five friends pooled their money to make a purchase. Can it happen that any two friends contributed less than a third of the cost of the purchase?

Problem 12.10. (MF – 2001.7.3): To build a standard house, there was not enough space. The architect changed the project, removed two entrances, and added three floors. At the same time, the number of apartments increased. He was delighted and decided to remove two more entrances and add three more floors. Could the number of apartments become even less than in the standard project? (The number of floors is the same at each entrance, and there is the same number of apartments on each floor at all entrances.)

Problem 12.11. (AT – 2016.3): Max and Leo came up with five natural numbers each, and all 10 numbers turned out to be different. The arithmetic mean of Max's

set is equal to the largest number in Leo's set. Can the arithmetic mean of Leo's set be equal to:

- a) the smallest number in Max's set;
- b) the largest number in Max's set?

Problem 12.12. (COM — 2009.7.3): Leo and Max were traveling together on a train. Each of them first looked out the window, then read the newspaper, then solved a crossword puzzle, and finally drank tea. For Leo, each subsequent activity took twice as much time as the previous one, and for Max, it took four times as much. They started looking out the window at the same time and finished drinking tea at the same time as well. What was Leo doing when Max started solving the crossword puzzle?

Problem 12.13. (COM — 2016.6.4): There are 27 students in the class, but not everyone came to the physical education lesson. The teacher divided those who came into two teams of equal size for playing pioneerball. In the first team, there were half of all the boys who came and a third of all the girls who came, and in the second team, there were half of all the girls who came and a quarter of all the boys who came. The rest of the students who came helped the referee. How many helpers could the referee have?

Problem 12.14. (COM — 2013.6.4): If each girl is given one chocolate and each boy is given two, there will be enough chocolates. But if each boy is given one chocolate and each girl is given two, there won't be enough. And if the girls are not given any at all, will each boy have enough with three chocolates?

Problem 12.15. (MF — 2010.7.4): In a singing competition, Rooster, Crow, and Cuckoo participated. Each member of the jury voted for one of the three performers. Woodpecker counted that there were 59 judges, and for Rooster and Crow together, 15 votes were cast; for Crow and Cuckoo together, 18 votes were cast; for Cuckoo and Rooster together, 20 votes were cast. Woodpecker counts poorly, but each of the four numbers he mentioned differs from the correct one by no more than 13. How many judges voted for Crow?

Problem 12.16. (MF — 1993.6.5): Alice, Beatrice, Clarice, and Dorice are sitting on a bench. If Clarice, who is sitting to the right of everyone, sits between Alice and Beatrice, then Beatrice will be on the far left. In what order are they sitting?

Problem 12.17. (COM — 2008.6.5): Beatrice has 130 wooden planks. He can make a toy windmill from 5 planks, a steamship from 7 planks, and an airplane from 14 planks. The airplane costs 19 gold coins, the steamship costs 8 gold coins, and the windmill costs 6 gold coins. What is the maximum number of gold coins Beatrice can earn?

Problem 12.18. (COM — 2009.7.5): Ali-Baba and 40 robbers are dividing the loot. The division is considered fair if any 30 participants get at least half of the loot together. What is the largest share that Ali-Baba can get in a fair division?

Problem 12.19. (COM — 2016.6.7): Jean lives in a multi-story building. At each entrance of the building, there is the same number of floors, with four apartments on each floor. Each apartment has a one-, two-, or three-digit number. Jean noticed that the number of apartments with a two-digit number in his entrance is ten times the number of entrances in the building. How many apartments can there be in this building?

Problem 12.20. (COM — 2006.6.7): Ilya Muromets remembers that to neutralize a 10-headed fire-breathing dragon, it is enough to have four fire extinguishers. And to neutralize a 16-headed dragon, seven fire extinguishers are enough. What is the smallest number of fire extinguishers needed to neutralize a 19-headed dragon?

Problem 12.21. (COM — 2006.7.6): At a sports holiday, seventh-grade students competed in pairs in the running according to the following rules. At the command, two people start running from the starting point in different directions along the circular track of the stadium. The moment of their meeting is considered the finish. Sasha and Yuri ran a lap in 45 seconds. Two Alenas started running at constant speeds (not necessarily equal), but when they had to run half a lap, one Alena increased her speed by 25%, and the other by 28%. It turned out that for the first half-lap, they ran 5 seconds more than for the second half-lap. Who has a better result: the girls or the boys?

