Tatiana Babicheva is an inspiring teacher who believes that teaching is not just about imparting knowledge but also about being a lifelong learner. This passion for growth and discovery led her to earn two PhDs: one in applied mathematics and its applications in Russia, and another in computer science from Université Paris-Saclay in France. She has published 10 books in Russian and French in the field of popular science, including mathematics at the competition level.

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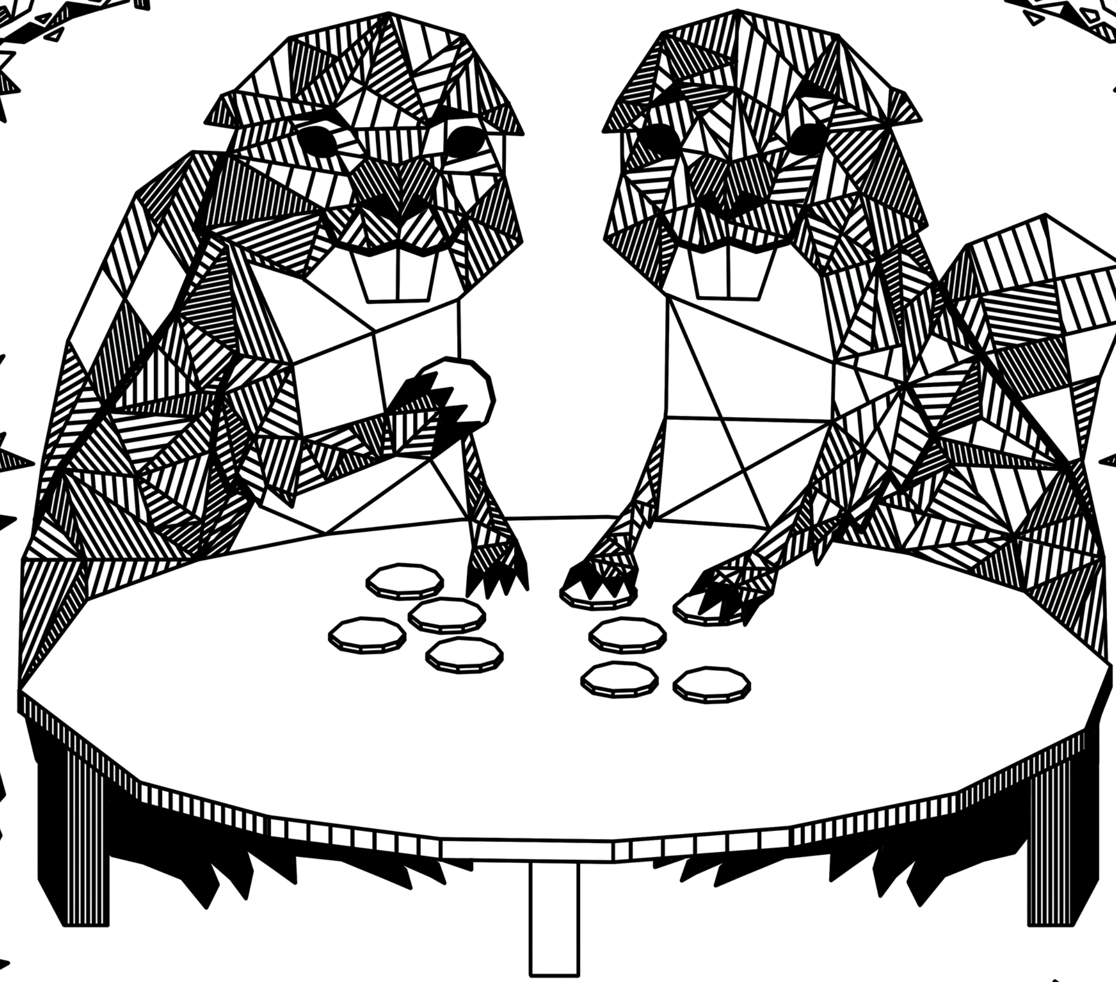
IDEAS AND METHODS

Mathematical Competitions. Levels A1-A2

2. Ideas and Methods



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Mathematical Competitions.
Levels A1-A2
Book 2. Ideas and Methods

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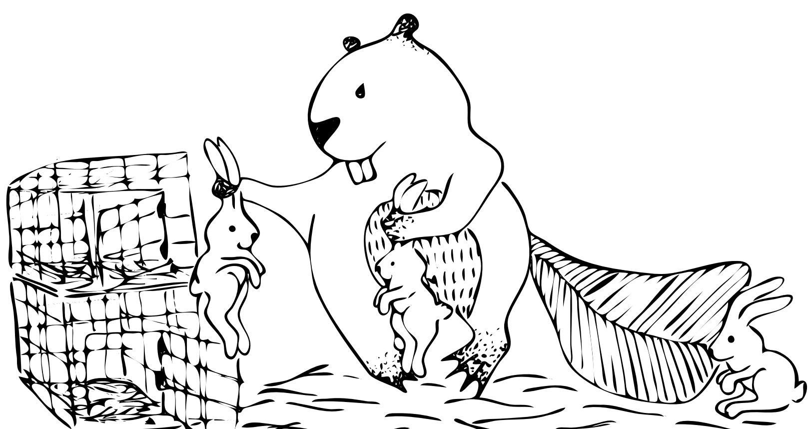
Dedication

> "Dedicated to our students, who learned the proof-based approach through the pages of this book. Your enthusiasm and engagement have truly made the journey enjoyable.

Introduction



Introduction to the series

Begin your preparation for Competition Mathematics with our carefully crafted series. These books are designed to inspire a love for problem-solving and foster critical thinking. They are ideal for both budding mathematicians and passionate enthusiasts.

Inside, you will find a wide range of challenges, puzzles, and problems. Each one is selected to enhance your mathematical abilities. Experience the challenge of solving complex equations and gain confidence by deciphering complex geometric puzzles. Every book has engaging content to stimulate your mind and expand your skills.

If you're preparing for regional competitions, national tournaments, or simply want to deepen your mathematical knowledge, this series is an invaluable resource. The books provide clear explanations, strategic insights, and numerous practice problems. They aim to build your confidence and equip you with the skills needed to tackle any mathematical challenge.

While school mathematics forms a foundation, this series goes beyond it without requiring advanced knowledge to understand the material. Our course covers a wide range of topics, reflecting the diverse nature of Olympiad problems. Solving a geometry problem may require knowledge of combinatorics, while a number theory problem might involve understanding invariants and the pigeonhole principle.

Olympiad problems are generally not restricted to specific grade levels, making these books suitable for high school students. Some of the problems included have been featured in the final stages of national math Olympiads for higher grades. The goal is to demonstrate how to solve problems using straightforward and elegant methods, avoiding unnecessary complexity.

We have categorized competition mathematics into levels similar to the international standards used for foreign language proficiency. This approach is based on the concept of the «language» of competition mathematics. Traditional grade-based divisions are often outdated, as understanding a topic might only require elementary-level math. Moreover, the topics in these books are interconnected. Without a grasp of a topic at level A1, understanding its expanded form at level A2 can be challenging.

Here's what to expect at each level:

Let's use an analogy with foreign languages:

Level A1. You understand (generally) foreign speech and can talk about family, activities, hobbies, travels, weather, and buying things. In short, the standard tourist set. Can you conjugate basic verbs and be familiar with different tenses? The question «How are you?» doesn't stump you. Congratulations! You have a good A1 level! This is enough for survival.

Similarly, in olympiad math — you can «survive» at beginner-level olympiads, understand what is required in problems, and formulate solutions. You likely won't need math knowledge beyond seventh grade to understand topics at this level. (The problem might be from an 11th-grade olympiad, but the solving method remains the same.)

At level A2, you can discuss preferences in art, cultural differences, and main social trends, etc. You form complex sentences («This is Peter, whose dad works at the bank. I've already told you about him»), can write to a friend on Facebook, describe a vacation, and understand the essence of any conversation in the language.

You can recognize and solve middle-level Olympiad problems. You will be able to avoid common mistakes and present your solutions effectively. Topics at this level typically require knowledge up to the eighth grade.

This series of books generally covers levels A1 and A2 of competition math: you will understand any problem from most competitions, formulate your solution, and even change the solution of ChatGPT to match the real competition problem. However, you are still far from being a native speaker.

What is in these Books?

This series uses a proof-based approach to problem-solving, which is usually reserved for advanced levels in countries like the USA and the UK. However, this method helps build a solid foundation in mathematics.

Each chapter is divided into four parts:

1. The first part covers the theoretical background and provides detailed solutions to typical problems.

2. The second part presents a problem set labeled by source. Olympiad problems are marked with notations like «Year.Grade/Round.Number.» For example, «ACM 2016.10A.5» is the fifth problem from the 10th-grade 10A variant of the ACM Olympiad 2016. Grade numbering may vary between countries, so adjust accordingly. Non-grade-specific Olympiads, like AIME, are marked by version (I or II) instead of grade.

You will encounter many problems from the Russian Olympiads (a country with a strong tradition in Olympiad mathematics) and various US mathematical competitions (such as AMC and AIME). We sincerely recommend not only finding the correct answer from the given AMC options but also approaching these problems from a proof-based perspective.

The problem number usually provides a sense of difficulty; generally, a higher number indicates a more challenging problem. However, this labeling doesn't always apply to some «independent» Olympiads, which can sometimes confuse genuine Olympiad participants.

3. The third part includes problems for independent solving, with some original problems introduced here.

4. Solutions are found in the fourth part.

The series consists of the following books:

1. Competitive Arithmetics
2. Ideas and Methods
3. Introduction to Discrete Mathematics
4. Introduction to Competitive Geometry
5. Competitive Number Theory
6. Competitive Geometry

This series is designed for both experienced Olympiad participants and newcomers to mathematical problem-solving. It offers a journey where theory and application meet, providing a rewarding experience. Welcome to a unique math adventure!

Introduction to this book

What does it mean to reason logically from a mathematical standpoint?

When is your «obvious» truly obvious to the reviewers?

Why is providing an example sometimes sufficient to solve a problem, while other times it is not?

In this book, we aim to answer all these questions.

We begin by presenting problems with fairly intuitive methods of solution that do not require specialized knowledge. Here, we cover topics such as examples and constructions, pouring, and weighing.

Next, we will introduce you to some of the fundamental methods for solving olympiad problems: proof by contradiction, the pigeonhole principle, the extreme principle, estimation plus example, exhaustive search, and double counting.

Consider this book your first guide to the proof-based problem-solving approach.

List of competitions used in this book

- «Математический праздник», in English mean «Mathematical festival». We note it in the book as «MF». The official site (in Russian) is https://olympiads.mccme.ru/matprazdnik/
- Городская устная математическая олимпиада для 6—7 классов, mean «City Oral Mathematical Olympiad for 6—7 grades». We note it in the book as «COM». The official site (in Russian) is https://olympiads.mccme.ru/ustn/
- Турнир городов, mean «Tournament of Towns». We note it in the book as «TOT». The official site is https://www.turgor.ru/en/
- Школьный этап Всероссийской олимпиады школьников, mean «first stage of All-Russian School Olympiad». We note it in the book as «1ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- Муниципальный этап Всероссийской олимпиады школьников, mean «second stage of All-Russian School Olympiad». We note it in the book as «2ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- Муниципальный этап Всероссийской олимпиады школьников (Москва), mean «second stage of All-Russian School Olympiad in Moscow». We note it in the book as «Mos2ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- Региональный этап Всероссийской олимпиады школьников, mean «third stage of All-Russian School Olympiad». We note it in the book as «3ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- Всероссийская олимпиада школьников, mean «All-Russian School Olympiad». We note it in the book as «ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- American Junior High School Mathematics Examination. We note it in the book as «AJHSME». The official site is https://www.maa.org/math-competitions/amc
- Московская математическая олимпиада, mean «Moscow Mathematical Olympiad». We note it in the book as «MMO». The official site (in Russian) is https://mmo.mccme.ru/
- Объединённая межвузовская математическая олимпиада школьников, mean United Interuniversity Mathematical Olympiad for schoolchildren. We note it in the book as «OMMO». The official site (in Russian) is

https://olympiads.mccme.ru/ommo/

- Junior Mathematical Olympiad. We note it in the book as «JMO». The official site is https://ukmt.org.uk/junior-challenges/junior-mathematical-olympiad
- Турнир им. Ломоносова, mean «Lomonosov Tournament». We note it in the book as «LT». The official site (in Russian) https://turlom.olimpiada.ru/
- Турнир Архимеда, mean «Archimedes Tournament». We note it in the book as «AT». The official site (in Russian) is http://www.arhimedes.org/
- Курчатов, mean «Kurchatov Competition». We note it in the book as «Kurchatov». The official site (in Russian) is https://olimpiadakurchatov.ru/
- Кружок МЦНМО, mean «Circle of Moscow Center for Continuous Mathematical Education». We note it in the book as «Mccme». The official site (in Russian) is https://mccme.ru/en/math-circles/circles-mccme/20232024/
- Кружок ВМШ 57 школы, mean Mathematical Circle of School 57. We note it in the book as «Circle 57». The official site (in Russian) http://school57.ru
- UK Maths Trust Pink Kangaroo. We note it in the book as «Pink Kangaroo». The official site is https://ukmt.org.uk/
- Белорусская республиканская математическая олимпиада, mean «Belarusian Republican Mathematical Olympiad». We note it in the book as «Belarus». The official site (in Russian) is https://olymp.bsu.by
- Кружок малого мехмата, means Circle for schoolchildren in the Faculty of Mechanics and Mathematics at Moscow State University. We note it in the book as «MMCircle». The official site (in Russian) is http://mmmf.msu.ru/
- The book of Altufova and Ustinov «Algebra and Number Theory. Collection of Problems for Mathematical Schools» in Russian. We note it in the book as «AU».
- UK Maths Trust Grey Kangaroo. We note it in the book as «Grey Kangaroo». The official site is https://ukmt.org.uk/
- Mid-Michigan Mathematical Olympiad. We note it in the book as «Michigan». The official site is https://users.math.msu.edu/users/mshapiro/michiganolympiad.html
- Formula of Unity. We note it in the book as «Formula». The official site is https://www.formulo.org/en/olymp
- Nederlandse Wiskunde Olympiade. The official site is https://www.vwo.be/vwo/

- Gauss Contest. The official site is https://cemc.uwaterloo.ca/resources/past-contests
- MATHCOUNTS competition. We note it in the book as «mathcounts». The official site is https://www.mathcounts.org/resources/mathcounts-minis
- HMMT. The official site is https://www.hmmt.org/
- Hamilton Mathematical Olympiad. We note it in the book as «HMO». The official site is https://ukmt.org.uk/

Contents

Balance Puzzles

> “ A mathematician is asked:
> — Does a crocodile have wings?
> — Of course! — he confidently replies.
> — How so?! Where do they have wings?!
> — Just their quantity is equal to zero.

—One joke that didn’t quite land

Theory and Practice

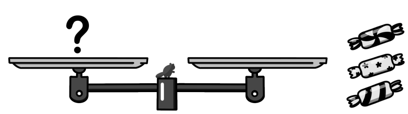
If you have participated in puzzle competitions before, you may have encountered weighing puzzles.

They usually involve finding, within a finite number of weightings, an object that differs in weight from the others. The search is carried out by comparing individual elements as well as groups of elements, either among themselves or with weights of a certain value. Different types of scales may be used, each with its own capabilities. The simplest case is using balance scales, which allow for comparing objects or groups of objects (with or without given weights, as specified in the problem statement).

Here's a simple problem on this topic.

Example 1.1. There are 3 externally identical candies, one of which is tasteless (lighter because it lacks filling). How can you identify the tasteless candy using balance scales without weights in just 1 weighing?

First, let's understand what balance scales without weights are and what they can do.

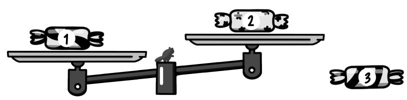


One of the most common mistakes students make in competitions is believing that even if we cannot see exactly how much heavier one side of the balance is, we can still determine which side tilted more during one of the weightings. Such a «solution» is given zero points.

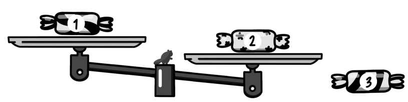
Unfortunately, most participants in competitions fail to understand how to correctly present their solution to a weighing problem. Of course, we are talking about proof-based competitions; in competitions like AMC, you would not encounter such a problem.

Solution: Let's number the candies and place candy number 1 and candy number 2 on different sides of the balance scales. There are three possible cases.

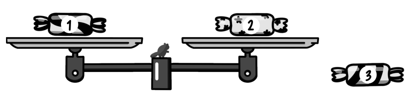
1. If the side with candy number 1 is heavier, then candy number 2 is tasteless.



2. If the side with candy number 2 is heavier, then candy number 1 is tasteless.



3. If the scales are balanced, then candy number 3 is tasteless.



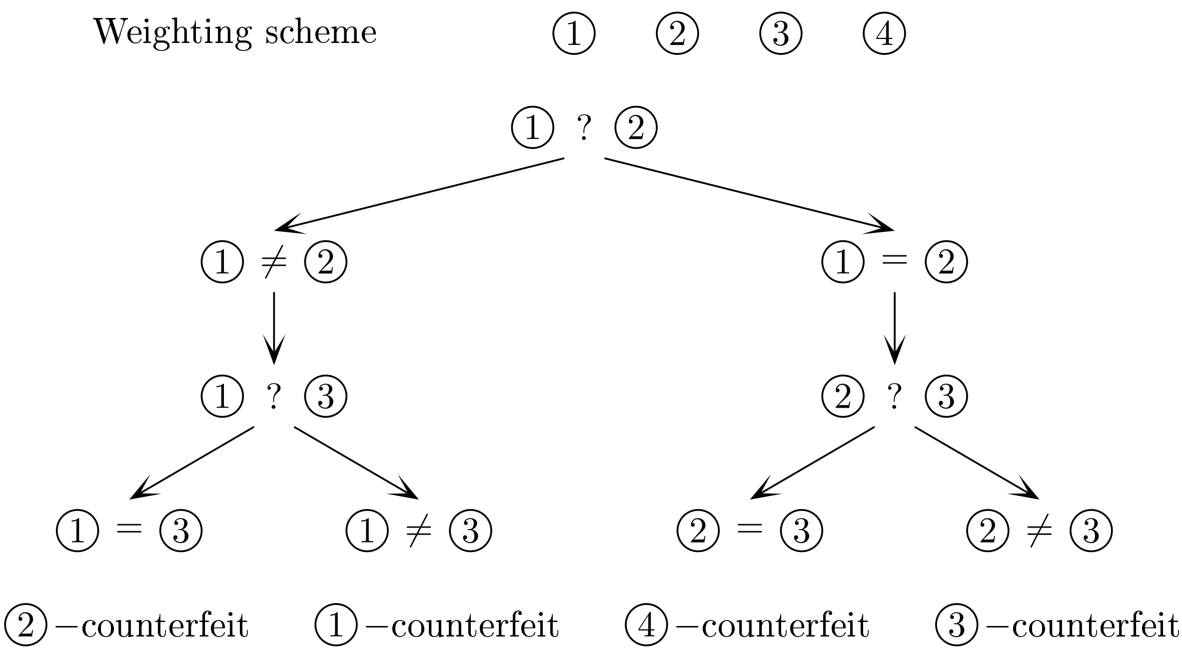
After considering all cases, the solution to the problem is complete. Remember to list all possible cases; missing to write down even the most obvious case, can cost you points. □

The similar problem, but about 9 coins, is quite recent. Its first publication was by E. D. Schell in the January 1945 issue of the American Mathematical Monthly. The classical problem setting of the «Coin Weight Problem» aims to solve the following mathematical problem: Among N coins, one coin has a different weight. With a balance scale, find the different coins with the fewest number of weightings.

Sometimes, it is unknown whether the object to be found is lighter or heavier than the others. Let's consider the following problem.

Example 1.2. Among four coins, exactly one is counterfeit (and it is unknown whether this coin is lighter or heavier than the genuine ones). How can you identify it using two weightings on balance scales without weights?

Solution: One of the methods for presenting the solution to such problems is a diagram of possible actions.



□

In weighing puzzles, the problem setting can be altered by introducing «faultiness» into the scales, which may indicate some distortion. For example, the scales may show balance in all cases where the weights on the individual sides differ by less than a certain amount. Additionally, a fixed set of weights may also be given.

Furthermore, weighing problems can also be given for ordinary scales that digitally display weight.

Example 1.3. The king was presented with the annual tribute — one sack of gold coins from each of the 10 provinces of his kingdom, but the secret service reported that one of the sacks contains counterfeit coins weighing 9 grams each (while all the others contain genuine coins weighing 10 grams each). The king has scales that show the exact weight, which he can use only once. How can the king find the sack with the counterfeit coins? (There are a large number of coins in each sack.)

Solution: To solve this problem, we can rely on the contradiction between the expected weight (if all coins were genuine) and the actual weight due to the counterfeit coins. For example, we can take a different number of coins from each sack: 1 coin from the first sack, 2 from the second, and so on. If there were no counterfeit coins, the scales would show a weight of 550 grams $\big((1+2+\ldots+10)\cdot 10\big)$. However, the presence of counterfeit coins will reduce it to $550-x$, where x is the number of the sack with the counterfeit coins. $\square$

If you encounter a type of problem you haven't seen before, don't worry; just give it a try!

Problem Set

Problem 1.1. (COM — 2002.7.1): In a set of 10 weights, any four weights outweigh any three of the remaining weights. Is it true that any three weights from this set outweigh any two of the remaining seven weights?

Problem 1.2. (MF — 2017.7.2): A pharmacist has three weights with which he measured 100 g of iodine for one customer, 101 g of honey for another, and 102 g of hydrogen peroxide for a third. He always placed the weights on one side of the scales and the goods on the other. Could it be that each weight is lighter than 90 g?

Problem 1.3. (2ARSO — 2005-2006.7.3): A seller has a beam balance with two pans. The scales can show weights from 0 to 5 kg. The product can only be placed on the left pan, and weights can be placed on either of the two pans. What is the minimum number of weights the seller needs to have to weigh any integer amount of sugar from 0 to 25 kg?

Problem 1.4. (AT — 2016.5): There are three gold bars weighing 3, 4, and 5 grams. Each bar is labeled with its weight, but the labels may be incorrect. The weights of the bars can be compared on balance scales without weights, but at the moment of weighing, an invisible gnome weighing 1 gram jumps onto one of the pans. How can you determine the correct weight of at least one bar using no more than two weightings?

Problem 1.5. (COM — 2017.6.6): Four externally identical coins weigh 1, 2, 3, and 4 grams. Can you determine the weight of each coin using four weightings on balance scales without weights?

Problem 1.6. (2ARSO — 2007-2008.7.2): In a piggy bank, there are 30 coins, including 2 counterfeit ones: one is lighter than the genuine ones by 0.5 g, and the other — by 1 g. How can you determine 14 genuine coins using two weightings on balance scales without weights?

Problem 1.7. (COM — 2008.6.7): A jeweler made 6 identical-looking silver jewelry pieces weighing 22, 23, 24, 32, 34, and 36 grams and instructed his apprentice to stamp the weight on each piece. Could the jeweler determine if the apprentice had mixed up the jewelry pieces using two weightings on balance scales without weights?

Problem 1.8. (2ARSO — 1998-1999.8.2): A counterfeiter has 40 externally identical coins, including 2 counterfeits that are lighter than the genuine ones and have identical weights. How can you select 20 genuine coins using two weightings on balance scales without weights?