Problem 12.22. (PVG 2016.5–6.1): Alice, Beatrice, Clarice, and Dorice played «Fool» (a card game) and played a total of 16 games. Each lost at least once. It is known that Alice lost the most, and Beatrice and Clarice together lost 9 times. How many times did Dorice lost?

Problem 12.23. (Lomonosov — 2016.5–6.2,7–8.1): How many weeks can a year overlap? We consider that a year overlaps a week if at least one day of this week falls on the given year.

Problem 12.24. (Lomonosov — 2017.5–6.3, 7–8.2): Our cat gave birth today! It is known that the two lightest ones weigh 80 g in total, the four heaviest ones weigh 200 g, and the total weight of all kittens is 500 g. How many kittens did the cat give birth to?

Problem 12.25. (Kurchatov — 2015.7.1): Five children were lined up, and they were given 111 candies. The children standing to the left of Dorice got 96 candies, to the right of Alice got 57, to the left of Fabrice got 69, and to the right of Beatrice got 75 candies. The fifth child is called Clarice. What is the name of the one who got the most candies, and how many candies did they get?

Problem 12.26. (2ARSO — 2015.7.4): A biologist sequentially placed 150 beetles in ten jars, and in each subsequent jar, he placed more beetles than in the previous one. The number of beetles in the first jar is at least half the number of beetles in the tenth jar. How many beetles are in the sixth jar?

Problem 12.27. (JMO — 2023.B1): The sum of four fractions is less than 1. Three of these fractions are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{10}$. The fourth fraction is $\frac{1}{n}$, where n is a positive integer. What values could n take?

Problem 12.28. (JMO — 2023.B3): As Rachel traveled to school, she noticed that, at each bus stop, one passenger got off and x passengers got on, where $x \geq 2$. After five stops, the number of passengers on the bus was x times the number of passengers before the first stop. How many passengers were on the bus before the first stop?

Problem 12.29. (JMO – 2016.B1): In a certain triangle, the size of each of the angles is a whole number of degrees. Also, one angle is 30° larger than the average of the other two angles.

What is the largest possible size of an angle in this triangle?

Problem 12.30. (JMO – 2015.B1): Let N be the smallest positive integer whose digits add up to 2015.

What is the sum of the digits of $N + 1$?

Problem 12.31. (AMC – 2023.10B.4): A box contains 28 red balls, 20 green balls, 19 yellow balls, 13 blue balls, 11 white balls, and 9 black balls. What is the minimum number of balls that must be drawn from the box without replacement to guarantee that at least 15 balls of a single color will be drawn?

(A) 75 (B) 76 (C) 79 (D) 84 (E) 91

Problem 12.32. (AMC – 2023.10B.7): Each piece of candy in a store costs a whole number of cents. Casper has exactly enough money to buy either 12 pieces of red candy, 14 pieces of green candy, 15 pieces of blue candy, or n pieces of purple candy. A piece of purple candy costs 20 cents. What is the smallest possible value of n ?

(A) 18 (B) 21 (C) 24 (D) 25 (E) 28

Problem 12.33. (AMC – 2014.10A.7): Nonzero real numbers x , y , a , and b satisfy $x < a$ and $y < b$. How many of the following inequalities must be true?

(I) $x + y < a + b$

(II) $x - y < a - b$

(III) $xy < ab$

(IV) $\frac{x}{y} < \frac{a}{b}$

- (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 12.34. (Mock AIME – 2006.2.1): A positive integer is called a dragon if it can be written as the sum of four positive integers $a, b, c,$ and d such that $a + 4 = b - 4 = 4c = d/4$. Find the smallest dragon.

Problem 12.35. (AMC – 2010.12A.6): A palindrome, such as 83438, is a number that remains the same when its digits are reversed. The numbers x and $x + 32$ are three-digit and four-digit palindromes, respectively. What is the sum of the digits of x ?

- (A) 20 (B) 21 (C) 22 (D) 23 (E) 24

Problem 12.36. (AMC – 2010.12A.5): Halfway through a 100-shot archery tournament, Chelsea leads by 50 points. For each shot, a bullseye scores 10 points, with other possible scores being 8, 4, 2, and 0 points. Chelsea always scores at least 4 points on each shot. If Chelsea's next n shots are bullseyes, she will be guaranteed victory. What is the minimum value for n ?