Problem 1.9. (COM — 2011.6.8): Among 63 coins, there are 7 counterfeit ones. All counterfeit coins have the same weight, as do all genuine ones, and the counterfeit coin is lighter than the genuine one. How can you determine 7 genuine coins using three weightings on balance scales without weights?

Problem 1.10. (COM — 2015.6.9): There are 13 gold and 14 silver coins, among which exactly one is counterfeit. It is known that if the counterfeit coin is gold, it is lighter than the genuine one because it is made of less gold, and if it is silver, it is heavier than the genuine one because it is made of cheaper and heavier metal. How can you find counterfeit coins using three weightings on balance scales without weights? (Genuine gold coins have identical weights, and genuine silver coins have as well identical weights that could differ from gold coins.)

Problem 1.11. (3ARSO — 2009.8.4): There are balance scales and 100 coins, some (more than 0 but less than 99) of which are counterfeit. All counterfeit coins weigh the same, as do all genuine coins, and the counterfeit coin is lighter than the genuine one. You can weigh these coins using the scales, but each weighting will cost you one of the coins paid before weighting. Prove that it is possible to guarantee the detection of a genuine coin.

Problem 1.12. (Bachet's weights problem) A trader had a 40-pound standard weight, which was dropped down accidentally and broken into four pieces, each weighing a different integral number of pounds. The trader then found that with the four weights thus obtained, he could measure any integral number of pounds from 1 to 40, placing the weights on one or both of the pans of his balance. What are the weights of the four pieces?

Problem 1.13. Given 81 coins, one of them fake and lighter, find the minimum number of weightings that will guarantee finding the fake coin.

Problem 1.14. There are 12 coins; one of them is fake. All real coins weigh the same. The fake coin is either lighter or heavier than the real coin. Find the fake coin and figure out whether it is heavier or lighter in 3 weightings on a balance scale.

Problem 1.15. There is a possibility that one of the ten identically looking coins is fake. The fake coin differs in weight from the original one. How can you decide using a balance scale if there is indeed a fake coin among these 10 coins? How many weightings would you need to determine that?

Problem 1.16. One of the 99 identically looking coins is fake. The fake coin differs in weight from the original one, but it is not known whether the fake coin is lighter or heavier than the rest. How can you determine, in two weightings, if the fake coin is lighter or heavier? What if you had 101 coins?

Problem 1.17. There are 64 stones of different weights (no two stones weigh the same). Find, in 94 weightings on a balance scale, the heaviest and the lightest of the stones.

Problem 1.18. You have eight similar coins and a beam balance. At most, one coin is counterfeit and hence underweight. How can you detect whether there is an underweight coin, and if so, which one, using the balance only twice?

Problem 1.19. (Formula) There are 27 cockroaches participating in cockroach racing. In each race, three cockroaches run. Each cockroach has his constant speed, not changing between the races. The speeds of all cockroaches are different. As a result of each race, we obtained only the order in which its participants had finished. We would like to know the two fastest cockroaches (in the correct order). Would 14 races be sufficient?

Problem 1.20. (Pink Kangaroo) Christina has eight coins whose weights in grams are different positive integers. When Christina puts any two coins in one pan of her balance scales, and any two in the other pan of the balance scales, the side containing the heaviest of those coins is always the heavier side. What is the smallest possible weight of the heaviest of the eight coins?

(A) 8 (B) 12 (C) 34 (D) 55 (E) 256

Skill Assessment Problems

Skill Assessment Problem 1.1. There are 9 externally identical candies, among which one is tasteless (lighter because it lacks filling). How can you identify the tasteless candy using balance scales without weights in 2 weightings?

Skill Assessment Problem 1.2. Among seven coins, there are 2 counterfeit ones (lighter, than the normal ones). Determine both counterfeit coins in 3 weightings on balance scales without weights.

Skill Assessment Problem 1.3. Among 9 coins, exactly one is counterfeit (unknown whether it is lighter or heavier than genuine). How can you determine it using three weightings on balance scales without weights?

Skill Assessment Problem 1.4. There are 9 kg of grain and balance scales with a 200 gram weight. How can you measure exactly 2 kg of grain using 3 weightings?

Skill Assessment Problem 1.5. The grain keeper beaver has 6 weights marked 1, 2, 3, 4, 5, and 6 kg. However, he suspects that the markings on two of the weights may be swapped. He cannot determine this information by visual inspection due to the weights being made of different materials. How can he determine whether the markings are correct using two weightings on balance scales, where any groups of weights can be compared?

Solutions of Skill Assessment Problems

Solution to Problem 1.1: Notice that with three candies, we can identify the tasteless one in a single weighing. Indeed, if we weigh the first two and they are different in weight, the lighter one is tasteless. If they are equal, then the tasteless one is the third candy.

Divide the nine candies into three groups of three. Our task is to determine in which group the lighter candy is located. Then, it will take only one more weighing to identify the tasteless candy from this group, which we know how to do. But this is the same problem: we have three groups, one of which is lighter, and we just weigh them in the same manner, and so on. □

Solution to Problem 1.2: Take two groups of three coins and compare them.

If they are equal, then each group has one fake coin. So, to identify a fake coin, one weighing is required for each group (see the solution to the previous problem). This makes a total of three weightings.

If they are not equal, two cases are possible: either the coin not in the groups is fake, or both fake coins are in the lighter group. Let's try to understand which case we have. The heavier group definitely does not contain fake coins, so we weigh one coin from this group with a coin not in the groups. If they are different in weight, then we have the first case; if they are equal, then we have the second case.

In the first case, we identified one fake coin from the lighter group in one weighing, and the second fake coin is the one that was not in the groups. This also requires three weightings in total.

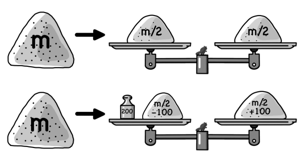
In the second case, we know for sure that both fake coins are in the lighter group. We can find the real coin with a higher weight in one weighing, and then the two remaining coins are fake. Again, this requires three weightings. □

Solution to Problem 1.3: Unlike the previous two problems, we don't know whether the fake coin is lighter or heavier. This adds an extra weight.

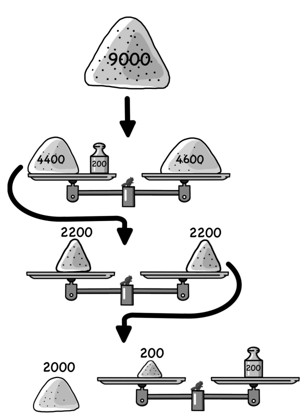
Again, divide the nine coins into three groups of three. Weigh two pairs of groups: the first with the second and the second with the third. Notice that since there is only one fake coin, two groups will be equal, and the third will not. Depending on whether the differing group is heavier or lighter, we can infer the nature of the fake coin.

Take the differing group and, using the method from the previous problem, identify the fake coin in one weighing. This makes a total of three weightings. □

Solution to Problem 1.4: Take a pile of grain weighing m. We can obtain two piles weighing either $m/2$, $m/2$ (pouring onto two pans without using the weight), or $m/2 - 100$, $m/2 + 100$ (the same, but with the weight on one pan).



We have 9000 g of grain. Divide the grain into piles of 4400 and 4600 g. Now «halve» $4400 \to 2200 + 2200$. The last weighting is to «subtract» 200 g from one pile. Thus, we have used three weightings.



□

Solution to Problem 1.5: Be careful! In this problem, the goal is not to determine which 2 weights are swapped, but simply whether the markings are correct or not.

For the first weighing, place the weights marked «6» and «1» on one side of the scales, and the weights marked «2» and «5» on the other side. If the scales are not balanced, it means the markings are definitely swapped, and the second weighing is unnecessary. If the scales are balanced, then the possible situations are:

- weights «6» and «1» are swapped;
- weights «2» and «5» are swapped;
- weights «3» and «4» are swapped.

For the second weighing, place the weights marked «6» and «2» on one side, and the weights marked «5» and «3» on the other. Reasoning similarly to the first weighing, if the scales are balanced, the possible situations are:

- weights «6» and «2» are swapped;
- weights «3» and «5» are swapped;
- weights «1» and «4» are swapped.

Thus, the possible error scenarios are inconsistent, so with these two weightings, we can guarantee whether the markings on any 2 weights are swapped or not. □

Are you enjoying the book so far?

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Water Jug Puzzles

> “Somewhere in Georgia, a train is traveling. In one of the compartments sits a mathematician and a local resident. A large flock of sheep passes by the window.
> — There are 7238 sheep in this flock, — the mathematician says automatically.
> — Wow! – the Georgian resident exclaims. — How did you know, genius? This is my flock, and there really are 7238 sheep in it. How did you count them so quickly?!
> — Very simple, – the mathematician replies. — I counted the number of legs and divided by 4.

—One joke that didn’t quite land

Theory and Practice

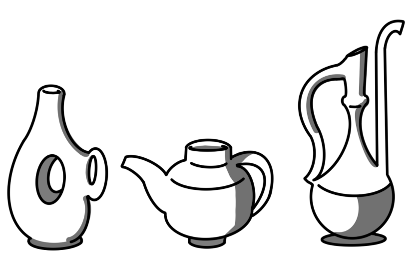
One of the Olympiad topics, the conditions of which raise many questions for students unfamiliar with this specificity, is related to water jug puzzles. Below is one of the most famous ones — the so-called Poisson problem. According to one popular story, the famous French mathematician, mechanic, and physicist Siméon Denis Poisson (1781—1840) solved this problem in his youth and later claimed that it was this problem that motivated him to become a mathematician.

This topic is also referred to as decanting problems or just jug-pouring problems. It can be known as Die Hard with a Vengeance puzzles. This amusing bucket problem was used in the action film Die Hard: With a Vengeance (1995). The characters John McClane and Zeus Carver (played by Bruce Willis and Samuel L. Jackson) solve the two bucket variant with two jugs and water from a public fountain in order to try to prevent a bomb from exploding by obtaining 4 gallons of water using only 5-gallon and 3-gallon jugs.

Example 2.1. Jean has 12 liters of milk and wants to drink half of it. But he only has 8-liter and 5-liter jags. How can he measure out 6 liters?

Solution: A typical solution from a participant unfamiliar with this topic might be: «Jean should pour half of the twelve-liter container into the eight-liter one. $12 \div 2 = 6$, so we have 6 liters.»

Unfortunately, this solution won't earn any points. As is often the case, the problem statement does not specify what the authors consider obvious — namely, that all the jags are opaque and of such shape that it is impossible to see how full they are.



The only actions available are pouring all the contents from one jag into another or pouring in as much liquid as the container can hold.

The standard method for presenting solutions to such problems is to construct a table showing the amount of liquid in each container at each step.

For example, the solution to this problem can be presented as follows:

Container Size	12	8	5
Step 0	12	0	0
Step 1	4	8	0
Step 2	4	3	5
Step 3	9	3	0
Step 4	9	0	3
Step 5	1	8	3
Step 6	1	6	5

This can be interpreted as follows:

In the first step, we completely fill the 8-liter vessel by pouring milk from the 12-liter vessel. This leaves $12 - 8 = 4$ liters of milk in the 12-liter vessel.

In the second step, we pour as much as we can from the 8-liter vessel into the 5-liter vessel. The 5-liter vessel can hold 5 liters, so the 8-liter vessel will have $8 - 5 = 3$ liters of milk remaining.

In the third step, we pour all the milk from the 5-liter vessel back into the 12-liter vessel. This gives $4 + 5 = 9$ liters of milk in the 12-liter vessel.

In the fourth step, we pour 3 liters from the 8-liter vessel into the 5-liter vessel.

In the fifth step, we pour as much as we can from the 12-liter vessel (which currently has 9 liters) into the 8-liter vessel. Since the 8-liter vessel is empty, it can take 8 liters, leaving $9 - 8 = 1$ liter of milk in the 12-liter vessel.

In the sixth step, we pour as much as we can from the 8-liter vessel (which now has 8

liters) into the 5-liter vessel, which currently has 3 liters. The 5-liter vessel can hold an additional $5 - 3 = 2$ liters. Therefore, we pour 2 liters from the 8-liter vessel, leaving $8 - 2 = 6$ liters in the 8-liter vessel, which we can finally drink. □

Consider the following problem.

Example 2.2. The evil wizard Crocobeaver is brewing a potion. In the final stage, he must add exactly 4 milliliters of dead water. Unfortunately, he only brought flasks of 3 milliliters and 5 milliliters to the dead water spring. What should Crocobeaver do to finish brewing his potion?

Solution: In this problem, the solution is very similar to the previous one. The only difference is that the volume of one of the vessels — the spring — is unlimited. We can choose to include or not include this «virtual» vessel in our table, and it should not affect the score obtained. If it is more convenient for you to include it, it is best to denote the unlimited spring with the symbol «infinity» — ∞.

Vessel Size	∞	3	5
Step 0	∞	0	0
Step 1	∞	0	5
Step 2	∞	3	2
Step 3	∞	0	2
Step 4	∞	2	0
Step 5	∞	2	5
Step 6	∞	3	4

When solving pouring problems, it is important to take into account the fact that if you repeat one of the previous steps at any point, then you are clearly overcomplicating something and going in the wrong direction. □

By Bézout's identity, which you will study in the Number Theory book, such puzzles have a solution if and only if the desired volume is a multiple of the greatest common divisor of all the integer volume capacities of jugs.

Dudeney in «Amusements in Mathematics» book provides an interesting historical account of the «pouring problems» or «Tartaglia's measuring puzzles.»

The first printed puzzle involving the measurement of a specific quantity of liquid by pouring between vessels of known capacities was introduced by Niccolò Fontana, famously known as «Tartaglia» (the stammerer, 1500—1559). His puzzle involves dividing 24 ounces of valuable balsam into three equal parts using only vessels with capacities of 5, 11, and 13 ounces. There are numerous solutions to this puzzle, each requiring six manipulations or pourings between the vessels.

Bachet de Méziriac later reprinted Tartaglia's puzzles in his «Problèmes plaisans et délectables» (1612). While it is widely believed that puzzles of this nature can only be solved through trial and error, Dudeney suggests that formulae can be developed to solve certain related cases. This area remains largely unexplored, presenting fertile ground for further investigation.

This type of puzzle is realized as mini-games on different sites. For example, in June 2024 there links work:

https://www.mathsisfun.com/games/jugs-puzzle.html

https://www.transum.org/Software/Investigations/jugs.asp

Problem Set

Problem 2.1. (MF — 2006.6.4): Esther stands on the bank of a river. She has two clay jugs: one holds 5 liters, and Esther remembers only that the other one holds either 3 or 4 liters. Help Esther determine the capacity of the second jug.

Problem 2.2. (ARSO — 2014.5.3): How can you measure 8 liters of water while standing near a river and having two buckets with capacities of 10 liters and 6 liters (8 liters of water should be in one bucket)?

Problem 2.3. (ARSO — 2014.6.3): How can you measure 2 liters of water while standing near a river and having two buckets with capacities of 10 liters and 6 liters? (Two liters of water should be in one bucket.)

Problem 2.4. (MMO — 1964.9.2): Water is poured evenly into n large capacity glasses. It is allowed to pour water from any glass to any other as much water as the latter contains. For which n is it possible to pour the water into one glass in a finite number of steps?

Problem 2.5. Considering three bowls with the following capacities: In bowl A (8 liters capacity), there are 5 liters of water. In bowl B (5 liters capacity), there are 3 liters of water. In bowl C (3 liters capacity), there are 2 liters of water. Can you measure exactly 1 liter, using only 2 pours?

Problem 2.6. (Dudeney's puzzle «Delivering the milk») A milkman one morning was driving to his dairy with two 10-gallon cans full of milk, when he was stopped by two countrywomen, who implored him to sell them a quart of milk each. Mrs. Green had a jug holding exactly 5 pints, and Mrs. Brown a jug holding exactly 4 pints, but the milkman had no measure whatever. How did he manage to put an exact quart into each of the jugs? It was the second quarter that gave him all the difficulty. But he contrived to do it in as few as nine transactions — and by a «transaction» we mean the pouring from a can into a jug, or from one jug to another, or from a jug back to the can. How did he do it?

Skill Assessment Problems

Skill Assessment Problem 2.1. In a barrel, there are $16\frac{1}{2}$ liters of jam. How can you pour out 6 liters from it using a 9-liter bucket and a 5-liter jug?

Skill Assessment Problem 2.2. It is said that the great sorcerer Johann Faust, while studying the works miraculously saved from the Library of Alexandria during its destruction, came across an ancient manuscript describing the recipe for making the elixir of immortality, for which tears of dragons, of crocodiles, and of cats were required. Having laboriously obtained all these ingredients, Faust returned to his laboratory with a 6-liter flask filled to the top with dragon tears, a 4-liter flask filled with crocodile tears, and a 5-liter flask filled with cat tears. He also had an empty 8-liter vessel at his disposal. The elixir of immortality requires exactly 6 liters, and the concentration of all its ingredients must be equal. Considering the effort expended to obtain all this, not a single drop could be spilled. Faust also understood that all these liquids mix very quickly to a uniform state. Help him solve this problem.

Skill Assessment Problem 2.3. The evil mage Crocobeaver is brewing a potion. In the final stage, he must add exactly 3 milliliters of dead water three times in a row. Unfortunately, this time, he only brought a flask with 9 milliliters of dead water to his tower, and he didn't have time to run to the spring again. He found empty vials of 5, 4, and 2 milliliters volume at his place. What should Crocobeaver do to finish brewing his potion?

Solutions to Skill Assessment Problems

Solution to Problem 2.1: If we don't want to deal with fractions, we can consider the volume not in liters but in half-liters. Then there are 33 half-liters in the barrel, 18 half-liters fit in the bucket, and 10 half-liters fit in the jug. The solution table will look like this.

Container Container Size	Barrel 33	Bucket 18	Jug 10
Step 0	33	0	0
Step 1	23	0	10
Step 2	23	10	0
Step 3	13	10	10
Step 4	13	18	2
Step 5	31	0	2
Step 6	31	2	0
Step 7	21	2	10
Step 8	21	12	0

The problem is solved. □

Solution to Problem 2.2: We will denote each type of tears by an abbreviation:

- d for dragon tears,
- c for crocodile tears, and
- k for cat tears. (kitten)

Step	4-liter	5-liter	6-liter	8-liter
Step 0	4(c)	5(k)	6(d)	0
Step 1	4(c)	5(k)	0	6(d)
Step 2	0	5(k)	4(c)	6(d)
Step 3	4(d)	5(k)	4(c)	2(d)
Step 4	4(d)	5(k)	0	4(c), 2(d)
Step 5	2(d)	5(k)	0	4(c), 4(d)
Step 6	0	5(k)	2(d)	4(c), 4(d)
Step 7	2(c), 2(d)	5(k)	2(d)	2(c), 2(d)
Step 8	2(c), 2(d)	5(k)	0	2(c), 4(d)
Step 9	0	5(k)	2(c), 2(d)	2(c), 4(d)
Step 10	0	3(k)	2(c), 2(k), 2(d)	2(c), 4(d)

Thus, after the series of transfers and mixes, the exact quantities required are obtained without spilling a single drop. □

Solution to Problem 2.3: The table will look like this.

Container Container Size	Flask 9	Test Tube 1 5	Test Tube 2 4	Test Tube 3 2
Step 0	9	0	0	0
Step 1	7	0	0	2
Step 2	3	0	4	2
Step 3	3	4	0	2
Step 4	0	4	3	2
Step 5	4	0	3	2
Step 6	6	0	3	0
Step 7	1	5	3	0
Step 8	1	3	3	2
Step 9	3	3	3	0

The complexity of the problem lies in the unusual formulation. «Three times in a row» means that Crocobeaver doesn't have time to measure new 3 milliliters after he has already added a portion to the potion. So, it is necessary to distribute the potion into the test tubes so that each of them contains exactly three milliliters at the same time. A solution where three milliliters are obtained several times and then poured into the potion will not earn points. Once we understand the problem, it's not difficult to solve. □

Examples and Constructions

> An engineer, a physics professor, and a mathematician travel by train. The engineer says, after seeing a black sheep:
> — Sheep are black in Scotland.
> The physics professor says:
> — You can only state that some sheep are black in Scotland.
> And the mathematician says:
> — You can only state that there is at least a sheep that has at least one black side.
>
> —One joke that didn't quite land

Theory and Practice

The title of this topic speaks for itself — in problem settings, we are usually asked to provide an example or construct a structure with specified properties. At the same time, presenting the solution usually does not pose any problems — it is sufficient to provide the required example and, perhaps, explain why it fits. Proof that it is the «most beautiful» or the only one is not required.

You may be asked to provide an example related to practically anything, starting from arithmetic constructions and ending with geometric ones, logical ones, and so on.

Let's consider a few problems.

Example 3.1. Find ten natural numbers whose sum and product are both equal to twenty.

Solution: In this problem, we are given the product of the ten numbers in advance, so the first thing to do is to factorize the number 20 into prime factors (recall that a prime number is a number that is divisible only by 1 and itself, and the number 1 itself is not considered prime): $20 = 2 \cdot 2 \cdot 5$. Thus, it becomes obvious that at least 7 factors are equal to 1. Then, by simple trial and error, we can obtain that

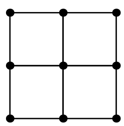
$$20 = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 \cdot 10 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 2 + 10.$$

This example is then written in the solution without describing how it was obtained. □

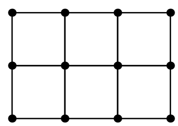
Let's consider a more «geometric» problem.

Example 3.2. Draw 8 identical squares such that exactly 15 points are vertices of the drawn squares.

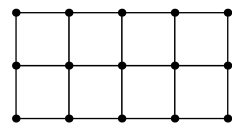
Solution: Each square has 4 vertices. Therefore, if the squares are located in different positions, there will be $4 \times 8 = 32$ vertices, which is more than twice the required number. In this case, let's try to cluster our squares. Group 4 squares into one large square (Figure a)). This will give us 9 vertices. Now we need to add 4 more squares, adding only 6 vertices. Add 2 squares next to it, completing the figure to a 2×3 rectangle (Figure b)). This adds 2 squares and 3 vertices. By adding 2 more squares in the same way, we obtain the required configuration — Figure c). This is the solution and the answer. □



a)



b)



c)

Let's return to the arithmetics.

Example 3.3. Find a natural number such that it is 2002 times greater than the sum of its digits.

Solution: Obviously, our number must be divisible by 2002, so it must have at least four digits. To avoid complicated equations with many unknowns, let's start by listing numbers divisible by 2002 and check them for satisfying the condition:

$$2002 \neq (2+0+0+2) \cdot 2002 = 4 \cdot 2002,$$

$$2002 \cdot 2 = 4004 \neq (4+0+0+4) \cdot 2002 = 8 \cdot 2002,$$
$$2002 \cdot 3 = 6006 \neq (6+0+0+6) \cdot 2002 = 12 \cdot 2002,$$
$$2002 \cdot 4 = 8008 \neq (8+0+0+8) \cdot 2002 = 16 \cdot 2002,$$
$$2002 \cdot 5 = 10010 \neq (1+0+0+1+0) \cdot 2002 = 2 \cdot 2002.$$

Finally, the next number satisfies the condition, and we get the answer:

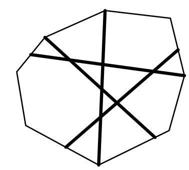
$$2002 \cdot 6 = 12012 = (1+2+0+1+2) \cdot 2002 = 6 \cdot 2002.$$

□

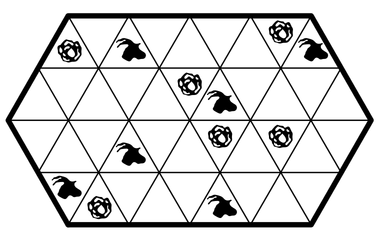
Problem Set

Problem 3.1. (MF — 1994.6.1): Among four people, there are no three with the same name, second name, or surname, but any two have either the same name, second name, or surname. Is this possible?

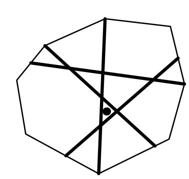
Problem 3.2. (MF — 2015.6.1): Four intersecting paths cross the yard (see the plan). Plant four apple trees so that there are an equal number of apple trees on each side of each path.



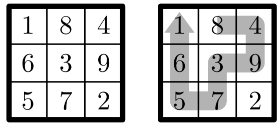
Problem 3.3. (MF — 2017.6.1;7.1): The farmer fenced off a plot of land and divided it into triangles with a side length of 50 m. In some triangles, he planted cabbage, and in others, he let goats graze. Help the farmer build additional fences along the grid lines of the least total length possible to protect all the cabbage from the goats.



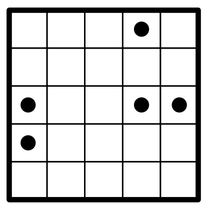
Problem 3.4. (MF — 2015.7.1): In a yard with four intersecting paths, there is one apple tree growing (see the plan). Plant three more apple trees so that there are an equal number of apple trees on each side of each path.



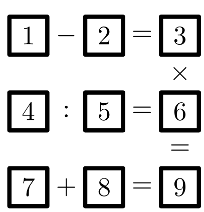
Problem 3.5. (MF — 2012.7.1): A 3×3 square is filled with digits, as shown on the left. It is allowed to move through the cells of this square, moving from one cell to an adjacent one (along the side), but it is not allowed to enter any cell twice. Leo walked as shown on the right and wrote down all the digits encountered along the way in order, resulting in the number 84937561. Draw another path to get a larger number (the larger, the better).



Problem 3.6. (AT — 2014.1) It is required to move each of the five chips to an adjacent cell so that in the end, there is no more than one chip in each row, each column, and on each main diagonal. Two cells are considered adjacent if they share a side. Show how to do this. Show the movements of the chips with arrows.

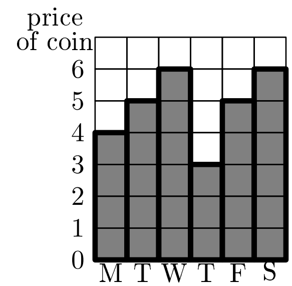


Problem 3.7. (AT — 2017.1): The cards with numbers $1, 2, 3, \ldots, 9$ are arranged on the figure so that four incorrect equalities are obtained (three horizontal, one vertical).

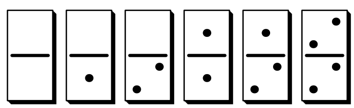


Rearrange these cards so that all equalities become correct. It is sufficient to provide the answer.

Problem 3.8. (AT — 2005.7.1) The figure shows how the exchange rate of the coins changed during the week. Leo had 30 dollars. On one of the days of the week, he exchanged all his dollars for coins. Then he exchanged all the coins for dollars. Then he exchanged all the received dollars for coins again, and finally, he exchanged all the coins back for dollars. Write down on which days he performed these operations if he had 54 dollars on Sunday. (It is sufficient to provide an example.)

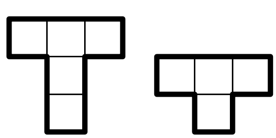


Problem 3.9. (MF — 2014.6.2): Arrange the six dominoes (see the figure) to form a 3×4 rectangle so that there are an equal number of dots in each of the three rows and each of the four columns. (Thicken the borders of the dominoes.)



Problem 3.10. (MF — 2010.6.3): Jean figured out how to stack a rectangular parallelepiped from identical cubes and cover it with three squares without gaps or overlaps. Do it yourself.

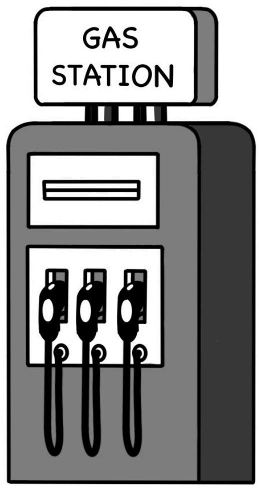
Problem 3.11. (MF — 2014.6.4): Draw a figure that can be cut into four pieces depicted on the left, as well as into five pieces depicted on the right. (The pieces can be rotated.)



Problem 3.12. (MF — 2007.6.4): In the Perfect City, there are six squares. Each square is connected by straight streets to exactly three other squares. No two streets in the city intersect. Out of the three streets departing from each square, one passes inside the angle formed by the other two. Draw a possible plan for such a city.

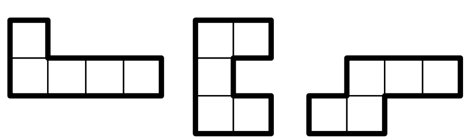
Problem 3.13. (MF — 2016.7.3): Combine three identical grid figures without symmetry axes into a figure with a symmetry axis.

Problem 3.14. (MF — 1998.6.5;7.3): Four gas stations A, B, C, and D are located around a circular road. The distance between A and B is 50 km, between A and C is 40 km, between C and D is 25 km, and between D and A is 35 km (all distances are measured along the circular road in the shortest direction).



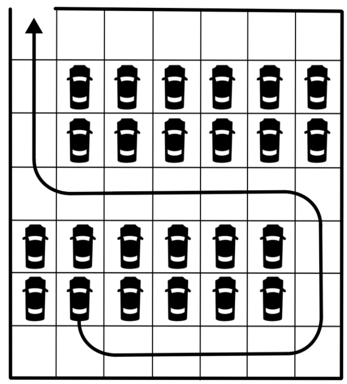
1. Provide an example of the arrangement of gas stations (indicating the distances between them) that satisfies the conditions of the problem.
2. Find the distance between B and C (indicate all possible options).

Problem 3.15. (MF — 2007.6.5;7.5): Draw how to combine the given three figures, using each one exactly once to form a figure with an axis of symmetry.



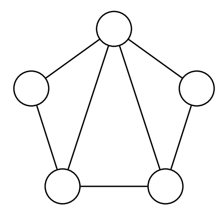
Problem 3.16. (MF — 2008.6.5): The parking station in Beaverville is a 7×7 grid, where each cell can accommodate a car. The parking station is surrounded by a fence,

with one side of the corner cell removed (this is the gate). Cars drive along a one-cell-wide path. Esther was asked to place as many cars as possible in the parking station in such a way that any car could leave while the others were parked. Esther placed 24 cars, as shown in the figure. Try to rearrange the cars differently to fit more cars.



Problem 3.17. (MF — 2014.7.5) Fit natural numbers into five circles such that if two circles:

- are connected by a line, then the ratio of the numbers in them is equal to 2 or 4;
- are not connected by a line, then the ratio of the numbers in them should not be equal to 2 or 4.



Problem 3.18. (MF — 2006.7.5) Grandpa called his grandson to the village: «Look at the extraordinary garden I planted! I have pears and apples growing there, and the apple trees are planted in such a way that exactly two pears grow at a distance of 10 meters from each apple tree.»
«So what's so interesting about that, you have half as many apple trees as pears,» — the grandson replied.
«But you guessed wrong,» said Grandpa. «I have twice as many apple trees in my garden as pears.»
Draw how the apples and pears could have grown in Grandpa's garden.

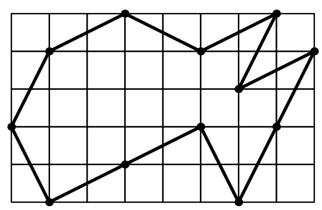
Problem 3.19. (MF — 2009.7.5): Draw two quadrilaterals with vertices at the grid nodes, from which you can construct:

a) both a triangle and a pentagon;

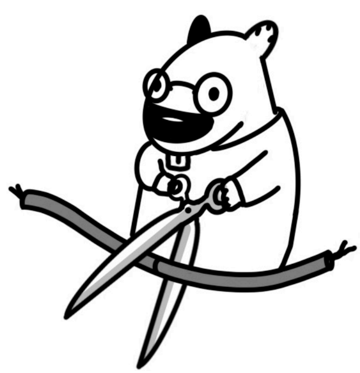
b) a triangle, a quadrilateral, and a pentagon.

Show how this can be done.

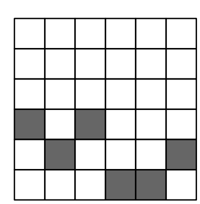
Problem 3.20. (MF — 2014.7.5): Max draws closed paths inside a 5×8 rectangle, following the diagonals of the 1×2 rectangles. The figure shows an example path passing through 12 such diagonals. Help Max draw the longest possible path.



Problem 3.21. (COM — 2005.7.1): The electrician had a piece of wire 25 m long, from which he planned to cut pieces of 1 m, 2 m, 3 m, 6 m, and 12 m. But in the morning, it turned out that someone had cut the wire into two pieces during the night. Can the electrician complete the planned work?



Problem 3.22. (MF — 2013.7.2): A part of the cells is shaded in the square, as shown in the figure. It is allowed to fold the square along any grid line and then unfold it back. Cells that coincide with shaded cells when folded are also shaded. Is it possible to shade the entire square with a) 5 or fewer, b) 4 or fewer, and c) 3 or fewer such folds?



If yes, enter the number of the fold in each cell, after which it will be shaded for the first time; draw the folding line and mark it with the same number. If not, prove it.

Problem 3.23. (MF — 2005.7.2): Can the numbers a) from 1 to 7; b) from 1 to 9 be arranged in a circle so that each of them is divisible by the difference of its neighbors?

Problem 3.24. (COM — 2011.6.4;7.2): Suppose several points are marked on the plane. Call a line *dishonest* if it passes exactly through three marked points and there are different numbers of marked points on each side of it. Can you mark 7 points and draw 5 dishonest lines for them?

Problem 3.25. (COM — 2006.6.4): Eight numbers were written around a circle. Then, between each pair of neighboring numbers, their sum was written, and the old numbers were erased. Could it happen that the numbers 11, 12, 13, 14, 15, 16, 17, 18 are now written around the circle?

Problem 3.26. (COM — 2004.7.4): There are 27 students in class. They were offered to attend clubs for singing, silence, and reading poetry. Each student wants to attend one or more of these clubs. It turned out that more than a third of the class wanted to attend each club. Is it always possible to distribute the children among the clubs so that each attends only one club and there are an equal number of children in each club?

Problem 3.27. (AT — 2012.4): Is it possible to cross out one of the natural numbers from 1 to 9, and arrange the remaining numbers at the vertices of a cube so that the sums of the numbers on each face of the cube are equal to each other but are not multiples of the crossed-out number?

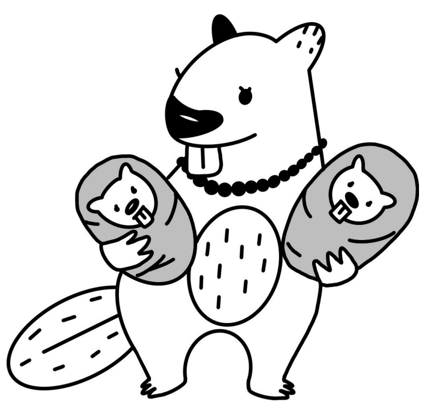
Problem 3.28. (MF — 2008.7.5): Esther cut out two identical figures from cardboard. She placed them, overlapping each other, on the bottom of a rectangular box. The bottom turned out to be completely covered. A nail was driven into the center of the bottom. Could the nail penetrate one piece of cardboard and not penetrate the other?

Skill Assessment Problems

Skill Assessment Problem 3.1. Write the number 1997 using ten twos and arithmetic operations.

Skill Assessment Problem 3.2. How can you distribute weights of $1, 2, \ldots, 9$ grams into three boxes so that the first one contains two weights, the second one contains three, the third one contains four, and the total weight of the weights in each box is the same?

Skill Assessment Problem 3.3. In the country of Beaverland, there live digital beavers. Every day in Beaverland, exactly one beaver is born, with the day number since the creation of the world written on its skin. The law states that any beaver whose skin number's digits sum to a multiple of 7 is assigned the status of a *beaverdemic*. One day, a beaver mom gave birth to twin beavers, but due to the digital system of the world, one was officially born a day earlier than the other. Could it be that both twins are *beaverdemics*?



Solutions to Skill Assessment Problems

Solution to Problem 3.1: To obtain full credit for this problem, it is sufficient to present the solution. In such problems, it is usually acceptable not to put arithmetic operation signs between numbers, which allows us to obtain the number 2222 using four twos. We need the number 1997, and we have already used four twos to obtain 2222. Now we need to subtract the number 225, which can be obtained using six twos:

$$1997 = 2222 - 222 - 2 - 2 \div 2.$$

□

Solution to Problem 3.2: This problem can be solved quite simply: first, we calculate the weight of each box. The total weight of all weights is 45 grams, so each box should contain weights totaling 15 grams. We can fill one box with weights of 9 and 6 grams, the second box with weights of 8, 5, and 2 grams, and the third box with weights of 1, 3, 4, and 7 grams. The answer is presented, and the problem is solved. □

Solution to Problem 3.3: Clearly, the problem asks to find two consecutive natural numbers, each of whose digit sums is divisible by 7. Note that this can only happen if the smaller number ends in several nines so that when adding one, the sum of digits changes by more than 1. Let n be the number of nines, then adding one will change the sum of digits by $k = 1 - 9n$, and k is divisible by 7 when n is 4. Hence, the smaller number can end in 9999, and to ensure that the sum of its digits is divisible by 7, we prepend 6 to the number, obtaining 69999. From this, the number of the second twin is 70000. Both numbers satisfy the conditions. □

Algorithms and Operations

> The best moment in a mathematician's life is when he has already derived the proof but hasn't found errors in the calculations yet.
>
> —One joke that didn't quite land

Theory and Practice

This topic is a logical continuation of the «examples and constructions» chapter, but while in «examples and constructions» we usually need to provide a «static example», in this topic, we are looking for a «dynamic» example, meaning we describe a sequence of actions that can lead us from the initial position to the desired one.

Let's consider an unexpected example of such a task proposed in the final exam in computer science in Russia.

Example 4.1. The «Doubler» performer has two commands with the following numbers:
1) subtract 1;
2) multiply by 2.
The first command decreases the number on the screen by 1, and the second one doubles it. Write down the sequence of commands in a program that transforms the number 17 into the number 135 and contains no more than 4 commands. Only indicate the command numbers.

For example, the program 212 is interpreted as:

- multiply by 2;
- subtract 1;
- multiply by 2.

It transforms the number 3 into the number 10.

Solution: Note that 135 is odd, and we can only increase the number by multiplying by 2, so at least the last of the commands must be subtracting 1. Also, note that there must be at least three commands 2; otherwise, we would get a number not exceeding $17 \cdot 2 \cdot 2 = 68 < 135$. Therefore, we have three commands 2 and one 1, which is the last one. Thus, the program that transforms 17 into 135 can be written as 2221. □

The process, from Latin «processus», means a step or progress. Problems related to algorithms and operations involve step-by-step execution of certain actions (usually a limited number of them). The solution to such problems is reduced to initiating a particular process — an algorithm, a sequence of steps, and it is important to demonstrate that it will be finite and lead to the desired result.

Example 4.2. In parliament, no one has more than three enemies. Prove that the parliament can be divided into two chambers so that each deputy in their chamber will have no more than one enemy.

Solution: Let's demonstrate the organization of the division process step by step. We will assign deputies to chambers one by one. Initially, we randomly divided the parliament into two chambers. Then, we find a deputy who has at least two enemies in their chamber and move him to the other chamber. The total number of enemy pairs sitting in the same chamber decreases. The process will be finite and stop because the number of hostile pairs is no more than the total number of people in the parliament multiplied by three (in fact, even less, but we'll cover this in the graph theory book). The termination of the process means the construction of the desired division. □

Unfortunately, sometimes erroneous reasoning is encountered. For the above problem, for example, one might give the following incorrect «solution».

Wrong solution: Place all deputies in the hall. Choose one deputy and send them to chamber A. Then, choose a deputy who has no enemies in A (while such deputies exist) and send them to A. Repeat this iteration until there are no such deputies left. Then, choose in the hall a deputy who has exactly one enemy and send them to A. Repeat this several times until such deputies are exhausted. Then, assign those deputies to chamber B who have three enemies in A. Since they do not quarrel with each other, this will not contradict the condition. Deputies remaining in the hall have exactly two enemies in A. But then in B they have no more than one enemy, and they can be sent to B. □

The mistake in the last «solution» is as follows: by sending deputy X to A, who has one enemy Y in A, and sending Z, who also has one enemy (let it again be Y), we end up with Y having two enemies in A.

Example 4.3. Three stakes are stuck into the ground, two empty, and the third one has disks with diameters of 3, 2, and 1 decimeters placed on it from bottom to top. The following is permitted: take the top disk from one stake and place it onto another stake, with the condition that only a disk of a smaller diameter can be placed on top of a disk with a larger diameter. The task is to move all the disks to another stake in the same order using these operations.

Solution: Represent the solution in the form of a table, where we will display the disks placed on each stake at each step (the leftmost digit denotes the top disk).

Step 0	123		
Step 1	23	1	
Step 2	3	1	2
Step 3	3		12
Step 4		3	12
Step 5	1	3	2
Step 6	1	23	
Step 7		123	

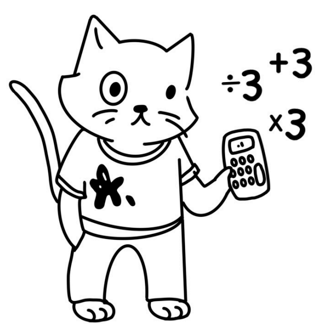
The result obtained at the last step completes the solution to the problem. □

The last problem is known as «Tower of Hanoi» and is widely used in different areas of life. For example, it is used in psychological research on problem-solving. In the 2011 film Rise of the Planet of the Apes, this puzzle, called in the film the «Lucas Tower», is used as a test to study the intelligence of apes.

Problem Set

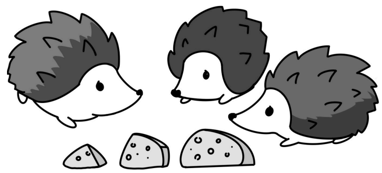
Problem 4.1. (COM — 2004.6.1): Two sticks are given. They can be attached to each other, and marks can be made. How can you determine using these operations which of the measures is greater — the length of the shorter stick or $2/3$ of the length of the longer stick?

Problem 4.2. (COM — 2002.6.1): The first of ten friends has 5 coins, the second has 10 coins, the third has 15 coins, and so on, with the tenth having 50 coins. They boarded a magic carpet plane, the flight of which cost 5 coins per person. Will they be able to pay the magic carpet plane company fairly if it doesn't give change and doesn't exchange money?



Problem 4.3. (MF — 2010.7.1): Leo has a calculator that allows him to multiply a number by 3, add 3 to a number, or (if the number is divisible by 3) divide it by 3. How can he obtain the number 11 from the number 1 using this calculator?

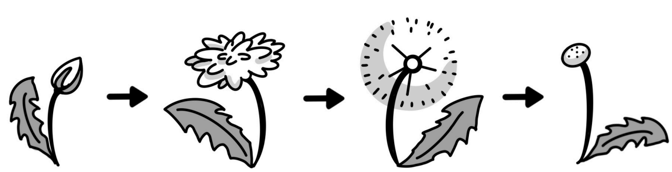
Problem 4.4. (MF — 1994.6.2): Find the number in the sequence $2, 6, 12, 20, 30, \ldots$ at: a) the 6th position; b) the 1994th position? Explain your answer.



Problem 4.5. (MF — 1998.6.2): Three hedgehogs shared three pieces of cheese weighing 5 g, 8 g, and 11 g. A fox, Alice, decided to help them. She can simultaneously cut and eat 1 g of cheese from any two pieces. Can Alice leave the hedgehogs with equal pieces of cheese?

Problem 4.6. (MF — 2014.6.3): A dandelion blooms in the morning, stays yellow for two days, turns white on the third day in the morning, and by evening it loses its petals. Yesterday, there were 20 yellow and 14 white dandelions in the meadow, and today there are 15 yellow and 11 white ones. How many:

a) Yellow dandelions were there in the meadow the day before yesterday;
b) White dandelions will be in the meadow tomorrow?



Problem 4.7. (MF — 2014.7.4): A dandelion blooms in the morning, stays yellow for three days, turns white on the fourth day in the morning, and by the evening of the fifth day, it loses its petals. On Monday afternoon, there were 20 yellow and 14 white dandelions in the meadow, and on Wednesday, there were 15 yellow and 11 white ones. How many white dandelions will be in the meadow on Saturday?

Problem 4.8. (COM — 2016.6.3): On the left bank of the river, 5 physicists and 5 chemists have gathered. Everyone needs to get to the right bank. There is a two-seat boat. At no time can there be exactly three chemists or exactly three physicists on the right bank? How can they all cross, making 9 trips to the right?

Problem 4.9. (COM — 2009.6.3): There are five batteries, three of which are charged and two are discharged. The camera works with two charged batteries. Show how to reliably turn on the camera in four attempts.

Problem 4.10. (COM — 2009.6.4): Jean is in a broken moon rover 18 km away from the Lunar base, where Esther is sitting. There is a stable radio connection between them. The air supply in the moon rover will last for 3 hours, and Jean also has a spacesuit with an air tank that will last for 1 hour. Esther has many air tanks, each with a 2-hour air supply. Esther cannot carry more than two tanks at a time (she uses one of them herself). The speed of movement on the Moon in a spacesuit is 6 km/h. Can Esther save Jean and not perish herself?

Problem 4.11. (COM — 2008.7.4): Twenty gentlemen met at the club. Some of them wore hats, and some did not. From time to time, one of the gentlemen would take off his hat and put it on one of those who did not have a hat at that moment. In the end, ten gentlemen calculated that each of them gave away their hat more times than they received it. How many gentlemen came to the club wearing hats?

Problem 4.12. (AT — 2017.4): Garland ball decorations are sold at the New Year's Fair. Each garland consists of 201 balls: some are red, and the rest are green. The balls are magical — upon the command of the Duty Snowman, they can change color: red ones become green, and green ones become red. In one step, he can change the color of any two-, three-, or four consecutive balls. For each recoloring, the Snowman charges 1 dollar. Max, who has 100 dollars, claims that he will definitely have enough money to turn any garland into a monochrome one. Is Max right?

Problem 4.13. (MF — 2015.7.5): There is a set of two cards: $\boxed{1}$ and $\boxed{2}$. During one operation, an expression is allowed to be composed using the numbers on the cards, arithmetic operations, and brackets. If its value is a non-negative integer, then it is written on a new card. (For example, having cards $\boxed{3}$, $\boxed{5}$, and $\boxed{7}$, you can compose the expression $\boxed{7}\boxed{5}/\boxed{3}$ and get the card $\boxed{25}$ or compose the expression $\boxed{3}\boxed{5}$ and get the card $\boxed{35}$.) How to get a card with the number 2015 in:

a) 4 operations;
b) 3 operations?

Problem 4.14. (MF — 1990.6.4;7.4): Arrange in a row: a) 5 prime numbers; b) 6 prime numbers such that the differences between adjacent numbers in each row are equal.