- (A) 38 (B) 40 (C) 42 (D) 44 (E) 46

Problem 12.37. (AMC – 2010.12A.20): Arithmetic sequences (a_n) and (b_n) have integer terms with $a_1 = b_1 = 1 < a_2 \leq b_2$ and $a_n b_n = 2010$ for some n . What is the largest possible value of n ?

- (A) 2 (B) 3 (C) 8 (D) 288 (E) 2009

Problem 12.38. (iTest – 2008.7): Find the number of integers n for which $n^2 + 10n < 2008$.

Problem 12.39. (iTest – 2008.23): Find the number of positive integers n that are solutions to the simultaneous system of inequalities:

$$4n - 18 < 2008$$

$$7n + 17 > 2008.$$

Problem 12.40. (iTest – 2008.16): In order to encourage the kids to straighten up their closets and the storage shed, Jerry offers his kids some extra spending money for their upcoming vacation. «I don't care what you do; I just want to see everything look clean and organized.»

While going through his closet, Joshua finds an old bag of marbles that are either blue or red. The ratio of blue to red marbles in the bag is $17 : 7$. Alexis also has some marbles of the same colors but hasn't used them for anything in years. She decides to give Joshua her marbles to put in his marble bag so that all the marbles are in one place. Alexis has twice as many red marbles as blue marbles, and when the twins get all their marbles in one bag, there are exactly as many red marbles and blue marbles, and the total number of marbles is between 200 and 250. How many total marbles do the twins have together?

Problem 12.41. (iTest – 2008.17): One day, when Wendy is riding her horse, Vanessa, they get to a field where some tourists are following Martin (the tour guide) on some horses. Martin and some of the workers at the stables each lead extra horses, so there are more horses than people. Martin's dog, Berry, runs around near the trail as well. Wendy counts a total of 28 heads belonging to the people, horses, and dogs. She counts a total of 92 legs belonging to everyone and notes that nobody is missing any legs.

Upon returning home, Wendy gives Alexis a little problem-solving practice, «I saw 28 heads and 92 legs belonging to people, horses, and dogs. Assuming two legs per person and four for the other animals, how many people did I see?» Alexis scribbles out some algebra and answers correctly. What is her answer?

Problem 12.42. (AMC – 2008.10B.11): Suppose that (u_n) is a sequence of real num-

bers satisfying $u_{n+2} = 2u_{n+1} + u_n$,

and that $u_3 = 9$ and $u_6 = 128$. What is u_5 ?

- (A) 40 (B) 53 (C) 68 (D) 88 (E) 104

Problem 12.43. (Cyprus MO – 2007.20): The mean value for 9 Math-tests that a student succeeded in was 10 (in scale 0-20). If we put the grades of these tests in increasing order, then the maximum grade of the 5th test is

- (A) 15 (B) 16 (C) 17 (D) 18 (E) 19

Problem 12.44. (Cyprus MO – 2006.23): Of 21 students taking Mathematics, Physics, and Chemistry, no student takes one subject only. The number of students taking Mathematics and Chemistry only equals four times the number taking Mathematics and Physics only. If the number of students taking Physics and Chemistry only equals to three times the number of students taking all three subjects, then the number of students taking all three subjects is

- (A) 0 (B) 5 (C) 2 (D) 4 (E) 1

Problem 12.45. (AMC – 2004.10A.15): Given that $-4 \leq x \leq -2$ and $2 \leq y \leq 4$, what is the largest possible value of $\frac{x+y}{x}$?

- (A) -1 (B) $-\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$ (E) 1

Problem 12.46. (AMC – 2000.12.12): Let A, M, C be nonnegative integers such that $A \cdot M \cdot C + A \cdot M + M \cdot C + A \cdot C = 2000$. What is the maximum value of $A + M + C$?

$$A \cdot M \cdot C + A \cdot M + M \cdot C + A \cdot C = 2000$$

Problem 12.47. (AHSME – 1992.18): The increasing sequence of positive integers a_1, a_2, a_3, \dots has the property that $a_1 + a_2 + a_3 + \dots + a_n = n^2$ for all $n \geq 1$. What is a_{100} ?

$$a_{n+2} = a_n + a_{n+1} \text{ for all } n \geq 1.$$

If $a_7 = 120$, then a_8 is

- (A) 128 (B) 168 (C) 193 (D) 194 (E) 210

Problem 12.48. (AHSME – 1989.11): Let a , b , c , and d be positive integers with $a < 2b$, $b < 3c$, and $c < 4d$. If $d < 100$, the largest possible value for a is