Problem 4.15. (MF — 1991.7.5): Two sequences are given:

$$2, 4, 8, 16, 14, 10, 2 \quad \text{and} \quad 3, 6, 12.$$

In each of them, each number is obtained from the previous one by the same rule. Find this rule. Find all natural numbers that remain unchanged (according to this rule). Prove that the number 2^{1991} becomes single-digit after several iterations.

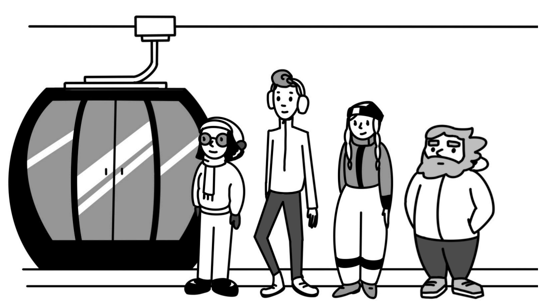
Problem 4.16. (COM — 2006.7.5): On an infinite sheet of checkered paper, x cells are colored black. Every second, all white cells with at least three out of four neighbors black become black, and all cells with at least three out of four white neighbors become white. The remaining cells remain unchanged. Can it happen after several seconds that there are exactly $\frac{3}{2}x$ black cells on the sheet of paper?

Problem 4.17. (MF — 1997.6.6): A family approaches a bridge at night. Dad can cross it in 1 minute, Mom in 2 minutes, the kid in 5 minutes, and Grandma in 10 minutes. They have only one flashlight. The bridge can only hold two people at a time. If two people cross, they move at the speed of the slower one. It's not allowed to move on the bridge without the flashlight. They can't shine the flashlight from afar. They can't carry each other on their backs. How can they cross the bridge in 17 minutes?

Problem 4.18. (MF — 1997.6.6): Koschei the Deathless captured 43 people and took them to an island. Ivan Tsarevich went to rescue them in a two-person boat. Koschei said to him:

– I'm tired of feeding these freeloaders; let them sail away from here on your boat. Keep in mind: from the island to the shore, only two can sail together, but back, one can manage alone. Before the crossing, I will tell each of them about at least 40 other captives that they are werewolves. You choose who I will say about whom. If a captive hears about someone being a werewolf, he won't get in the boat with him, but he can stay on the shore. I will enchant them so that they remain silent on land, but in the boat, they will tell each other about all the werewolves they know. As long as there is at least one captive left on the island, you cannot sail with them. Only when all 43 are on the other side, you can sail back for one of them. If you fail to arrange their crossing, you will stay with me forever.

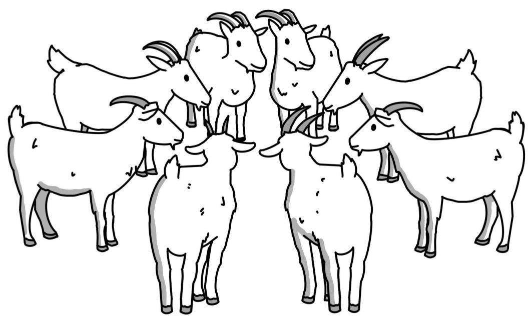
Does Ivan have a way to pass the test and return home with the captives?



Problem 4.19. (COM — 2014.6.6): Four people weighing 50, 60, 70, and 90 kg approach the cabin of a cable car leading up the mountain. There is no attendant, and the cabin operates automatically, shuttling back and forth only with a load of 100 to 250 kg (in particular, it does not operate empty), provided that passengers can be seated on two benches so that the weights on the benches differ by no more than 25 kg. How can they all go up the mountain?

Problem 4.20. (MF — 1993.6.7): Leo stands with a large bag of coins in the corner of an empty rectangular cave with cells colored in a chessboard pattern, sized $m \times n$

cells. From any cell, he can take a step to any of the four neighboring cells (up, down, right, or left). In doing so, he must either place 1 coin in that cell or take 1 coin from it, provided it is not empty. After Leo's walk through the cave, is it possible that there is exactly 1 coin on each black cell and no coins on white cells?



Problem 4.21. (COM — 2003.6.7): Eight goats of different heights stand in a circle. Each of them can jump over two neighboring goats counterclockwise. Prove that regardless of the initial arrangement of the goats, they can be arranged by height.

Problem 4.22. (COM — 2005.7.6): A stack of cards lies on the table with the backs facing up. It is required to rearrange them in reverse order (again with the backs facing up) using the following operation several times: two neighboring cards are taken from any place in the stack, flipped as a whole, and placed back in their original position. For what number of cards in the stack can this be done?

Problem 4.23. (COM — 2012.7.7): Sugar powder is stored in the warehouses of two stores; the first warehouse has 16 tons more than the second one. Every night at exactly midnight, the owner of each store steals a quarter of the sugar from their competitor's warehouse and transfers it to their own warehouse. After 10 nights, the thieves were caught. In which warehouse was there more sugar at the moment of their capture, and by how much?

Problem 4.24. (COM — 2003.7.7): Eight volumes of the «Encyclopedia of Goats» were stacked. It is allowed to take either the third book from the top or the bottom one from the stack and place it on top. Prove that regardless of the initial arrangement of the volumes, they can be stacked in order of their numbers.

Problem 4.25. (Kurchatov — 2016.8): Three hundred students from no fewer than four schools came to the Olympiad. Prove that they can be divided into teams of three each so that in each team, either all three students are from the same school or all three are from different schools.

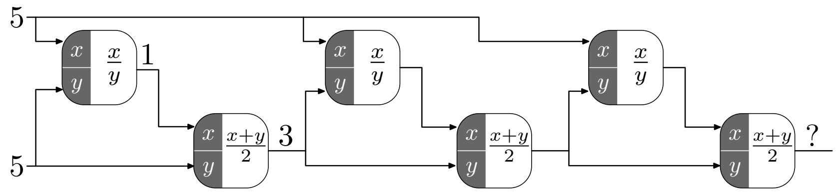
Problem 4.26. (MF — 1991.5-6.2): An electrician was called to repair a garland of four connected lamps, one of which burned out. If at least one of the lamps is burned out, the garland will not work at all. It takes 10 seconds to unscrew any bulb from the garland and 10 seconds to screw it back in. The time spent on other actions is negligible. What is the minimum time the electrician can definitely find the burned-out bulb if he has one spare bulb?

Problem 4.27. (LT — 1986.8.4): In a company of k people ($k > 3$), each person has received a piece of news known only to them. In one phone call, two people share with each other all the news they know. Prove that after $2k - 4$ conversations, they can all know all the news.

Problem 4.28. (HMO) Naomi has a broken calculator. All it can do is either add one to the previous answer or square the previous answer (it performs the operations correctly). Naomi starts with 2 on the screen. In how many ways can she obtain the answer of 1000?

Problem 4.29. (HMMT) Neo has an infinite supply of red pills and blue pills. When he takes a red pill, his weight will double, and when he takes a blue pill, he will lose one pound. If Neo originally weighed one pound, what is the minimum number of pills he must take to make his weight 2015 pounds?

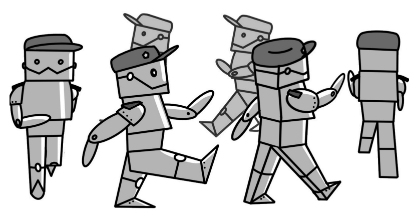
Problem 4.30. (mathcounts) Six function machines are connected, as shown. Three of the function machines are «dividers», while three of the function machines are «averagers». Each divider takes two inputs (on the left) and sends their quotient to the output (on the right). Given that the initial inputs are $x = 5$ and $y = 5$, what is the final result emitted by the right-most function machine? Express your answer as a missed number.



Skill Assessment Problems

Skill Assessment Problem 4.1. Three stakes are stuck into the ground, two empty, and the third one has disks with diameters of 4, 3, 2, and 1 decimeters placed on it from bottom to top. The following is permitted: take the top disk from one stake and place it onto another stake, with the condition that only a disk of a smaller diameter can be placed on top of a disk with a larger diameter. The task is to move all the disks to another stake in the same order using these operations.

Skill Assessment Problem 4.2. Entrepreneur Max started trading. Every morning, he buys goods for a certain portion of his money (possibly all the money he has). After lunch, he sells the purchased goods for twice the price he bought them. How should Max trade so that after 5 days he has exactly 25000 dollars, if he initially had 1000 dollars?



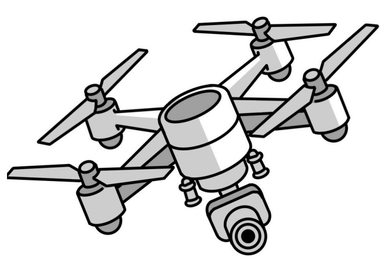
Skill Assessment Problem 4.3. A general decided to modernize the army of robots. To do this, he arranged robot soldiers in a circle and started walking around the circle, turning every tenth soldier into a drone, after which the drone would take off and fly away. However, due to his forgetfulness, he went around the circle more than once and stopped when he noticed that only one soldier was left. What was the number of these soldiers if there were a total of 20 soldiers?

Solutions to Skill Assessment Problems

Solution to Problem 4.1: Let's represent the solution in the form of a table, where we'll display the disks mounted on each stuck at each step (the leftmost digit indicates the top disk).

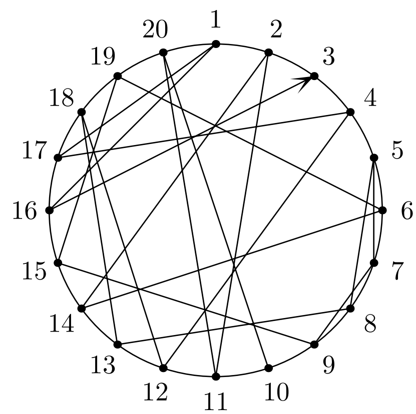
Step 0	1234		
Step 1	234	1	
Step 2	34	1	2
Step 3	34		12
Step 4	4	3	12
Step 5	14	3	2
Step 6	14	23	
Step 7	4	123	
Step 8		123	4
Step 9		23	14
Step 10	2	3	14
Step 11	12	3	4
Step 12	12		34
Step 13	2	1	34
Step 14		1	234
Step 15			1234

The constructed table completes the solution to the problem. □



Solution to Problem 4.2: We can solve the problem by working backwards. After the fifth day, he must have 25000 dollars. If he had 13000 dollars after the fourth day, by buying goods for 12000 and selling them for 24000, he would have the desired amount. Similarly, if he had 7000 dollars after the third day, he would buy goods for 6000 dollars and sell them for 12000. And 7000 dollars can be obtained from 4000 dollars by buying goods for 3000. Finally, 4000 dollars can be obtained in two days by buying goods with all the money both times. □

Solution to Problem 4.3: The sequential order of transformations according to the algorithm is shown in the figure, from which it follows that the soldier left would be numbered 3. □



Logic Problems

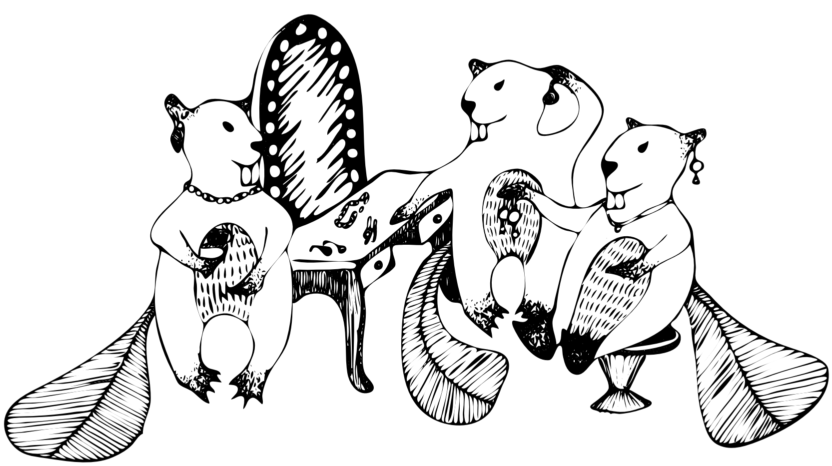
> “ Another wonderful illustration of applying mathematical methods to zoology.
> Theorem: A crocodile is longer than it is wide.
> Proof. Take an arbitrary crocodile and prove two auxiliary lemmas.
> Lemma 1: A crocodile is longer than it is green.
> Proof. Look at the crocodile from above — it is long and green. Look at it from below — it is long but not as green (actually, it is dark gray). Therefore, lemma 1 is proven.
> Lemma 2: A crocodile is greener than it is wide.
> Proof. Look at the crocodile from above again. It is green and wide. Look at it from the side: it is green but not wide. This proves lemma 2.
> The statement of the theorem obviously follows from the proven lemmas.
> The converse theorem («A crocodile is wider than it is long») is proven similarly.
> At first glance, both theorems imply that the crocodile is square. However, since the inequalities in their formulations are strict, a true mathematician will draw only one correct conclusion: CROCODILES DO NOT EXIST!
>
> —One joke that didn’t quite land

Theory and Practice

Often, in school olympiads, there are problems that can be grouped into a common category known as «logical problems». They often cause frustration among participants because many of these logical questions require a clear explanation of the answer, even though the problem might seem obvious or evident. The simplest way to avoid losing points due to insufficiently correct formatting is to write the solution to the problem in such detail that even a first-grader can understand it («I couldn't do it. I couldn't reduce it to the freshman level. That means we don't really understand it.», R. Feynman). Here is a chance that the examiners may get irritated because of you «over-explaining» the problem, but they will not be able to deduct your points, so be as specific and detailed as possible when writing the answer.

Let's consider some classic logic problems.

Example 5.1. Beavers Tata, Didi, and Dita have pierced ears. They have three pairs of earrings among them — red, green, and blue (they can be for the right ear or for the left ear). Tata has earrings of the same color in different ears. Didi definitely doesn't have any red earrings. Dita has a green earring in the right ear and a different color earring in the left ear. Who has a blue earring in the left ear?



Solution: Let's determine the color of Didi's earring in the right ear. It's not red, and since Dita has a green earring on the right, then Didi has a blue earring on the right. Thus, Tata has the remaining red earring in the right ear. Since Tata's earring colors are the same, then she also has a red earring in the left ear. What about Dita? Since she has a green earring in the right ear and the red one has already been taken, she must have a blue earring in the left ear. Answer: Dita. □

One of the classic methods of formatting logic problems is to create tables. Let's consider the following problem.

Example 5.2. Three friends — Jean, Alex, and Serge — teach mathematics, physics, and literature in schools in New York, Los Angeles, and Paris. Jean doesn't work in Los Angeles, Alex doesn't work in New York, a New York resident teaches literature, a Los Angeles resident doesn't teach physics, and Alex doesn't teach mathematics. What subject does each of them teach, and in which city?

Solution. Let's construct a table of pairwise correspondences between names, cities, and subjects. We will combine them into one table below.

	J	A	S	NY	LA	P
Ma						
Ph						
L						
NY						
LA						
P						

We'll fill in the table with facts from the problem statement; for this, we'll mark minuses in the cells where the statement is false and pluses in those where the statement is true. Since our conditions are mutually exclusive (i.e., a person cannot live in different cities simultaneously), when we put a plus in a cell, we must put minuses in the remaining cells in the 3×3 sub-table in the same row and column. After putting a plus according to the condition that the literature teacher lives in New York, we'll fill in the empty cells of the third row and the first column of the right sub-table with minuses. As soon as we put a minus after the condition «the Los Angeles resident doesn't teach physics», we realized that there is only one empty cell left in the Los

Angeles column of the right sub-table, so we put a plus there. After the initial filling, we see such a table (Figure a).

	J	A	S	NY	LA	P
Ma		−		−	+	−
Ph				−	−	+
L				+	−	−
NY		−				
LA	−					
P						

a) After the initial filling

	J	A	S	NY	LA	P
Ma	−	−	+	−	+	−
Ph	−	+	−	−	−	+
L	+	−	−	+	−	−
NY	+	−	−			
LA	−	−	+			
P	−	+	−			

b) Solved problem

We already have a lot of information: the literature teacher lives in New York, the mathematics teacher lives in Los Angeles, and the physics teacher lives in Paris.

Let's reason further. Alex doesn't live in New York, Alex isn't a mathematician, the mathematician lives in Los Angeles $\to$ Alex doesn't live in Los Angeles $\to$ Alex lives in Paris. We put a plus. Now we know that Jean lives in New York and Serge lives in Los Angeles. Combining our knowledge, we get the final table (Figure b). □

Another important topic in logic problems is problems involving knights (truth-tellers), knaves (liars), and spies (who can tell both truths and lies, possibly with imposed restrictions). Let's consider a problem on this topic.

Example 5.3. Out of three people, A, B, and C, one is a knight, another is a liar, and the third is a spy. A said: «I am a spy». B said: «A and C sometimes tell the truth». C said: «B is a spy». Who is the liar, who is the knight, and who is the spy?

Solution: A says that he is a spy, so he cannot be a knight because, in that case, he would lie.

Let's assume A is a liar. Then he never tells the truth, so B lied when he said that A and C sometimes tell the truth because if he meant that each of them sometimes tells the truth, then he lied. Therefore, B is a spy. Thus, C is a knight, and C tells the truth. This option fits. But we must check all possible options.

Let's assume A is a spy. Then C definitely lies because B cannot also be a spy. So, C is a knave because we already have a spy. Then B is a knight. But C never tells the truth, so B lied, hence he is not a knight. Contradiction.

Thus, we have obtained the only answer: A is a liar, B is a spy, C is a knight. $\square$

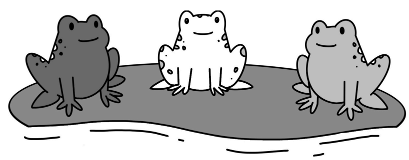
Problem Set

Problem 5.1. (MF — 2010.6.2): In the Land of Forestia, only elves and gnomes live. Gnomes lie about their gold but tell the truth otherwise, while elves lie about gnomes but tell the truth otherwise. Once, two residents of Forestia said:

A: All my gold I stole from the Dragon.

B: You are lying.

Determine whether each of them is an elf or a gnome.



Problem 5.2. (COM — 2011.6.2): In the Island of Multicolored Frogs, some inhabitants always tell the truth, while others always lie. Three islanders said the following:

Bre: There are no blue frogs on our island.

Ke: Bre is a liar. He himself is a blue frog!

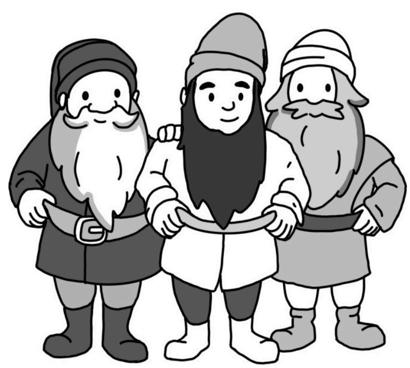
Keks: Of course, Bre is a liar. But he is a red frog.

Are there any blue frogs on this island?

Problem 5.3. (COM — 2005.6.2): The house has the shape of a square divided into nine identical square rooms. In each room, there is either a knight who always tells the truth or a liar who always lies. Each inhabitant of the house stated: «Among my neighbors, there are more knights than liars». It is known that among the residents of the house, there are both knights and liars. How many knights are among them? (Rooms with a common wall are considered neighbors.)

Problem 5.4. (COM — 2003.6.2): Each of the three friends always either tells the truth or always lies. They were asked the question: «Is there at least one liar among the other two?» The first friend answered, «No,» the second friend answered, «Yes.» What did the third friend answer?

Problem 5.5. (COM — 2014.7.2): In a row stand 2014 people, and one of them is named Arthur. Each of those standing in the row is either a knight, who always tells the truth, or a liar, who always lies. Each person except Arthur said, «There are exactly two liars between me and Arthur». How many liars are in this row if it is known that Arthur is a knight?



Problem 5.6. (COM — 2013.7.2): In the family of merry gnomes, there is a father, a mother, and a child. The names of the family members are Alpha, Beta, and Gamma. Two gnomes made two statements each during dinner. Gamma said, «Beta and Alpha are of different sexes. Beta and Alpha are my parents». Alpha said, «I am Gamma's father. I am Beta's daughter». Determine the names of each family member if it is known that each gnome spoke the truth once and joked once.

Problem 5.7. (MF — 2003.6.3): On the island, there are knights who always tell the truth and liars who always lie. A traveler met three islanders and asked each of them, «How many knights are among your companions?» The first replied, «None». The second said, «One». What did the third person say?

Problem 5.8. (MF — 2015.6.3): A mathematician with five children entered a pizzeria.

Alice: I want it with tomatoes and no sausage.

Beatrice: I'll have it without tomatoes.

Clarice: I want it with tomatoes. But no mushrooms!

Dorice: And I without mushrooms. But with sausage!

Fabrice: I want it with mushrooms.

Dad: Well, with such picky eaters, one pizza definitely won't be enough...

Can the mathematician order two pizzas and treat each child with the one they asked for, or will he have to order three pizzas?

Problem 5.9. (MF — 2011.6.3;7.3) Before the football match between the teams «North» and «South», five predictions were made:

a) There will be no draw;

b) South will score a goal;
c) North will win;
d) North will not lose;
e) There will be exactly 3 goals scored in the match.

After the match, it turned out that exactly three predictions were correct. What was the final score of the match?

Problem 5.10. (COM — 2013.6.3): Mrs. Owless opened a school, and on September 1, there were three lessons in each of the first three grades: Flying, Descending, and Fooling. The same subject cannot be taught in two classes at the same time. Flying was the first lesson in class 1B. The Fooling teacher praised the students in 1B: «You are doing even better than in 1A». Descending was not the second lesson in class 1A. In which class Fooling was the last lesson?

Problem 5.11. (COM — 2012.7.3): Four children said the following about each other:

Alice: Three solved the problem: Beatrice, Clarice, and Dorice.

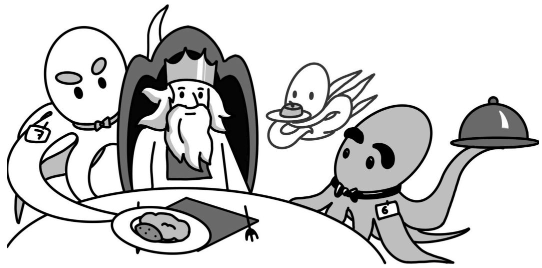
Beatrice: Three did not solve the problem: Alice, Clarice, and Dorice.

Clarice: Alice and Beatrice lied.

Dorice: Alice, Beatrice, and Clarice told the truth.

How many children actually told the truth?

Problem 5.12. (AT — 2015.3): Once upon a time, on the Island of Knights (who always tell the truth) and Knaves (who always lie), a traveler arrived. Stepping ashore, he met a procession of four islanders carrying 12 red and 4 blue balls (four each). Each of them made one statement. The first said, «I have fewer red balls than blue ones». The second said, «I have no fewer blue balls than red ones». The third said, «I have an equal number of blue and red balls». The fourth said, «I have no more than one red ball». Can you indicate how many knights could have been among them?



Problem 5.13. (MF — 2009.7.3): The underwater king has octopuses with six, seven, or eight legs serving him. Those with 7 legs always lie, while those with 6 or 8 legs always tell the truth. Four octopuses met. The blue one said, «Together we have 28 legs», the green one said, «Together we have 27 legs», the yellow one said, «Together we have 26 legs», and the red one said, «Together we have 25 legs». How many legs does each one have?

Problem 5.14. (MF — 2009.6.4): If an octopus has an even number of legs, it always tells the truth. If it has an odd number, it always lies. Once, a green octopus said to a dark blue one:

– I have 8 legs. And you only have 6.

– But I have 8 legs, said the dark blue one, offended. And you only have 7.

– The dark blue one really has 8 legs, supported the purple one and boasted: – And I have as many as 9!

– None of you has 8 legs, intervened the striped octopus in the conversation. Only I have 8 legs!

Which octopus had exactly 8 legs?

Problem 5.15. (MF — 1996.6.4): Three people, A, B, C counted a pile of balls of four colors. Each of them correctly distinguished two colors, and two others could be confused: one confused red and orange, another orange and yellow, and the third yellow and green. The results of their counts are given in the table. How many balls of each color were there, actually?

	red	orange	yellow	green
A	2	5	7	9
B	2	4	9	8
C	4	2	8	9

Problem 5.16. (COM — 2008.6.4): In the School of Sorcery, there are 13 students. Before the clairvoyance exam, the teacher seated them at a round table and asked them to guess who would receive the clairvoyant diploma. Each of them modestly kept silent about themselves and their two neighbors, but about all the others wrote: «None of these ten will receive it!» Of course, all those who passed the exam guessed correctly, and all the other students made a mistake. How many wizards received the diploma?

Problem 5.17. (MF — 1991.7.4): Know-It-All came to visit the twin brothers, Max and Leo, knowing that one of them never tells the truth, and asked one of them, «Are you Max?» «Yes,» the one replied. When Know-It-All asked the same question to the second one, he received an equally clear answer and immediately determined who was who. Who was called Max?

Problem 5.18. (MF — 1998.7.4): On the Island of Contrasts, both knights and liars live. Knights always tell the truth; liars always lie. Some residents claimed that there is an even number of knights on the island, while others claimed that there is an odd number of liars. Can the number of inhabitants on the island be odd?

Problem 5.19. (AT — 2013.4): King Pea has three sons: the eldest is Max, the middle one is Leo, and the youngest is Jean. The king wants to marry his eldest son to Princess Esther. It is known that two of King Pea's sons are knights (always tell the truth), and one is a liar (always lies), but few people know who is who. Princess Esther wants to find out whom she is being proposed to marry: a knight or a liar. Can she find out by asking Jean one question? (Jean can only answer «yes» or «no» to questions; he knows who among his brothers is a knight and who is a liar).

Problem 5.20. (AT — 2014.4): At a conference on mathematical physics, knights and liars (knights always tell the truth, liars always lie) gathered around a round table, and it is known that there are an equal number of liars among the physicists and mathematicians. Each participant at the conference was asked the question, «Who is your neighbor on the right: a physicist or a mathematician?» Summing up, the chairman noticed: «It's interesting that there are 34 of us here, with an equal number of physicists and mathematicians, yet each one claims that their neighbor on the right is a mathematician.» Determine whether the chairman was a knight or a liar.

Problem 5.21. (COM — 2016.7.4): Jean has 5 coins, and exactly one of them is fake. Only Esther knows which one is fake. Jean can choose three coins, give one of them to Esther, and in return, find out about the other two and whether there is a fake coin among them. Jean knows that Esther will tell the truth about a real coin and lie about a fake one. How can Jean determine the fake coin among all five, asking no more than three questions?

Problem 5.22. (COM — 2002.7.4): In Russia, there is a 5 point grade scheme in school, 5 showing great performance and 2 showing terrible performance. Max came to take a computer test. Six questions appeared on the screen, each of which had to be answered «yes» or «no». After answering all the questions, the computer calculates the number of correct answers and gives a grade: a «2» if there are no more than two correct answers, a «3» if there are three, a «4» if there are four, and a «5» if there are five or six.

Max did not know the answer to any of the questions. Nevertheless, based on previous experience, he knew the following: the first and last questions require opposite answers; there are no three consecutive questions with the same answer; there are no strictly alternating positive and negative answers; the sequence of answers to the first three questions is never exactly the same as the sequence of answers to the last three questions.

Help Max avoid getting a «2».

Problem 5.23. (MF — 2011.6.5): A dragon locked six gnomes in a cave and said, «I have seven hats of seven rainbow colors. Tomorrow morning, I will blindfold you and put a hat on each of you, and hide one hat. Then I will take off the blindfolds, and you will be able to see the hats on each other's heads, but I will not allow you to communicate. After that, each of you secretly tells me the color of the hidden hat. If at least three of you guess correctly, I will release you. If fewer do, I will have you for lunch.» How can the gnomes agree in advance on what to do to save themselves?

Problem 5.24. (AT — 2017.5): Baba Yaga tests Alice. On a checkered board of size 5×9, she marked an invisible 2×2 square with ink. Alice is allowed to choose several cells and ask Baba Yaga if there is at least one marked cell among them, to which Baba Yaga must answer truthfully: «yes» or «no». Can Alice find the marked square by asking no more than 5 questions?

Problem 5.25. (MF — 2005.6.6): In the Land of Emptiness, there live three tribes: elves, goblins, and halflings. An elf always tells the truth, a goblin always lies, and a halfling alternates between telling the truth and lying. Once, several inhabitants of the Land of Emptiness were feasting at a round table, and one of them pointed to his left neighbor, saying, «He is a halfling». The neighbor said, «My right neighbor lied». Exactly the same phrase was then repeated by his left neighbor, and so they continued to say «My right neighbor lied» many times in a row, and maybe they still do. Determine which tribes the feasters belonged to if it is known that a) there were nine; b) there were ten inhabitants of the Land of Emptiness. Explain your reasoning.

Problem 5.26. (MF — 2012.6.6): It is known that the Jackal always lies, the Lion tells the truth, the Parrot simply repeats the last answer heard (and if asked first, it will answer randomly), and the Giraffe gives an honest answer, but to the previous question asked to it (and to the first question, it answers randomly). Wise Hedgehog stumbled upon the Jackal, the Lion, the Parrot, and the Giraffe in the fog and decided to find out their order. By asking each one in turn, «Are you the Jackal?» he only understood where the Giraffe was. By asking them all in the same order, «Are you the Giraffe?» he was able to understand where the Jackal was, but complete clarity did not come. Only after the first one answered «yes» to the question «Are you the Parrot?» did the Hedgehog finally understand the order of the animals. So, in what order were they asked?

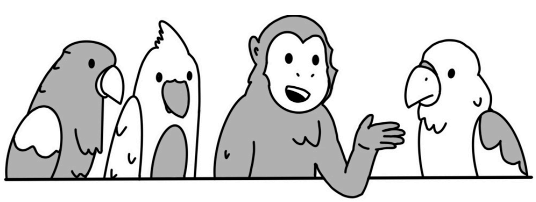
Problem 5.27. (COM — 2010.6.6): On the island of Rightland, all residents can make mistakes, but juniors never contradict seniors, and when seniors contradict juniors, they (seniors) are not wrong. The following conversation took place between residents A, B, and C:

A: B is the tallest.

B: A is the tallest.

C: I am taller than B.

Does it follow from this conversation that the younger a person is, the taller they are (for the three speakers)?



Problem 5.28. (AT — 2012.6): Around the round table, there are 38 parrots and a monkey. It is known that each of them is either always lying (called «liars») or always telling the truth (called «truth-tellers»). Monkey asked each parrot the same question: «Is your right neighbor a truth-teller or a liar?» (the survey went around the circle). The first two parrots (to the right of Monkey) answered: «my right neighbor is a liar». The next two: «my right neighbor is a truth-teller», the next two: «my

right neighbor is a liar», and so on. After the survey, Monkey said, «Among us, there are no fewer than 9 truth-tellers». How many truth-tellers were there, actually?

Problem 5.29. (AT — 2017.6): At the bus stop, where buses with numbers $164, 171, 258, 285, 365, 367, 377, 577$ stop, a teacher (who knows the number of the bus needed) and three of his students (who do not know it) arrived. The teacher suggested playing a game.

He secretly told each of them one of the digits of the bus number: he told Alice the first digit, Beatrice the second one, and Clarice the third one, and asked them to guess the number of the bus needed (the children know who told the first digit, the second one, and the third one). After that, a conversation took place between the children:

Alice: I don't know the number, but I understand that the others don't know it either.

Beatrice: I don't know the number, but now Clarice must know it.

Clarice: Yes, I know the number, and you two helped me to figure it out.

Indicate the number of the bus needed.

Problem 5.30. (COM — 2014.6.7): The Liar always lies, the Trickster tells the truth or lies as he wishes, and the Changer alternates between telling the truth and lying. A traveler met the Liar, the Trickster, and the Changer, who know each other. Can he determine who is who by asking them questions?

Problem 5.31. (COM — 2010.6.9): In a certain state, there are citizens of three types:

a) *the fool* considers everyone else fools and himself smart;
b) *the modest smart* knows everyone else correctly and considers himself a fool;
c) *the confident smart* knows everyone else correctly and considers himself smart.

In the parliament, there are 200 deputies. The Prime Minister conducted an anonymous survey of parliamentarians: How many smart people are currently in the room?

Based on the survey data, he could not determine the number of smart people. But then the only deputy who did not participate in the survey returned from the trip. He filled out a questionnaire about the entire parliament, including himself, and after reading it, the Prime Minister understood everything. How many smart people could be in the parliament (including the traveler)?

Problem 5.32. (MF — 2007.7.6): Jean is walking along the streets of the city at one of the intersections where a treasure is buried. At each intersection, he is informed via radio whether he is approaching or moving away from the treasure (compared to the previous intersection). The radio either always tells the truth or always lies (but Jean does not know if it lies or not). Can Jean accurately determine where the treasure is buried if the city plan looks like the figure below? (Intersections are marked with dots.)

a) b)

Problem 5.33. (Canadian mathematics competition) In downtown Gausseille, there are three buildings with different heights: The Euclid (E), The Newton (N) and The Galileo (G). Only one of the statements below is true.

1. The Newton is not the shortest.
2. The Euclid is the tallest.
3. The Galileo is not the tallest.

Ordered from the shortest to tallest in height, the buildings are ...

Problem 5.34. (mathcounts) Seven friends are riding the bus to school:

- Cha and Bai are on 2 different buses.
- Bai, Abu, and Don are on 3 different buses.
- Don, Gia, and Fan are on 3 different buses.
- Abu, Eva, and Bai are on 3 different buses.
- Gia and Eva are on 2 different buses.
- Fan, Cha, and Gia are on 3 different buses.
- Cha and Eva are on 2 different buses.

What is the least possible number of buses on which the friends could be riding?

Problem 5.35. (Junior Kangaroo) In Carl's pencil case, there are nine pencils. At least one of the pencils is blue. In any group of four pencils, at least two have the same color. In any group of five pencils, at most three have the same color. How many pencils are blue?

A 1 B 2 C 3 D 4 E More information needed

Problem 5.36. (Grey Kangaroo) Twelve color cubes are arranged in a row. There are 3 blue cubes, 2 yellow cubes, 3 red cubes, and 4 green cubes, but not in that order. There is a yellow cube at one end and a red cube at the other end. The red cubes are all together within the row. The green cubes are also all together within the row. The tenth cube from the left is blue. What color is the cube sixth from the left? A green B yellow C blue D red E red or blue

Problem 5.37. (Math Kangaroo, Canada, 2007, Grade 11 and 12) An island is inhabited by knights and liars. Each knight always tells the truth, and each liar always lies. Once, an islander A, when asked about himself and another islander B, claimed that at least one of A and B was a liar. Which of the following sentences is true?

A A is not able to make the above statement.

B Both are liars.

C Both are knights.

D A is a liar while B is a knight.

E B is a liar while A is a knight.

Problem 5.38. Pop Quiz, a popular internet puzzle.

1. The next question with the same answer as this one is:

   A 2 B 3 C 4 D 5
2. The first question with answer C is:

   A 1 B 2 C 3 D 4
3. The last question with answer A is:

   A 5 B 6 C 7 D 8
4. The number of questions with answer D is:

   A 1 B 2 C 3 D 4
5. The answer occurring the most is (if tied, first alphabetically):

   A A B B C C D D
6. The first question with the same answer as the question following it is:

   A 2 B 3 C 4 D 5
7. The answer occurring the least is (if tied, last alphabetically):

   A A B B C C D D
8. The highest possible score on this test is:

   A 5 B 6 C 7 D 8

Skill Assessment Problems

Skill Assessment Problem 5.1. In the train compartment, an astronomer, a poet, a prose writer, and a playwright were traveling. Their names were Alice, Beatrice, Clarice, and Dorice. Each of them brought a book written by one of the passengers in the compartment. Alice and Beatrice exchanged books and immersed themselves in reading. The poet read a play. The prose writer, a very young person who released his first book, said that they never read anything about astronomy. Beatrice took one of Dorice's works on the journey. None of the passengers brought or read books written by themselves. What did each of them read? Who was who?

Skill Assessment Problem 5.2. Beaver, Crocodile, and Macaque from the island of knights and liars met, and two of them said:

Beaver: «We are all liars».

Crocodile: «Exactly one of us is a knight».

Who among these three is the knight, and who is the liar?

Solutions to Skill Assessment Problems

Solution to Problem 5.1: First, it is worth determining which books people of which professions were reading. From the fact that the poet reads a play and the prose writer does not read books on astronomy, we obtain the following table:

	astr	poet	prose	play
astr	−	−	+	−
poet	−	−	−	+
prose	−	+	−	−
play	+	−	−	−

Did you notice that all the owners of professions lined up in a circle behind each other? Specifically, if they are arranged in the order of who reads whom, then the following picture emerges:

$$\text{astr} \to \text{prose} \to \text{poet} \to \text{playwright} \to \text{astr.}$$

Another clue can be noticed: Dorice wrote several works, and the prose writer only wrote the first book. If Beatrice bought one of Dorice's works for the journey and Alice and Beatrice exchanged books, then Alice reads Dorice.

Another conclusion can be made from the fact that Alice and Beatrice exchanged books. Since Alice did not have his own book, then after the exchange, Beatrice could not receive it, so Beatrice did not read Alice's book. So, who did they read? Alice already reads Dorice; Beatrice also does not read his own book; therefore, Beatrice reads Clarice.

We have a similar picture for surnames:

$$\text{Alice} \to \text{Dorice and Beatrice} \to \text{Clarice.}$$

Given that Dorice is not a prose writer, we get a table with all possible variants:

astr	prose	poet	playwright
D	B	C	A
C	A	D	B
B	C	A	D

Our conclusions can be expressed in the form of statements.

Solution 1:

- Alice (poet) reads Dorice (playwright).
- Clarice (prose writer) reads Alice (poet).
- Beatrice (astronomer) reads Clarice (prose writer).
- Dorice (playwright) reads Beatrice (astronomer).

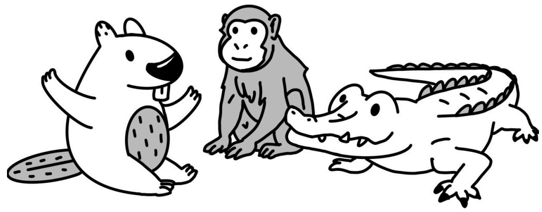
Solution 2:

- Clarice (poet) reads Alice (playwright).
- Beatrice (prose writer) reads Clarice (poet).
- Dorice (astronomer) reads Beatrice (prose writer).
- Alice (playwright) reads Dorice (astronomer).

Solution 3:

- Dorice (poet) reads Beatrice (playwright).
- Alice (prose writer) reads Dorice (poet).
- Clarice (astronomer) reads Alice (prose writer).
- Beatrice (playwright) reads Clarice (astronomer).

It is easy to verify that in all variants, the system of statements is consistent. □



Solution to Problem 5.2: Let's assume that Beaver is a knight; therefore, his statement is true, and he is a liar. Contradiction. So Beaver is a liar, and his statement is false. The statement «It is false that we are all liars» means «Among us there are knights», not «We are all knights».

Let's assume that Crocodile is a liar, then his statement is false, and the true statement is «The number of knights among us is not equal to one». Since from Beaver's statement it follows that there are knights among Beaver, Crocodile, and Macaque, and Beaver and Crocodile are liars, then there is exactly one knight, and it is Macaque. We again come to a contradiction since Crocodile, the liar, told the truth. Therefore, the Crocodile is the knight, and the Macaque is the liar. □

Proof by Contradiction

> “A mathematician organizes a raffle in which the prize is an infinite amount of money paid over an infinite amount of time. Of course, with the promise of such a prize, his tickets sell like hot cake. When the winning ticket is drawn, and the jubilant winner comes to claim his prize, the mathematician explains the mode of payment: «1 dollar now, 1/2 dollar next week, 1/3 dollar the week after that...»
>
> —One joke that didn’t quite land

Theory and Practice

The name of this method («Proof by Contradiction») essentially speaks for itself. It is a particular case of a method known as «reductio ad absurdum» (Latin), or «reduction to absurdity».

If we need to prove a certain statement A, we will assume that it is false, meaning the negation of A is true. After that, we somehow need to arrive at a contradiction, which would mean that our assumption was incorrect, and the problem is solved.

This method of proof is based on the truth of the law of double negation in classical logic. The law of double negation is a principle underlying classical logic, according to which «if it is not the case that not A, then A is true». The law of double negation is also called the law of double negation elimination.

We often encounter reasoning based on proof by contradiction in real life. Suppose, at some point, you didn't want to go to school and told your parents at home that your throat hurts. Your parents look at your throat and say, «If your throat really hurt, it would be red. It's not red, so it doesn't hurt».

This principle can be very useful in solving mathematical problems because it allows us to establish the truth of a statement by demonstrating that its negation leads to a contradiction. Here are a few examples of how it can be applied:

One classic example involves a cruel king who wanted to execute his minister without upsetting the kingdom's people. He announced that he, being a kind king, would forgive the minister or trust in their ancient gods. He wrote «death» and «life» on two identical-looking pieces of paper, and the minister would choose one. Depending on what was written on it, his fate would be decided. The minister, being wise, realized that both papers said «death», yet he managed to survive by swallowing the chosen paper. To determine his fate, the people looked at the remaining paper, which read «death». This proved that he was fortunate to have picked the paper with «life» written on it.

Example 6.1. Can 44 beavers in a kindergarten be divided into 9 groups so that the number of beavers in different groups is different?

Solution: Trying to come up with an example of such a division, we quickly reach a dead end. Thus, we conclude that it is probably impossible to divide the beavers into groups in this way. Let's assume the opposite, that such a division exists. Let's sort our groups by the number of beavers in them in ascending order (since there are no two groups with the same number of beavers). In the group with the smallest number of beavers, there are at least 1 of them, in the second smallest — at least 2, and so on. That is, in total, we must send at least $1 + 2 + \ldots + 9 = 45$ beavers to the kindergarten, which is more than 44. This contradiction completes the solution to the problem. □

Example 6.2. Integer points on a line are colored red and blue. Prove that there is a segment with both ends and the midpoint colored in the same color.

Solution: Let's assume the opposite, i.e., let's assume that such a segment does not exist. Consider all points with even coordinates, except 0. Either among them, there is a point of the same color as 0, or they are all of a different color than 0. In the second case, we immediately come to a contradiction by considering the points 2, 4, and 6. So, we have points of the same color as 0 and $2x$, where $x \in \mathbb{Z}$. Then $-2x$, x, and $4x$ must be of a different color. But x is the midpoint of the segment with ends $-2x$ and $4x$, leading to a contradiction. □

Example 6.3. There are 25 boys and 25 girls sitting around a round table. Prove that someone sitting at the table has both neighbors as boys.

Solution: Let's assume the opposite, i.e., let's assume that each person at the table has neighbors who are either of the opposite gender or girls, i.e., each person is seated next to at least 1 girl. Then, at most 2 boys can sit next to each other, and at least one girl must sit next to each girl. Thus, if we consider groups of boys and girls sitting next to each other around the table, we get that in each group, there are at most 2 boys and at least 2 girls. Since, due to the roundness of the table, the number of groups of boys and girls is the same, and the number of boys and girls at the table is the same, we conclude that in each group of boys and girls, there are exactly 2 people, but 25 is not divisible by 2. This contradiction completes the solution to the problem. □

Example 6.4. Among any ten of sixty animals, there are at least three of the same species. Prove that among all of them, there are 15 animals of the same species.

Solution: Let's assume the opposite, i.e., let's assume that the first part of the condition is met even if there are no more than 14 representatives of each species. Let's call animals that represent their species alone «singles», in contrast to «groups» where there will be at least 2 animals. Since the first part of the condition must be met, there can be no more than 4 «groups», otherwise we could select a group of 10 animals with 2 of each species. On the other hand, if there are no more than 3 groups, then there will be at least $60 - 3 \cdot 14 = 18$ singles, and by choosing 10 singles, we immediately arrive at a contradiction. Thus, there are exactly 4 groups, in which case there are at least $60 - 4 \cdot 14 = 4$ singles, and by choosing 2 animals from each group and 2 singles, we arrive at a contradiction. Contradictions obtained in all possible distributions of animals complete the solution to the problem. □

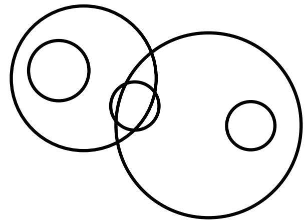
Problem Set

Problem 6.1. (COM — 2012.6.8): Can 100 weights with masses $1, 2, 3, \ldots, 99, 100$ be divided into 10 piles of different masses in such a way that the heavier the pile, the fewer weights it contains?

Problem 6.2. (COM — 2005.6.8): At the World Congress of Wise Men, stargazers sit in a row opposite to alchemists behind a long table, and at the head of the table sits the Most Honorable Sage. On the first day of the congress, it turned out that opposite each alchemist sat a stargazer with a longer beard than his. On the second day, the alchemists agreed to sit at the table in order of increasing beard length from the end of the table to the Most Honorable Sage. But the stargazers also agreed among themselves to sit in order of increasing beard length from the end of the table to the Most Honorable Sage. Prove that on the second day, opposite each alchemist will sit a stargazer with a longer beard than his.

Problem 6.3. (MMO — 1977.10.4): Every point of the number line, whose coordinate is an integer, is colored either red or blue. Prove that there is a color with the following property: for each natural number k, there are infinitely many points of this color whose coordinates are divisible by k.

Problem 6.4. (LT — 2014.6.3): A forester was counting pine trees in the forest. He walked around 5 circles, as shown in the figure, and counted exactly 3 pine trees inside each circle. Is it possible that the forester never made a mistake?



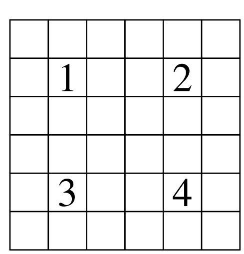
Problem 6.5. (Michigan) The plane is painted in two colors. Show that there is an isosceles right triangle with all vertices of the same color.

Problem 6.6. (Pink Kangaroo) There are some squares and triangles on the table. Some of them are blue, and the rest are red. Some of these shapes are large, and the rest are small. We know that
1. If the shape is large, it's a square;
2. If the shape is blue, it's a triangle.

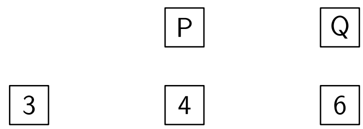
Which of the statements A-E must be true?

(A) All red figures are squares.
(B) All squares are large.
(C) All small figures are blue.
(D) All triangles are blue.
(E) All blue figures are small.