- (A) 2367 (B) 2375 (C) 2391 (D) 2399 (E) 2400

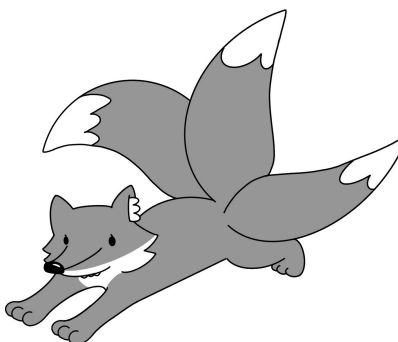
Problem 12.49. (AHSME – 1963.19): In counting n colored balls, some red and some black, it was found that 49 of the first 50 counted were red. Thereafter, 7 out of every 8 counted were red. If, in all, 90 % or more of the balls counted were red, the maximum value of n is:

- (A) 225 (B) 210 (C) 200 (D) 180 (E) 175

Problem 12.50. (AHSME – 1952.30): When the sum of the first ten terms of an arithmetic progression is four times the sum of the first five terms, the ratio of the first term to the common difference is:

- (A) 1 : 2 (B) 2 : 1 (C) 1 : 4 (D) 4 : 1 (E) 1 : 1

Skill Assessment Problems



Skill Assessment Problem 12.1. In the magical fox forest, there are 296 foxes, three-tailed and five-tailed. It is known that among any 99 foxes, there is at least one three-tailed fox, and among any 199 foxes, there is at least one five-tailed fox. How many three-tailed foxes live in the forest?

Skill Assessment Problem 12.2. Three-tailed and five-tailed foxes (together, there were less than 15 of them) came to a gathering in the clearing in the magical forest. The total number of tails was 53. If instead of three-tailed foxes, five-tailed foxes came, and instead of five-tailed ones, three-tailed ones came, then together they would have fewer tails than now, but no more than one and a half times. How many three-tailed foxes are currently in the clearing?

Solutions of Skill Assessment Problems

Solution to Problem 12.1: If among any 99 foxes, there is at least one three-tailed fox, it means that there cannot be more than 98 five-tailed foxes in total, otherwise, we could take as the selected set of 99 foxes only the five-tailed ones and get a contradiction. Similarly, there cannot be more than 198 three-tailed foxes. Therefore, the total number of foxes is no more than $98 + 198 = 296$. Equality is achieved when there are exactly 98 five-tailed foxes and 198 three-tailed foxes. \square

Solution to Problem 12.2: This problem is easiest to solve using semi-brute force methods. From the condition that there are currently fewer than 15 foxes, it follows that if all foxes were three-tailed, there would be no more than $14 \cdot 3 = 42$ tails. Hence, the remaining tails, at least $53 - 42 = 11$, are brought by increasing the number of tails by 2 for some of the foxes, turning them into five-tailed ones. Thus, there are at least 6 five-tailed foxes in total. Let's consider possible options for distributing foxes in the clearing initially.

1. If there were 6 five-tailed foxes, then the three-tailed foxes have a total of $53 - 6 \cdot 5 = 23$ tails; 23 is not divisible by 3, so this option is impossible.
2. If there were 7 five-tailed foxes, then the three-tailed foxes have a total of $53 - 7 \cdot 5 = 18$ tails. Thus, there were 6 three-tailed foxes in total. Then, if we replace all three-tailed foxes with five-tailed ones and vice versa, we get $7 \cdot 3 + 6 \cdot 5 = 51$ tails, which is less than 53 but not more than one and a half times, so this option is possible.
3. If there were 8 five-tailed foxes, then the three-tailed foxes have a total of $53 - 8 \cdot 5 = 13$ tails; 13 is not divisible by 3, so this option is impossible.
4. If there were 9 five-tailed foxes, then the three-tailed foxes have a total of $53 - 9 \cdot 5 = 8$ tails; 8 is not divisible by 3, so this option is impossible.
5. If there were 10 five-tailed foxes, then the three-tailed foxes have a total of $53 - 10 \cdot 5 = 3$ tails. Thus, there was 1 three-tailed fox. Then, if we replace all three-tailed foxes with five-tailed ones and vice versa, we get $10 \cdot 3 + 1 \cdot 5 = 35$ tails, which is less than 53 but more than one and a half times, so this option is impossible.

Thus, currently, there are 6 three-tailed foxes in the clearing. \square

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