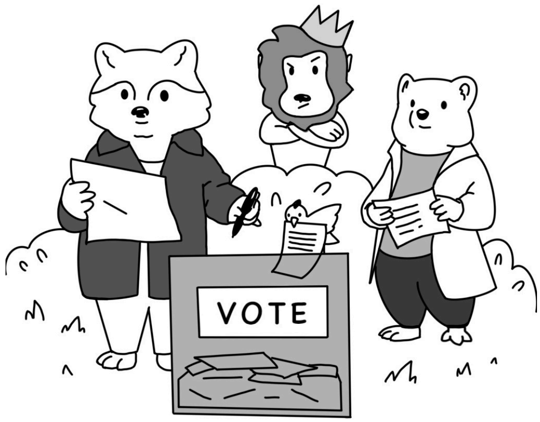
Problem 6.7. (JMO) The integers 1 to 4 are positioned in a 6 by 6 square grid as shown and cannot be moved. The integers 5 to 36 are in the 32 empty squares. Prove that no matter how this is done, the integers in some pair of adjacent squares (i.e., squares sharing an edge) must differ by at least 16.



Problem 6.8. (AJHSME) Five cards are lying on the table, as shown. Each card has a letter on one side or a whole number on the other side. Jane said, «If a vowel is on one side of any card, then an even number is on the other side.» Mary showed Jane was wrong by turning over one card. Which card did Mary turn over? (Each card number is the one with the number on it. For example, card 4 is the one with 4 on it, not the fourth card from the left/right).



Skill Assessment Problems



Skill Assessment Problem 6.1. In the forest, the animals decided to overthrow the king lion and staged a revolution. In the presidential elections of the forest, each of the voting animals puts on the ballot the names of the 10 most worthy candidates in their view. There are 11 ballot boxes for voting on the main clearing of the forest. After the election day, it turned out that each ballot box contains at least one ballot, and for any choice of 11 arbitrary ballots, one from each box, there is a candidate whose name appears on each of the selected ballots. Prove that at least one ballot box contains only the name of the same candidate.

Skill Assessment Problem 6.2. The nodes of a square grid are colored in two colors. Prove that there exists a right-angled triangle with vertices of the same color.

Skill Assessment Problem 6.3. Ten friends sent each other holiday postcards, so that each sent 5 postcards. Prove that there are two friends who sent postcards to each other.

Solutions to Skill Assessment Problems

Solution to Problem 6.1: Let's assume the opposite. If the condition of the problem is not fulfilled, then for each candidate in each urn, there will be at least 1 ballot where he is not listed. Let's take an arbitrary ballot from the first urn. It contains the names of 10 presidential candidates. Then, in the second urn, there will be a ballot where the name of the first candidate listed in this list is not written; in the third urn — a ballot where the name of the second candidate is not listed, and so on, up to the eleventh urn, where the name of the tenth candidate is not listed. Therefore, if we take all the mentioned ballots together with the one chosen from the first urn, the candidate whose name appears in each of them will not be found. This contradiction completes the solution of the problem. □

Solution to Problem 6.2: Let's assume the opposite — suppose such a triangle does not exist. Let's find two nodes of the same color in one row. This can be done.

Consider an arbitrary node — let it be of color A. Let's try to find a node of the same color in the same row with it. If this cannot be done, then all other nodes in this row are of color B, and in this case, we can take two nodes of the second color.

Let's try to find a node of the same color in the same column with one of the found monochromatic nodes. If this can be done, the problem is solved.

Otherwise, we have two columns, each of which, except for one node, is colored in the same color. This also leads to the existence of the desired triangle. This contradiction completes the solution of the problem. □

Solution to Problem 6.3: Let's assume the opposite, i.e., that such people do not exist. This would mean that in any pair of people, only one card was sent or none at all. There are a total of

$$10 \cdot 9 \cdot \frac{1}{2} = 45$$

pairs, which means that no more than 45 cards were sent. However, there were exactly $10 \cdot 5 = 50$ cards sent in total — a contradiction. □

Pigeonhole Principle

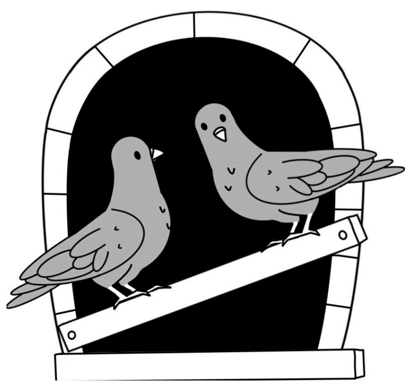
> A mathematician is asked to solve a problem: «Given a gas stove, a water tap, and a kettle. The task is to boil water.»
> This is easy, — he replies. — First, we pour water into the kettle. Then we light the fire and place the kettle on the stove.
> Okay, now a new task, — they say to him. — The task is to boil the kettle with water already poured into it.
> Well, this is even easier! We pour the water out of the kettle and reduce the task to the previous one.
>
> —One joke that didn't quite land

Theory and Practice

Although the pigeonhole principle appears as early as 1624 in a book attributed to Jean Leurechon, it is commonly referred to as Dirichlet's box principle or Dirichlet's drawer principle, or just Dirichlet's principle.

The solutions to problems using the pigeonhole principle usually rely on the method of proof by contradiction.

Typically, students engaged in mathematical olympiads encounter pigeonhole principles in the early years of secondary school. It can be stated as follows: «If there are N pigeonholes and at least $N+1$ pigeons, then there must be at least two pigeons in one of the pigeonholes». Let's use the method of proof by contradiction — suppose «this is not the case», meaning there are **less than** two pigeons in each pigeonhole, i.e., 1 or 0 pigeons. Then, in N pigeonholes, the maximum number of pigeons would be $N \cdot 1 = N$, which is less than $N + 1$.



A natural generalization is the following statement: «If there are N pigeonholes and at least $kN + 1$ pigeons, then there must be at least $k + 1$ pigeons in one of the pigeonhole.»

In reality, you are unlikely to encounter a problem where you actually have to place pigeons in pigeonholes. In each specific problem, you need to understand what plays the role of pigeons and what plays the role of pigeonholes.

Example 7.1. Given 6 integers. Prove that among them, we can select two whose difference is divisible by 5.

Solution: The numbers given in the problem hint at its solution using the pigeonhole principle. The number of «pigeons» should be one more than the number of «pigeonholes», and 6 is one more than 5. Therefore, the «pigeons» are the numbers themselves. We need to understand how the «pigeonholes» are chosen. There should be 5 «pigeonholes», and the problem mentions divisibility by 5. There are exactly 5 possible remainders when dividing by 5: 0, 1, 2, 3, 4. Thus, we assign each «pigeon» number to a «pigeonhole» corresponding to its remainder when divided by 5. By the pigeonhole principle, some two «pigeons» are in the same «pigeonhole», which means that some two numbers have the same remainder when divided by 5, implying that their difference is divisible by 5. □

Example 7.2. The numbers from 1 to 9 are divided into three groups. Prove that at least one of the groups has a product of numbers not less than 72.

Solution: Let's use proof by contradiction. Suppose what we need to prove is false. This means that in each group, the product will be less than 72, which is equivalent to being less than or equal to 71. However, 71 is a prime number, so it cannot be the product of the specified numbers. Hence, the product of all the numbers is less than or equal to 70^3. On the other hand, the product of all numbers is equal to $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 = (3 \cdot 4 \cdot 6) \cdot (2 \cdot 5 \cdot 7) \cdot (8 \cdot 9) = 72 \cdot 70 \cdot 72$, which is greater than 70^3. This contradiction completes the proof. □

Example 7.3. Prove that among any six people, there will either be three pairwise acquaintances or three pairwise strangers.

Solution: In this case, «pigeons» are people. «Pigeonholes» represent acquaintances or strangers. Let's represent 6 people as points on a plane and connect two points with a red segment if the corresponding people are acquainted, otherwise with blue. The task is to prove the existence of at least one «pigeonhole» with three «pigeons».

Consider an arbitrary person. They must be connected by a segment of one color to each of the remaining 5 people. By the Pigeonhole Principle, with at least three connections, there will be a «pigeonhole» with three «pigeons» (let's say red). Then, if any two people within this group are also connected by a red segment, we will see a red trio. Otherwise, these three individuals form a blue trio. □

Example 7.4. Prove that in any company, there will be two people with the same number of acquaintances from within that company.

Solution: Let's find «pigeons» and «pigeonholes» again. It's not hard to guess that people will be «pigeons» and the number of their acquaintances will be «pigeonholes». Each person can have from 0 to $n-1$ acquaintances, where n is the number of people in the company. So, the pigeonholes will have numbers from 0 to $n-1$. It seems like there are equal numbers of «pigeons» and «pigeonholes», so why doesn't the Pigeonhole Principle work here? It does, but with a slight modification. Notice that in pigeonholes 0 and $n-1$, pigeons cannot sit simultaneously. Indeed, then there would be a person who is not acquainted with anyone and a person acquainted with everyone, which is impossible. Therefore, we actually have $n-1$ non-empty pigeonholes. That is, the pigeonhole principle still works in this problem.

This contradiction completes the solution of the problem. □

Problem Set

Problem 7.1. (COM — 2017.7.2): Four girls from 7th grade and four girls from 8th grade came to the club: three named Alice, three named Beatrice, and two named Clarice. Could it be that each of them has at least one classmate with the same name who came to the club?

Problem 7.2. (COM — 2011.7.3): While tidying up the children's room before the guests arrived, Mom found 9 socks. Among any four socks, at least two belonged to the same child, and among any five socks, no more than three had the same owner. How many children could there be, and how many socks could belong to each child?

Problem 7.3. (MF — 1994.6.7): Among any ten out of sixty schoolchildren, there will be three classmates. Is it necessarily true that among all sixty schoolchildren, there are: a) 15; b) 16 classmates?

Problem 7.4. (MF — 1994.7.6): In one of the schools, a club on astronomy was held 20 times. At each session, exactly five schoolchildren were present, and no two schoolchildren met at the club more than once. Prove that at least 20 schoolchildren attended the club in total.

Problem 7.5. (TOT — 1988/1989.7.3): Prove that from any seven natural numbers (not necessarily consecutive), one can select three numbers whose sum is divisible by three.

Problem 7.6. (Mos2ARSO — 2009.7.3): There are 25 students in the class. It is known that for any two girls in the class, the number of male friends from this class does not coincide. What is the maximum number of girls that can be in this class?

Problem 7.7. (LT — 2007.7.2): Max has several coins in his pocket. If Max randomly pulls out 3 coins from his pocket, among them, there will definitely be a coin worth «1 ruble». If Max randomly pulls out 4 coins from his pocket, among them, there will definitely be a coin worth «2 rubles». Max pulled out 5 coins from his pocket. Name these coins.

Problem 7.8. (LT — 1985.7.6): Given 25 numbers. The sum of any four of them is positive. Prove that the sum of all of them is also positive.

Problem 7.9. (LT — 1985.7.8): There are 25 people in the class. It is known that among any three of them, there are two friends. Prove that there is a student with at least 12 friends.

Problem 7.10. (Belarus — 1966.8.5): Thirty teams participate in a football championship. Prove that at any moment of the competition, there are two teams that have played the same number of matches by that moment.

Problem 7.11. (Nederlandse Wiskunde Olympiade) In a box, there are 100 cards that are numbered from 1 to 100. The numbers are written on the cards. While being blindfolded, Lisa is going to draw one or more cards from the box. After that, she will multiply the numbers on these cards. Lisa wants the outcome of the multiplication to be divisible by 6. How many cards does she need to draw to make sure that this will happen?

Problem 7.12. (Formula) There is a pile of identical cards; each card contains numbers from 1 to 12. Bill took one card and secretly marked 4 numbers on it. Mark can do the same operation with some other cards. After that, the boys show their cards to each other. Mark wins if he has a card where at least two marked numbers coincide with Bill's numbers. Find the smallest number of cards Mark should use to win the game and find the way to fill them.

Problem 7.13. (mathcounts) How many people must be in the room to guarantee that at least two of them have the same birthday or at least one of them was born in February?

Skill Assessment Problems

Skill Assessment Problem 7.1. Prove that it is impossible to completely cover an equilateral triangle with two smaller equilateral triangles.

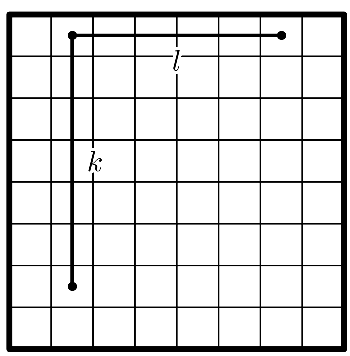
Skill Assessment Problem 7.2. In an 8×8 grid, integer numbers are placed such that any two numbers in adjacent cells differ by no more than 4. Prove that among these numbers, there are at least 2 equal ones.

Skill Assessment Problem 7.3. On a chessboard, more than a quarter of the squares are occupied by chess pieces. Prove that at least two neighboring squares (either side-by-side or corner-to-corner) are occupied.

Solutions to Skill Assessment Problems

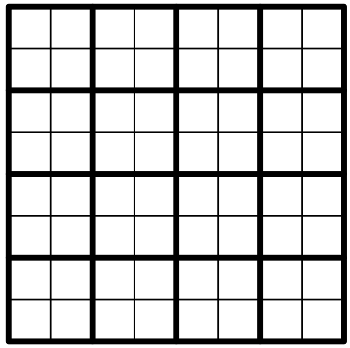
Solution to Problem 7.1: Notice that the largest segment that can be covered by a regular triangle is the length of its side. Consider 3 vertices-pigeons of the original regular triangle and 2 smaller regular triangles-pigeonholes. Then, by the pigeonhole principle, some two vertices must be covered by the same triangle, which is impossible, as required to be proven. □

Solution to Problem 7.2: Suppose the opposite. Then, in the grid, there are no two equal numbers — all numbers are different. Let's consider the smallest and largest of them — they will differ by at least 63. But one can reach from one of them to the other through adjacent cells in at most $k + l \leq 7 + 7 = 14$ moves (see the figure below).



Since the difference between adjacent numbers is no more than 4, making ≤ 14 moves, the difference will be no more than $4 \times 14 = 56$. This contradiction completes the proof of the problem. □

Solution to Problem 7.3: Let us divide the board into 2×2 squares (as shown in the figure below).



Since there are a total of 64 squares on the chessboard and more than a quarter of them are occupied, there are at least 17 occupied squares. Since there are only 16 squares of size 2×2, and there are 17 pieces, by the pigeonhole principle, in some squares, there are at least two pieces, and therefore they are adjacent either side-by-side or corner-to-corner. $\square$

Games in Mathematics

> «Students nowadays are so clueless», the math professor complains to a colleague.
> «Yesterday, a student came to my office hours and wanted to know if General Calculus was a Roman war hero…»
>
> —One joke that didn't quite land

Theory and Practice

It is usually quite simple to understand that the problem is about some game; it is explicitly stated in the problem statement. Typically, there are two players in the game.

By a move, one understands certain actions allowed by the rules of the game, and, as a rule, the execution of a move is mandatory. Moves are made in turns. With very rare exceptions, at some point, the game ends — after a certain, pre-known number of moves or due to some position reached in the game. After that, the result of the game is determined: victory for one of the players or a draw. However, draws in games that may appear in mathematical Olympiad problems are not so common. The question in such problems is usually formulated as follows: what will be the result if both sides play correctly? It is said that player A wins with optimal play by both sides, or wins forcibly, if there is a strategy that allows him to make moves in such a way that the result of the game will be a win for player A in any case. It is easy to understand that, in this case, there is no strategy for player B to guarantee that he will not lose. It is said that with optimal play by both sides, the game ends in a draw if each player has a strategy that results in a draw or his own victory in any case.

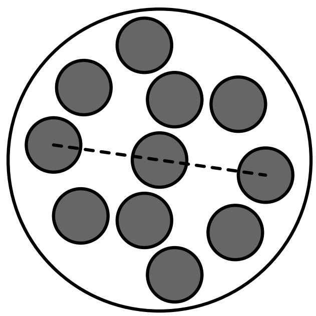
During the game or when trying to formalize the solution, the concept of a «best» move is often applied. But there is a problem: in most games, it is almost impossible to clearly define what this «best» move means without analyzing the position to the very end. For example, in chess, there is the concept of a «sacrifice of a piece», which at first glance may seem like a «bad» move because the material is lost. But in perspective, this move can turn out to be quite «good» as it will lead to a position with a higher evaluation in a few moves.

Thus, after making a move, it is often impossible to immediately determine whether it is the «best» one. Therefore, this concept can only be applied in games where we can calculate the opponent's moves to the very end, and in all other cases, one should actively avoid this slippery concept.

The most common strategy in competitions is the **symmetric strategy** and its variations. In this case, the player with the winning strategy adheres to some symmetry: for each move of the opponent, he responds with a symmetric move.

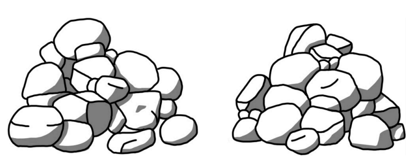
Example 8.1. Suppose there is a round table and an unlimited number of identical round coins. Two players take turns putting coins on the table, and they cannot place them on top of each other. The player who has no place to put a coin loses. Who wins with optimal play?

Solution: Let's present a winning strategy for the first player, i.e., we will guide their moves. As the first move, we place a coin in the very center of the table. On any move of the second player, we will respond with a move that is centrally symmetric, i.e., symmetric with respect to the center of the table (shown in the figure below).



Let's prove that such a strategy will work: it is sufficient to show that the first player can always make a move. Due to the strategy, after the first player's move, the arrangement of coins on the table will always remain centrally symmetric. This means that if the second player puts a coin somewhere on the table, a centrally symmetric place remains free — and it is exactly where we will place the coin. □

Let's give an example of another problem where symmetry is less pronounced.



Example 8.2. Suppose there are two piles, each containing 20 stones. On each turn, a player can take any number of stones, but only from one pile. The player who

cannot make a move loses. Who wins with optimal play?

Solution: Let's prove that the second player has a winning strategy, i.e., we will guide their moves. On any move of the first player, we will take the same number of stones from the other pile. Then, after any move of the second player, the number of stones in both piles will be equal. This means that after any move of the first player, the number of stones in the piles will be different, and the second player will be able to make their move.

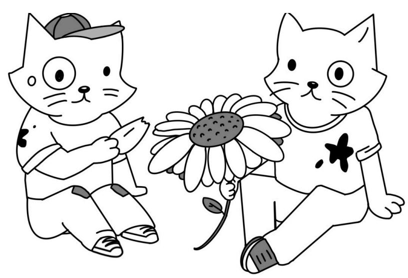
It is easy to see the solution to this problem when the number of stones in the piles is unequal — in this case, the first player wins, who by their first move equalizes the number of stones, reducing the problem to the previous one.

As you can notice, the symmetry in the above problem is no longer geometric; the symmetry was in the equality of the piles. □

When speaking about each player making the «best» move or following the «correct» strategy, it's worth noting that in the latter case, the only correct move for the first player is indeed equalizing the number of stones in the piles, while all other moves would be incorrect. Of course, one might argue that «what if the other player makes a mistake, I can still win even by playing incorrectly», and even provide concrete examples like the 2006 World Chess Championship match between Kramnik and Topalov, where both players made a mistake by missing mate. However, this would not be relevant to mathematics — in the mathematical model describing the game, there is no room for error and the emotional factors inherent in chess.

Problem Set

Problem 8.1. (COM — 2005.7.2): Jean and Esther are playing the game «Bulls and Cows». Jean has chosen a four-digit number with distinct digits, and Esther tries to guess this number. To do this, Esther proposes her own four-digit numbers (also with distinct digits), and Jean tells her for each of them how many «bulls» (i.e., digits that not only are present in Esther's number and in Jean's number but also are in the same positions) and «cows» (digits that are present in both numbers but are in different positions) there are. Esther proposed the numbers 5860, 1674, 9432, and 3017, and for each number received the answer «2 cows». What number did Jean choose?



Problem 8.2. (AT — 2014.3): Max and Leo play the following game: they take turns plucking petals from a daisy with 64 petals. On each turn, it is allowed to pick any odd number of petals less than 16, and it is prohibited to repeat previously made moves. (For example, if Leo picks 3 petals on his turn, then neither Max nor Leo is allowed to pick 3 petals in the future.) The player who plucks the last petal wins. Max starts. Who will win, no matter how the opponent plays?

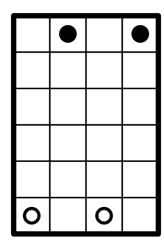
Problem 8.3. (COM — 2003.6.4): Jean and Esther are playing «Battleship» on an 8×8 grid according to the following rules. Esther places 16 single-cell ships so that they do not touch (not even by corners). With each turn, Jean names one cell on the grid, and if there is a ship on this cell, the ship is considered destroyed. Prove that regardless of the arrangement of the ships, Jean will be able to destroy at least one ship in 4 turns.

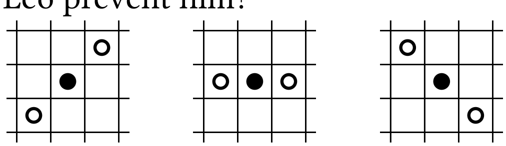
Problem 8.4. (*Symmetric Strategy*) Who wins in the following situations when playing correctly?

1. There are two stacks of coins on the table: one stack has 30 coins, and the other has 20. On each turn, it is allowed to take any number of coins from one stack. The player who cannot make a move loses.
2. A number 1 is written on the board. Two players take turns adding any number from 1 to 5 to the number on the board and replacing it with the sum. The player who first writes the number 30 on the board wins.
3. Two players take turns placing rooks on a chessboard on squares not already occupied by rooks. The player who, on their turn, has all the squares of the board under attack by placed rooks wins.

Problem 8.5. (COM — 2017.6.7): Two pirates, Alice and Beatrice, each having 74 gold coins, decided to play the following game: they will take turns laying coins on the table, putting one, two, or three coins at a time, and the winner is the one who places the hundredth coin on the table. Alice starts. Who can win such a game, regardless of the opponent's actions?

Problem 8.6. (MF — 1994.7.5): On a 4×6 grid, there are two black chips (Max's) and two white chips (Leo's, see the figure). Max and Leo take turns moving any of their chips one cell forward (vertically). Max starts. If, after a move, any of the guys' black chips ends up between two white chips horizontally or diagonally (as shown in the lower figures), it is considered «killed» and is removed from the board. Max wants to move both of his chips from the top row of the board to the bottom row. Can Leo prevent him?





Problem 8.7. (AT — 2013.6): Max and Leo play a game: they color cells of a 4×4 square board. Max makes his turn first, then Leo, then Max again, and so on, until a 2×2 square is completely colored. The player who colored the last cell in such a square loses. Which of the boys can always win?

Problem 8.8. (MF — 2014.7.6): Two numbers are written on the board: 2014 and 2015. Jean and Esther take turns, starting with Jean. On each turn, one can either:

- Decrease one of the numbers by its non-zero digit or by the non-zero digit of the other number;
- Divide one of the numbers in half if it is even.

The player who first writes a single-digit number wins. Who can win, no matter how the opponent plays?

Problem 8.9. (AT — 2015.6): Alice and Beatrice found a wallet with 12 coins of denominations $1, 2, 3, 4, \ldots, 12$ beecoins. They decided to divide the found money according to the following rules.

1. Beatrice takes out two coins from the wallet (whichever he wishes) and shows them to Alice.
2. Alice decides how many and which coins to give to Beatrice (one, two, or none). All coins not given to Beatrice are returned to the wallet.

If the sum in the wallet is not divisible by 3, then the division ends, and Alice takes all the remaining coins from the wallet. If the sum is divisible by 3, then the process is repeated.

a) Can Alice act in such a way as to definitely get more money than Beatrice?

b) What is the largest sum she can expect to get regardless of Beatrice's moves?

Problem 8.10. (COM — 2014.7.6): Max and Leo play on a 7×7 board. They take turns placing numbers from 1 to 7 in the cells of the board so that no row or column contains the same digit twice. Max takes his turn first. The player who cannot make a move loses. Who can win, no matter how the opponent plays?

Problem 8.11. (COM — 2009.7.6): A cannibal is a fantastic chess piece that can move like a chess king — to an adjacent cell vertically or horizontally but cannot move diagonally. Two cannibals stand on opposite corner squares of a chessboard and take turns moving. A cannibal who moves to a square where another cannibal is already standing is allowed to have his awful meal. Who eats whom in the correct game, and how should they play to achieve that?

Problem 8.12. (COM — 2003.7.6): Two players are playing tic-tac-toe on an infinite sheet of grid paper. The player who first manages to place five identical symbols in a row (horizontally or vertically) wins. Prove that the second player can play in such a way as not to lose.

Problem 8.13. (COM — 2016.7.7): Alice laid out 2016 matches on the table and suggested to Max and Leo to play a game, taking matches from the table one by one: Max can take either 5 matches or 26 matches in his turn, while Leo can take either 9 or 23 matches. Before the game started, Alice left, and when he returned, the game was already over. There were two matches left on the table, and the one who could not make the next move lost. After some thought, Alice understood who had played first and who had won. Find it out yourself!

Problem 8.14. (COM — 2010.7.7): Postman Jean did not want to deliver the package. Then Esther suggested playing the following game: with each move, Jean writes letters from left to right, alternating **M** and **P** letters as he wishes, until there are exactly 11 letters in the row. After each of his moves, Esther, if she wants, swaps any two letters. If, in the end, the written «word» is a palindrome (i.e., it reads the same forwards and backward), then Jean delivers the package. Can Esther play in such a way as to definitely get the package?

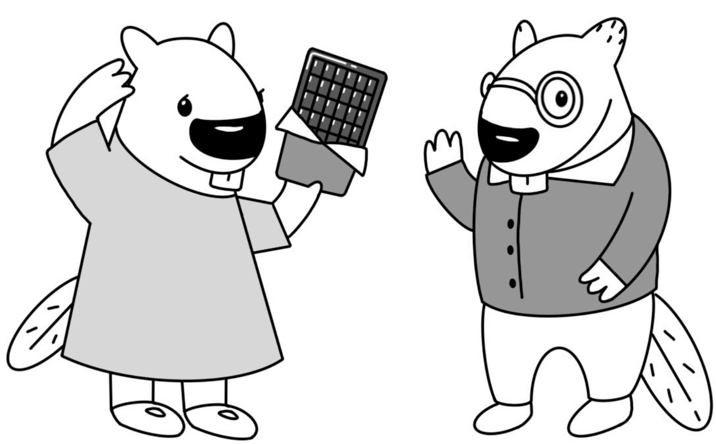
Problem 8.15. (Formula) Two people are playing a game with the following rules. In turn, they name an eight-digit number, containing no zeros and whose sum of digits is divisible by 9. Also, every next number must begin from the last digit of the previous number. It is forbidden to repeat numbers. The player who cannot name a number loses. Who can win no matter how the other person plays, the person who starts, or his rival?

Skill Assessment Problems

Skill Assessment Problem 8.1. On an 8×8 board, two players take turns coloring cells so that no three-cell corners appear. The player who cannot make a move loses. Who wins with the correct game?

Skill Assessment Problem 8.2. (MechMathCircle — 2012/2013.Games.5) Two players are playing the following game. Each player, in turn, crosses out 9 numbers (of their choice) from the sequence $1, 2, \dots, 100, 101$. After eleven such crosses, there will be 2 numbers left. The first player is awarded as many points as the difference between these remaining numbers. Prove that the first player can always score at least 55 points, no matter how the second player plays.

Skill Assessment Problem 8.3. Given a chocolate bar of size 6 by 12 pieces. Beavers Jean and Esther decided to play the following game: you can take any rectangular piece and break it along the line, dividing the pieces into two smaller rectangular pieces. The resulting pieces cannot be eaten! The player who cannot make a move loses, and the winner of the game receives the entire chocolate bar. Beavers really love 99-percent chocolate and really want to win. Who wins with the correct game if Esther concedes the right of the first move to Jean?



Solutions to Skill Assessment Problems

Solution to Problem 8.1: Let's use a symmetric strategy. We will play for the second player and make moves symmetrically about the center of the board. Then, for every move made by the first player, we will have a response. Indeed, if this is not the case, meaning that a corner of three cells appears, then due to symmetry, this corner must have existed before our move, which cannot be. This means that with the correct game, the second player wins. □

Solution to Problem 8.2: The first player, with their first move, crosses out the 9 central numbers from 47 to 55. Then, the strategy for the first player is as follows: when the second player chooses number a among the others, the first player, with their next move, crosses out number b such that $|a-b|=55$.

Alternatively, after the first move, all the remaining numbers are paired as follows:

$$(1;56),(2;57),\ldots,(a;a+55),\ldots,(46;101).$$

The second player, with their move, crosses out any 9 numbers, and the first player, with their next move, «cleans up» after them, ensuring that no numbers are left without a pair. Thus, in the end, there will always be a pair of numbers with a difference of 55. □

Solution to Problem 8.3: We will provide a winning strategy for Jean. With the first move, Jean breaks the chocolate bar into two pieces of size 6×6. For each of Esther's moves with one of the parts, Jean makes the same move with the other part using a symmetric strategy. Since Jean always has a response to any of Esther's moves, he wins.

Note that in this problem, it does not matter what strategy the beavers use: the game will end in favor of Jean in any case because with each move, the total number of pieces increases by one. Initially, there was 1 piece, and in the end, there will be 72. This means that there will be exactly 71 moves made throughout the game, and Jean will make the last move and win. □

Extreme Principle

> Definition from the dictionary for mathematicians: Recursion (noun) — see recursion.
>
> —One joke that didn't quite land

Theory and Practice

The title of the «extreme» principle speaks for itself — to solve the problem, you need to find and consider some «extreme» object. What does «extreme» mean? This term can be a little confusing, especially if it's a problem involving numbers rather than a geometric problem. «Extreme» usually means the largest or smallest number that expresses some property present in the problem. For example, it could be the greatest distance between points if a set of points is given, the smallest of the triangle areas, the smallest angle, and so on.

Example 9.1. There are 11 weights weighing $1, 2, \ldots, 11$ grams. Five of them are bronze, five are silver, and one is gold. All the bronze ones together weigh less than all the silver ones by 30 grams. How much does the gold weigh?

Solution: The difference in weight between the five heaviest weights ($11+10+9+8+7=45$) and the five lightest weights ($1+2+3+4+5=15$) is exactly 30 grams, which means the difference of 30 grams can be achieved in only one way, and therefore, the gold weight is the remaining one, weighing six grams. □

Example 9.2. Prove that for any natural number n, the expression

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$$

is not an integer.

Solution: This problem seems to be a problem in number theory, which is true. It can be attributed to the subsection «extreme principle in number theory». Let's choose the largest power of two among the numbers from 1 to n, and let it be 2^k. Then, among the remaining numbers, there is no longer any divisible by 2^k — the next number divisible by 2^k is 2^{k+1}, and it is already greater than n, since we considered the largest power of two. Therefore, all the remaining numbers from 1 to n are divisible at most by the $k-1$ power of two.

Let's bring the numbers to a common denominator — the least common multiple of the numbers from 1 to n. Then, the denominator of the resulting fraction will be a number of the form $x \cdot 2^k$, where x is an odd number. In the numerator, there will be a sum of $n-1$ even numbers and an odd number x, which gives the term $\frac{1}{2^k}$, so the numerator will be odd. But an odd number cannot be divided by an even one, so the fraction cannot be an integer.

How do we guess in this problem that we need to consider the largest power of two? The problem talks about divisibility, and bringing to a common denominator is a reasonable step. By looking at specific cases for small n, you can see that in the numerator, there are all even terms except one — the one that came from the power of two. □

Example 9.3. (OMMO — 2015-2016.9): The wrestling Federation assigns each competition participant a qualification number. It is known that in wrestling matches if the qualification numbers of the wrestlers differ by more than 2, the one with the smaller number always wins. A tournament for 256 wrestlers is held according to the Olympic system: at the beginning of each day, the fighters are divided into pairs, and the loser is eliminated from the competition (there are no draws). What is the largest qualification number the winner can have?

Solution: Note that a wrestler with the number k can only lose to a wrestler with a number not greater than $k+2$, so after each round, the smallest number cannot increase by more than 2 numbers. In the tournament with 256 participants, there are 8 rounds ($256 = 2^8$), and after each round, the number of participants decreases by half. Therefore, the number of the tournament winner does not exceed $1+2\cdot 8 = 17$.

Suppose that the wrestler with the number 17 can win. Then, in the first round, the wrestlers with numbers 1 and 2 must be eliminated. This is possible only if the wrestler with the number 1 loses to the wrestler with the number 3, and the wrestler with the number 2 loses to the wrestler with the number 4. Therefore, after the first round, the wrestlers with numbers 3 and 4 will remain. Similarly, after the second round, the wrestlers with numbers 5 and 6 will remain; after the third — 7 and 8, ..., after the seventh — 15 and 16. Hence, in the final fight, the wrestlers with numbers 15 and 16 will meet. This contradicts the assumption that the wrestler with the number 17 can win. Therefore, the wrestler with the number 16 can win in the final bout. □

Example 9.4. Prove that there exist 100 consecutive integers among which exactly 7 are prime.

Solution: It is easy to understand that among the first 100 natural numbers there are more than 7 primes.

It is easy to give an example of 100 consecutive composite numbers. For example, here are 100 consecutive composite numbers: $101!+2$, $101!+3, \ldots, 101!+101$. The first of them is divisible by 2, the second by 3, ..., and the last by 101.

We will «shift» a row of 100 numbers to the right along the number line; in this process, one number will be removed from the left, and one will be added to the right. During such a process, the number of composite numbers can change by at most 1. Initially, there were more than 7 prime numbers, and in the end, there were 0. Therefore, due to discrete continuity, there exists a set with exactly 7 prime numbers, as required to prove. □

Let's consider the application of this method to problems related to geometric figures.

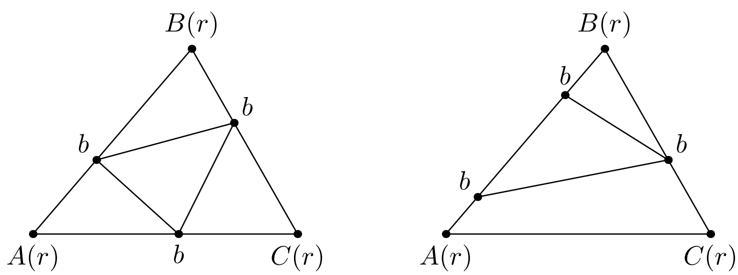
Example 9.5. (Rosenthal, Extreme principle//Kvant Journal, №9 (1988)) On the line, there is a set of points M such that each point in M is the midpoint of a segment connecting two other points in M. Prove that the set M is infinite.

Solution: **First method.** In this problem, in addition to the extreme principle, proof by contradiction will be needed. Suppose that the set M is finite. Then, it is possible to identify the extreme points on the line. However, they cannot be in the middle of any segment. This contradiction confirms the validity of the proposition being proved.

Second method: Consider all the lengths of the segments between points in M. Choose the segment AB of the greatest length. Obviously, there are no points outside the segment AB. But then, neither A nor B can be midpoints of any segments. Contradiction. □

Example 9.6. (problems.ru, problem 35135) On the plane, some points are colored blue and red in such a way that no three points of the same color lie on the same line (there are at least three points of each color). Prove that some three points of the same color form a triangle, and there are no more than two points of the other color on each side of this triangle.

Solution: Consider the triangle ABC with the smallest area, with all vertices of the same color — without loss of generality, we can assume they are red. Let's prove that such a triangle is the desired one. Suppose the opposite: then there are at least 3 blue points on its sides. Two cases are possible: each blue point lies on its own side, or two blue points lie on one side, and one lies on the other side. In both cases (shown in the figure below), the area of the triangle with vertices at the blue points will be smaller than the area of triangle ABC, which cannot be: triangle ABC has the smallest area among single-color triangles. This contradiction completes the proof of the problem.



□

Let's end with a problem suitable for students older than seventh grade.

Example 9.7. Foxes of the Fairy Forest love to visit each other, but they don't like it when someone else walks through their gardens. For this reason, the land in the Fairy Forest is divided in such a way that every two plots are adjacent to each other (at least at one point). Alice decided to find a straight path that would pass through all the plots (i.e., have at least one common point with all the gardens). Can he do it? (A fox's garden has the shape of a not necessarily convex polygon, and all gardens are located in the same plane.)

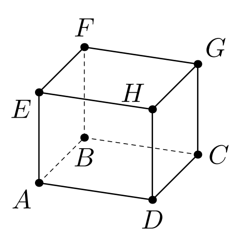
Solution: Consider a straight road passing through the Fairy Forest, project all the polygon gardens onto it, and assign a left-right direction to it. On the road, we get several segments, any two of which have a common point. From the left ends of these segments, take the rightmost one. The point obtained in this way belongs to all the segments, so the desired path for Alice is the perpendicular drawn through it to the road, and it will pass through all the given polygon gardens. □

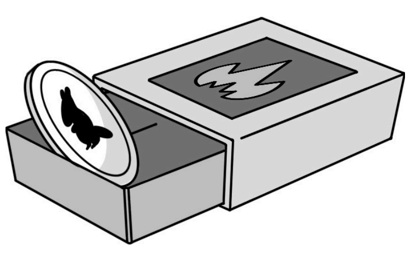
Problem Set

Problem 9.1. (COM — 2008.6.3): Once upon a time, the land of Tarnie was ruled by King Etianr. In order to simplify the thinking of the Tarniens, he devised a simple language for them. Its alphabet consisted of only six letters: **A**, **I**, **T**, **R**, **N**, **E**, but their order differed from that used in the English language. Words in this language were all sequences using each of these letters exactly once.

Etianr issued a complete dictionary of the new language. According to the alphabet, the first word in the dictionary turned out to be «Tarnie». What word followed Etianr's name in the dictionary?

Problem 9.2. (MF — 2000.7.5): Natural numbers are placed in the vertices of the cube $ABCDEFGH$ in such a way that the numbers in neighboring (along an edge) vertices differ by no more than one. Prove that there must be two diametrically opposite vertices whose numbers differ by no more than one. (Pairs of diametrically opposite vertices of the cube: A and G, B and H, C and E, D and F.)





Problem 9.3. (COM — 2008.7.7): Max collects coins. In his collection, there are 27 coins, each with a different diameter, mass, and year of issue. Each coin is stored in a

separate matchbox. Can Max stack these matchboxes into a $3 \times 3 \times 3$ parallelepiped in such a way that any coin is lighter than the coin below it, smaller than the coin to its right, and older than the one in front of it?

Problem 9.4. (COM — 2013.6.9): There are 27 students in a class. Each of them participates in at most two clubs, and for any pair of students, there exists a club where they both participate. Prove that there is a club where at least 18 students participate.

Problem 9.5. (COM — 2013.7.9): Each student in the class participates in at most two clubs, and for any pair of students, there exists a club where they both participate. Prove that there is a club where at least two-thirds of the entire class participates.

Problem 9.6. (COM — 2011.7.9): Computers №1, №2, №3, ..., №100 are connected in a ring (the first with the second, the second with the third, ..., the hundredth with the first). Hackers have prepared 100 viruses, numbered them, and at different times launched each virus in random order on the computer with the same number. If a virus hits an uninfected computer, it infects it and moves to the next computer in the chain with a higher number until it reaches an already infected computer (from computer №100 the virus moves to computer №1). Then the virus dies, and this computer is restored. No computer is infected by two viruses simultaneously. How many computers will be infected after all 100 viruses complete their attack?

Problem 9.7. (Nederlandse Wiskunde Olympiade) On a school trip, twenty students will be abseiling. In each round, one student will descend the mountain. Hence, after twenty rounds, all students will have gone down the mountain safely. In the first round, cards bearing the numbers 1 to 20 are distributed among the students. The students getting the number 1 will go down first. In the round 2, cards bearing the numbers 1 to 19 are distributed among the remaining students. The student receiving the number 1 is next to descend. They continue in this way until there is only one student left in round 20, who automatically gets a card bearing the number 1. By an amazing coincidence, no student gets the same number twice. In the first round, Sara gets a card with the number 11. What is the sum of the numbers on the cards received by Sara?

Skill Assessment Problems

Skill Assessment Problem 9.1. (N. Agakhanov) Prove that the numbers from 1 to 16 can be written in a line, but cannot be written in a circle so that the sum of each pair of neighboring numbers is a square of a natural number.

Skill Assessment Problem 9.2. At an exhibition, some artists met, some of whom are personally acquainted with each other. It turned out that any two of them, having the same number of acquaintances, do not have common acquaintances. Prove that there is an artist who is acquainted with exactly one of the exhibition visitors.

Skill Assessment Problem 9.3. (Rozental) On the plane, a set of points M is given such that each point in M is the midpoint of a segment connecting two other points in M. Prove that the set M is infinite.

Skill Assessment Problem 9.4. Can the very strange old lady Mrs. Owless draw 2024 segments on a sheet of paper so that the ends of each segment lie inside some other two segments? (The boundary points of the segment are not considered inside.)

Skill Assessment Problem 9.5. Four chess players from Beaverland and 7 foreign chess players played in a round-robin tournament, where each played two games with each other. The winner of the game received one point, the loser received zero, and for a draw, each received half a point. All participants scored a different number of points, and the sum of all points of Beaverland chess players turned out to be equal to the sum of all points of foreign chess players. Prove that there was at least one Beaverlander among the top three winners.

Solutions to Skill Assessment Problems

Solution to Problem 9.1: For the number 16, there is only one suitable neighbor, so the required arrangement in a circle is impossible (each number in the circle would have 2 neighbors).

An example demonstrating the possibility of such an arrangement in a row is:

$$16, 9, 7, 2, 14, 11, 5, 4, 12, 13, 3, 6, 10, 15, 1, 8.$$

□

Solution to Problem 9.2: Let X be an artist with the maximum number of acquaintances, denoted by N. Each of these N artists has at least one acquaintance with X and no more than N acquaintances in total. Moreover, no two of them have the same number of acquaintances. Therefore, their number ranges from 1 to N. If there is no visitor among them who has only one acquaintance, then someone will have $N + 1$ acquaintances, which contradicts the initial assumption. Hence, such a visitor must exist. □

Solution to Problem 9.3: Suppose the opposite — let M be finite. Fix the position of the plane and select the leftmost point. If there are several leftmost points, choose the bottom one. Obviously, it cannot be the midpoint of any segment. This is a contradiction. □

Solution to Problem 9.4: Consider an arbitrary arrangement of 2024 segments. Draw an arbitrary line not perpendicular to any of the segments — the existence of such a line is guaranteed by the finite number of directions of the original segments on one side and the infinite possible directions of the line on the other. Project the ends of the segments onto this line and consider the end of the furthest segment (either one). Obviously, it cannot lie inside any other segment. This is a contradiction. □

Solution to Problem 9.5: Count the total number of games played. There are 110 games in total, as each player played against each other twice ($11 \cdot 10 \cdot \frac{1}{2} \cdot 2 = 110$). Thus, 110 points were scored. Therefore, each group scored 55 points. The best player from Beaverland scored at least 14.5 points (since $14.5 + 14 + 13.5 + 13 = 55$). If the top three places were taken by foreign players, they scored at least $15 + 15.5 + 16 = 46.5$ points. Then, the remaining 4 foreign players scored a total of 8.5 points, which is less than the 12 points they would have scored only in games against each other. This is a contradiction. $\square$

Brute Force

> Three math statisticians went hunting. Suddenly, a huge wild boar runs out at them. The first statistician shoots and hits 5 centimeters to the left. The second one shoots and hits 5 centimeters to the right. The third one says with satisfaction: «Excellent, we hit it!»
>
> —One joke that didn’t quite land

Theory and Practice

How many ways are there to choose three apples from the basket? How many options are there for the school schedule? It is combinatorics that deals with such questions. We will begin to study it in the book dedicated to discrete mathematics and continue in a separate book. In combinatorial problems, we are usually interested in how many combinations, satisfying certain conditions, can be formed from a given finite set of objects.

In simple cases, we can simply list all the combinations we need and count them. However, listing should never be haphazard! Examples of correct enumeration include listing numbers in ascending order or listing words in alphabetical order; with such enumeration, no option will escape our attention, and, on the other hand, the possibility of repeating options will be excluded. It is important: if you are solving a combinatorial problem and trying to determine, for example, the number of ways to arrange children in a row by enumeration, then even if you miss exactly one case out of a million, you will receive 0 points for the entire problem. Therefore, never attempt to enumerate options if you have already tried this method and realized that there will be more options than, for example, can fit on one page.

Example 10.1. How many two-digit numbers can be formed using the digits $1, 2, 3$?

Solution: We list the numbers in ascending order:

$$11, 12, 13, 21, 22, 23, 31, 32, 33.$$

There are a total of 9 numbers. □

Example 10.2. For tomorrow, we need to prepare lessons in mathematics, Russian language, and geography (in any order). How many ways can we prepare the lessons for tomorrow?

Solution: Let's encode our subjects with letters: **M** — mathematics, **R** — Russian, **G** — geography. Then, for example, **MRG** means that we do mathematics first, then

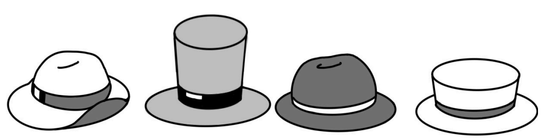
Russian, then geography. Let's list the options in alphabetical order: **GMG**, **GRM**, **MGR**, **MRG**, **RMG**, **RGM**. There are 6 options. Thus, the lessons for tomorrow can be prepared in six ways. □

Example 10.3. How many ways are there to arrange two red and two blue balls in a row? The balls do not differ except for color.

Solution: Let's denote the red ball by **R** and the blue one by **B**. Let's list the possibilities in inverse alphabetical order: **RRBB**, **RBRB**, **RBBR**, **BRRB**, **BRBR**, **BBRR**. In total, there are 6 possibilities. □

Example 10.4. The store sells white, black, and green fabric. You need to buy fabric of two different colors. How many options are there to choose from?

Solution: There are 3 options in total: white and black, white and green, black and green. □



Example 10.5. Euler's problem (or the problem of derangements, well known in combinatorics). Four gentlemen gave their hats to the waiter at the entrance of the restaurant and received them back when they left. How many options are there in which each of them receives a wrong hat?

Solution. Let's number our gentlemen from 1 to 4. We will denote the combination of hat selection as follows: the digit will indicate the ownership of the taken hat, and its ordinal number will indicate the person who took it. Let's list all possible options in ascending order of numbers: 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321. In total, there are 9 options. □

Example 10.6. How many ways are there to arrange three different flowers in two vases?

Solution: Let's denote our flowers as 1, 2, and 3. One vase will be on the left side of the comma, and the other on the right side: **nothing,** 123; 1, 23; 2, 13; 3, 12; 12, 3; 13, 2; 23, 1; 123, nothing. In total, there are 8 options. □

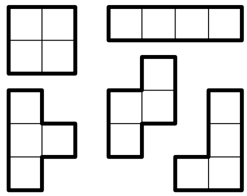
Enumerative methods can be used not only in combinatorics. For example, earlier in this section, when solving logic problems, we sometimes divided the solution path into 2 or more branches — this can also be called exhaustive enumeration if you consider all possible branches. For example, let's consider Max. Max can be a liar, a trickster, or a knight; let's divide the further solution into 3 possible cases. If you mention any of the branches but do not fully consider them, the solution ceases to be exhaustive and can only be evaluated as zero points.

Also, for example, exhaustive enumeration methods are actively used in computer science and related fields. For instance, a cipher is considered cryptographically secure if there is no significantly faster «hacking» method than an exhaustive enumeration of all encryption keys. Moreover, in modern mathematics and computer science, there still exists a large class of problems that cannot be solved significantly faster than by exhaustive enumeration methods.

Exhaustive enumeration may also appear in problems related to combinatorial geometry.

Example 10.7. A tetromino is a connected shape consisting of 4 cells, cut out of a grid paper along the grid lines. Shapes that can be overlapped using rotation and mirror reflection are considered equivalent. How many different tetrominoes exist?

Solution: All possible tetrominoes are shown in the figure. There are 5 of them.



□

Problem Set

Problem 10.1. (MF — 2004.6.1): A grasshopper jumps forward 80 cm or backward 50 cm along a straight line. Can it move exactly 1 m 70 cm away from the starting point in less than 7 jumps?

Problem 10.2. (MF — 2003.6.6): Numbers from 1 to 6 are placed on the faces of a cube. The cube is thrown twice. The first time, the sum of the numbers on the four lateral faces was 12, and the second time it was 15. What number is written on the face opposite to the one where the number 3 is written?

Problem 10.3. (MF — 2002.6.6): Leo wrote down all the numbers of the month in a row:

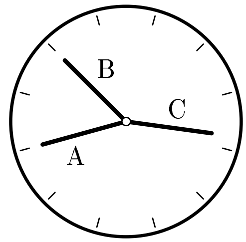
$$123456789101112\ldots$$

and colored three days (the birthdays of his friends), none of which are consecutive. It turned out that all the uncolored segments consisted of the same number of digits. Prove that the first day of the month is colored.

Problem 10.4. (COM — 2005.6.7): On the way to the Christmas holiday, several boys helped Santa Claus carry gifts. Each of the boys carried three gifts, and Santa Claus carried the remaining 142 gifts on his sleigh. Santa Claus divided all the gifts equally among all these boys and 14 girls. How many boys could there be?

Problem 10.5. (AT — 2012.1): Jean noticed that the date of the Archimedes Tournament, written with eight digits (22.01.2012), has an interesting property: by rearranging the first four digits, you can get the year number. What other dates in this year have the same property?

Problem 10.6. (MF — 2013.7.4): Jean saw strange clocks in the museum (shown below).



They differ from ordinary clocks in that there are no numbers on their dial, and it's generally unclear where the top of the clock is; moreover, the second, minute, and hour hands are of equal length. What time did the clocks show? (Hands A and B in the figure point exactly at the hour marks, and hand C is just a bit short of reaching the hour mark.)

Problem 10.7. (COM — 2006.7.4): A military band was rehearsing in the square. To perform the anthem, the musicians lined up in a square, and to perform a lyrical song, they rearranged themselves into a rectangle. At the same time, the number of rows increased by five. How many musicians are in the band?

Problem 10.8. (MF — 2016.7.6): In the contest «Come on, monsters!» there are 15 dragons lined up. The number of heads differs by 1 for neighbors. If a dragon has more heads than both of its neighbors, it is considered cunning; if it has fewer heads than both neighbors, it is considered strong; the rest (including those on the edges) are considered ordinary. There are exactly four cunning dragons — with $4, 6, 7,$ and 7 heads, and exactly three strong dragons — with $3, 3,$ and 6 heads. The first and last dragons have an equal number of heads.

a) Give an example of how this could have happened.
b) Prove that the number of heads for the first dragon is the same in all the possible options.

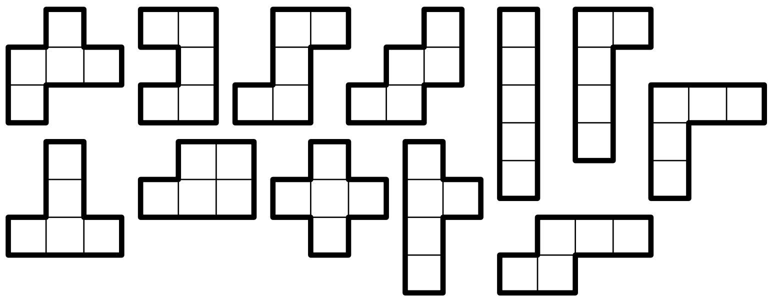
Skill Assessment Problems

Skill Assessment Problem 10.1. A pentomino is a connected shape consisting of 5 cells, cut out of grid paper along the grid lines. Shapes that can be overlapped using rotation and mirror reflection are considered equivalent. How many different pentominoes exist?

Skill Assessment Problem 10.2. List all decreasing sequences of four natural numbers whose sum is 15.

Solutions to Skill Assessment Problems

Solution to Problem 10.1: All possible pentominoes are shown in the diagram. There are 12 pentominoes in total.



□

Solution to Problem 10.2: The following partitions of the number 15 into the sum of four decreasing numbers are possible:

$$9+3+2+1=15; 8+4+2+1=15; 7+5+2+1=15;$$
$$7+4+3+1=15; 6+5+3+1=15; 6+4+3+2=15.$$

□

Pairings and Groups

> “ Max and Leo are sitting on the rails smoking. Max asks Leo:
> — Leo, I’ve wanted to ask for a long time, why do the train’s wheels are round, but when the train is moving, they clatter?
> — Oh, you, Max! It’s elementary! The formula for a circle is pi-r-squared! So, this very square is what’s clattering.
>
> —One joke that didn’t quite land

Theory and Practice

This topic can be considered as one of the special cases of the «examples and constructions» topic, as well as a subtopic of certain combinatorial problems.

While in problems related to «examples and constructions» the solution usually involves a counterexample, in this topic, on the contrary, the task typically requires proving that constructing such an example is impossible.

Example 11.1. Is it possible to draw a closed 13-segment polyline where each segment intersects exactly one of the remaining segments?

Solution: If such a thing were possible, all segments of the polyline could be paired with intersecting ones. However, this would imply an even number of segments, leading to a contradiction. □

Example 11.2. Max wrote natural numbers from 1 to 6 on the faces of a cube. Leo didn't see the cube but claimed that at least twice adjacent numbers on the cube would be separated by only one edge. Is Leo correct?

Solution: Let's divide the cube faces into pairs of opposites, resulting in 3 pairs. How many adjacent numbers are there? There are 5 pairs: 1 and 2, 2 and 3, 3 and 4, 4 and 5, 5 and 6. Thus, a maximum of 3 pairs out of these can be on non-adjacent faces, meaning that at least 2 pairs of numbers will remain adjacent and separated by only one edge. □

Example 11.3. Given a row of 64 weights placed on a table, where the weight difference between any two adjacent weights is 1 g. Is it always possible to divide the weights into two piles with equal weights and the same number of weights?

Solution: Let's group all weights into sets of four consecutive ones.

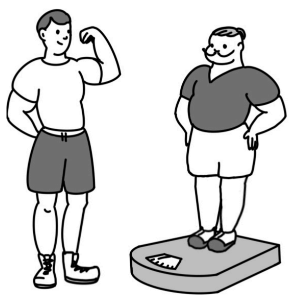
If, for each set, we put the first and last weights into the first pile and the second and third into the second pile, then the weight of each pile will remain equal as the sum of the first and last weights in each set is always exactly equal to the sum of the second and third weights. □

Example 11.4. (COM — 2015.7.3): From the natural numbers 1 to 100, 50 distinct numbers are chosen. It turns out that the sum of any two of them is not equal to 100. Is it always true that among the chosen numbers, there will be a perfect square?

Solution: Suppose that among the chosen numbers we have the number 100, then the problem setting affirms the true. Otherwise, if the number 100 is not chosen, then all numbers from 1 to 99, except 50, can be divided into 49 pairs in such a way that the sum of the numbers in each pair is 100: $1+99, 2+98, \ldots, 49+51$. If both numbers from any pair were chosen, they would sum up to 100, which contradicts the condition. Therefore, at most, one number can be chosen from each pair. Since there are exactly 50 numbers, it follows that from each pair, exactly one number should be chosen, as well as the number 50. In the pair $36+64$, both numbers are perfect squares, and at least one of them will be among the chosen numbers. □

Example 11.5. (AU): From the numbers 1 to $2n$, $n+1$ numbers are chosen. Prove that among the chosen numbers, there are two numbers, one of which divides the other.

Solution: To solve this problem using the Pigeonhole Principle, we need to understand what serves as pigeonholes and what serves as pigeons in the problem. It's clear that numbers are the pigeons. But what about the pigeonholes? Let's create a group (pigeonhole) for each odd number and distribute our $2n$ numbers among these pigeonholes. In the pigeonhole numbered k, we put the number k itself and all numbers of the form $k \cdot 2^1$, $k \cdot 2^2$, ..., $k \cdot 2^l \leqslant 2n$. It's easy to see that each of our numbers falls into exactly one pigeonhole — if the number is odd, it falls into the corresponding pigeonhole by definition, and if it's even, we divide it by two until the result becomes odd. For example, for $n = 6$, we create 6 groups — $1, 3, 5, 7, 9, 11$. Group 1 will contain numbers $1, 2, 4, 8$, group 3 — numbers $3, 6, 12$, group 5 — numbers 5 and 10, and groups 7, 9, and 11 will each contain one number. Now, each pigeon is assigned to its own box. Let all pigeons out. If we take any $n+1$ pigeons, by the Pigeonhole principle, at least one of the n pigeonholes will contain at least two pigeons. Remembering that each pigeonhole, according to the selection condition, may contain numbers that divide each other, we conclude the proof. □



Example 11.6. (MF — 1993.5,6.8) There are 100 heavyweight athletes weighing from 1 to 100 kg in a sports club. What is the smallest number of teams into which they can be divided so that no team has two athletes, one of whom weighs exactly twice as much as the other?

Solution: One team is clearly not enough (then the team would include athletes weighing 1 and 2 kg, which doesn't meet the required condition).

We will demonstrate how to manage with two teams. If we find any of the athletes weighing the same as another, we will assign them to the same team.

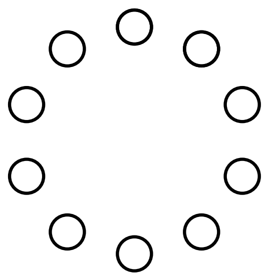
Then, for each athlete weighing $2a$, there is at most one athlete from the team weighing $4a$ (for those weighing more than 50 kg, older teammates do not fit into the 100-kg standards at all), and at most one weighing a (and then only in the case of an even weight since weight standards assume integer values only — such discrimination).

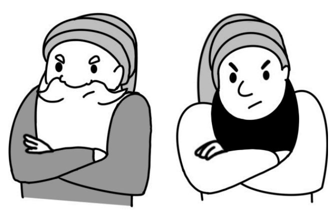
Taking the above into account, we can organize the distribution into teams as follows: We form chains of athletes with weights a,$2a$,$4a$, . . . and distribute them alternately into two teams. Obviously, all athletes can be distributed into similar chains with non-intersecting elements (the correctness of this statement is elementary to prove by contradiction — otherwise, for some weight, two different ones, each weighing twice as much, would be found — it's absurd). Some chains will consist of only one element ($51, 53, 55$ and so on) — it doesn't matter which team they end up in. The longest chain will consist of powers of two ($1, 2, 4, 8, \ldots$) — for example, those standing in even places will be assigned to one team and the rest to the other.

Thus, all athletes will be distributed into two teams, and the condition of the problem will be satisfied. $\square$

Problem Set

Problem 11.1. (MF — 1998.6.3): Arrange the numbers from 1 to 10 in the circles (vertices of a regular decagon) so that for any two neighboring numbers, their sum is equal to the sum of the two numbers symmetric to them with respect to the center of the circle.





Problem 11.2. (AT — 2015.4): One fine day, each of the 2015 dwarfs got offended by exactly one other dwarf, and each dwarf was offended by exactly one other dwarf. Alice needs to arrange the dwarfs into three groups so that no dwarf in any group is offended by another dwarf in the same group. Is this always possible? Justify your answer.

Problem 11.3. (COM — 2013.7.4): Two magicians show the viewer a trick. The viewer has 24 cards numbered from 1 to 24. He selects 13 cards and hands them to the first magician. The first magician returns two of them. The viewer adds one of the remaining 11 cards to the two returned cards, shuffles them, and hands these three cards to the second magician. How can the magicians agree so that the second magician can always determine which of the three cards the viewer added?

Problem 11.4. (MF — 2009.7.4): A stingy knight keeps gold coins in 77 chests. One day, while recounting them, he noticed that if any two chests are opened, the coins in them can be equally distributed between these two chests. Then he noticed that if any 3, 4, ..., or 76 chests are opened, the coins in them can also be redistributed so that there are equal numbers of coins in all the opened chests. Then he heard a knock on the door, and the old miser didn't have time to check whether it was possible to redistribute all the coins equally among all 77 chests. Can he give an accurate answer to this question without looking into the chests?

Problem 11.5. (Grey Kangaroo) There are 4 teams in a football tournament. Each team plays exactly once. In each match, the winner gets 3 points, and the loser gets 0 points. In the case of a draw, both teams get 1 point. After all matches have been played, which of the following total number of points is it impossible for any team to have obtained?

A4 B5 C6 D7 E8

Problem 11.6. (Pink Kangaroo) Each of the children in a class of 33 children likes either PE or Computing, and 3 of them like both. The number of children who like only PE is half as many as those who like only Computing. How many students like Computing?

(A)15 (B)18 (C)20 (D)22 (E)23

Problem 11.7. (Formula) The numbers $1, 2, \ldots, 10$ are placed on a circle in some order. Prove that there are 3 adjacent numbers whose sum is not less than 18.

Problem 11.8. (Grey Kangaroo) A total of 2021 colored koalas are arranged in a row and are numbered from 1 to 2021. Each koala is colored red, white, or blue. Amongst any three consecutive koalas, there are always koalas of three colors. Sheila guesses the colors of five koalas. These are her guesses: Koala 2 is white; Koala 20 is blue; Koala 202 is red; Koala 1002 is blue; Koala 2021 is white. Only one of her guesses is wrong. What is the number of the koala whose colour she guessed incorrectly?

(A)2 (B)20 (C)202 (D)1002 (E)2021

Problem 11.9. (Pink Kangaroo) Alice, Belle, and Cathy have an arm-wrestling contest. In each game, two girls wrestled while the third rested. After each game, the winner played the next game against the girl who had rested. In total, Alice played 10 times, Belle played 15 times, and Cathy played 17 times. Who lost the second game?

(A) Alice

(B) Belle

(C) Cathy

(D) Either Alice or Belle could have lost the second game

(E) Either Belle or Cathy could have lost the second game

Problem 11.10. (Pink Kangaroo) A total of 2021 balls are arranged in a row and are numbered from 1 to 2021. Each ball is colored in one of four colors: green, red, yellow, or blue. Among any five consecutive balls, there is exactly one red, one yellow, and one blue ball. After any red ball, the next ball is yellow. The balls numbered 2 and 20 are both green. What color is the ball numbered 2021?

(A) Green

(B) Red

(C) Yellow

(D) Blue

(E) It is impossible to determine

Skill Assessment Problems

Skill Assessment Problem 11.1. Which type of number is more common among all three-digit numbers: those with the middle digit greater than both outer digits or those with the middle digit smaller than both outer digits?

Skill Assessment Problem 11.2. To break the curse of beaver-mania, the hedgehog needs to find the sum of all possible six-digit numbers consisting of $1, 2, 3, 4$. Help the hedgehog.

Solutions to Skill Assessment Problems

Solution to Problem 11.1: Let's consider why this problem is placed in this topic.

Let's notice the following facts. If we consider a three-digit number with digits $\overline{abc}$, then subtracting it from 999, we get the number $\overline{(9-a)(9-b)(9-c)}$.

Then, if in the first number, the middle digit was greater than both outer digits, then in the second number, it will be smaller than both outer digits, and vice versa. Thus, dividing the numbers into pairs $100-899, 101-898, \dots, 499-500$, we get that in each pair, either both numbers do not belong to any of the categories, or one of them belongs to the first category and the other to the second.

Outside these groups are the numbers from 900 to 999. In none of these numbers can the middle digit be greater than the first digit from the extremes, 9. At the same time, for example, the number 909 has the property that its middle digit is smaller than both extremes. Thus, the total number of numbers with the middle digit smaller than both extremes is greater than those with the middle digit greater than both extremes. □

Solution to Problem 11.2: Let's form pairs of the form

$$\overline{abcdef} \text{ and } \overline{(5-a)(5-b)(5-c)(5-d)(5-e)(5-f)}.$$

Then all numbers have exactly one pair. The sum of two numbers in a pair is 555555. How many numbers are there in total?

Let's subtract 1 from each digit and arrange the numbers in ascending order: $000000, 000001, 000002, 000003, 000010, \dots$

These are the numbers where there are no more than 6 digits in the number system with base 4, that is, numbers less than $1000000_4 = 4^6_{10}$. There are $4^6 = 4096$ such numbers (including 0). Thus, there are 2048 pairs of such numbers, and the sum of all such numbers is $2048 \cdot 555555 = 1137776640$. □

Double Counting

> A math teacher irritably tells the students during a seminar:[2] How many times have I told you that there is no bigger or smaller half, and yet the bigger half of you still hasn't grasped this?
>
> —One joke that didn't quite land

Theory and Practice

In some problems, you can get the required equation if you calculate the same quantity in two ways. The difficulty is to figure out exactly what value to calculate in two ways (and which ones exactly!).

Example 12.1. Thirty students, among whom were seventh graders and eighth graders, exchanged handshakes. It turned out that each seventh grader shook hands with eight eighth graders, and each eighth grader shook hands with seven seventh graders. How many seventh graders and how many eighth graders were there?

Solution: Let x be the number of seventh graders, and y be the number of eighth graders; then $x + y = 30$. The second equation is obtained if we calculate the total number of handshakes in two different ways. On the one hand, the number of handshakes is equal to $8x$, since each seventh grader «initiates» 8 handshakes. On the other hand, the number of handshakes is equal to $7y$, since each eighth grader «initiates» 7 handshakes. Therefore, $8x = 7y$. Solving the resulting simultaneous equations, we find: $x = 14$, $y = 16$. □

Example 12.2. (LT — 1985) There were 7 empty boxes. Into some of them, 7 empty boxes were placed, and so on. As a result, there were 10 non-empty boxes. How many boxes were there in total?

Solution: Let x be the total number of boxes. Let's calculate the number of boxes that are contained in some other box in two different ways. On the one hand, it is equal to $x - 7$, since each box, except for the initial seven, is contained in some other box. On the other hand, their number is equal to $10 \cdot 7 = 70$, since each of the 10 non-empty boxes contains exactly 7 (and there are no boxes in any empty box). Therefore, $x - 7 = 70$, hence $x = 77$. This problem could also be solved differently. After each addition of seven boxes into another box, the number of non-empty boxes increases by 1. Initially, there were 0 of them, and in the end, there were 10, so the number of boxes increased by $10 \cdot 7 = 70$, i.e., $x = 7 + 70 = 77$. □

Example 12.3. (Kozlova) Alice, Beatrice, and Clarice were solving problems. To speed things up, they bought candy and agreed that for each solved problem, the girl who solved it first would receive four candies, the second solver would receive two, and the last solver would receive one. The girls claim that each of them solved all the problems and received 20 candies, with no simultaneous solutions. Is this possible?

Solution: After solving each problem, the total number of candies each girl received increased by $4+2+1=7$. Therefore, the total number of candies received by the girls must be divisible by 7, but $20 \cdot 3 = 60$ is not divisible by 7. Therefore, the girls are mistaken. $\square$

The next problem is also related to the recently discussed extreme principle.

Example 12.4. (3ARSO — 1995.9.3): The cells of a 15×15 square table are colored in red, blue, and green. Prove that there are at least 2 rows in which there are an equal number of cells of at least one color.

Solution: Suppose the opposite: this would mean that each color has a different number of cells in each row. Thus, the minimum number of cells of any color would be

$$0+1+2+\ldots+14 = 14 \cdot 15 \cdot \frac{1}{2},$$

and the total number of colored cells would then be at least

$$14 \cdot 15 \cdot \frac{1}{2} \cdot 3 = 21 \cdot 15.$$

However, there are exactly $15 \cdot 15$ cells in total, which is less. This contradiction completes the solution to the problem. $\square$

The following problem will also touch on the extreme principle as well as the theme of this chapter and also involve an estimation plus an example.

Example 12.5. (MMO — 2007.8.3): Sixteen teams participated in a football championship. Each team played against each other once, with 3 points awarded for a win, 1 point for a draw, and 0 points for a loss. We call a team successful if it scores at least half of the maximum possible number of points. What is the maximum number of successful teams that could be in the tournament?

Solution: Suppose that all 16 teams could be successful. Since each team plays 15 games in total, the maximum possible number of points that one team can score is $15 \cdot 3 = 45$. Since only integer points can be scored, for a successful performance, each team must have at least 23 points. Then, in total, all teams must score at least $16 \cdot 23$ points. Let's try to estimate the total number of points scored by all teams differently. In one game, the teams together score either 2 or 3 points, so in any case, no more than 3 points are scored. There are a total of $16 \cdot 15 \cdot \frac{1}{2}$ games, so the total number of points scored is no more than

$$16 \cdot 15 \cdot \frac{1}{2} \cdot 3 = 16 \cdot 22.5,$$

which leads to a contradiction.

An example with 15 successful teams: Team number 16 loses all its matches. The remaining teams play as follows: teams with even numbers win against teams with odd numbers, and vice versa. Then, each of them wins 8 out of 15 games and becomes successful. □

Problem Set

Problem 12.1. (MechMathCircle — 2015/2016.7ББ.3): Is it possible to place some number of asterisks in the cells of a 10×10 square such that each 2×2 square contains exactly two asterisks, and each 3×1 rectangle contains exactly one asterisk?

Problem 12.2. (MechMathCircle — 2015/2016.7ББ.4): Jean cut out many identical squares and wrote the numbers $1, 2, 3$, and 4 in the corners of each of them in an arbitrary order. Then Esther stacked the squares into a pile and wrote the sum of the numbers in each of the four corners of the stack. Is it possible that the sum in each corner of the stack is a) 2015; b) 2016?

Problem 12.3. (LT — 1978.1): a) Is it possible to number the edges of a cube with the numbers -6, -5, -4, -3, -2, -1, 1, 2, 3, 4, 5, 6 such that for each vertex of the cube, the sum of the numbers of the edges incident to it is the same?
b) Can the edges of the cube be numbered with natural numbers from 1 to 12 in the same way?

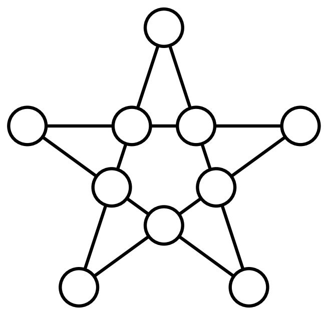
Problem 12.4. (MechMathCircle — 2015/2016.7ББ.6): Is it possible to arrange seven non-negative integers around a circle such that the sum of any three consecutive numbers is equal to 1 for some triple, 2 for some triple, and so on up to 7 for some triple?

Problem 12.5. (Mccme — 2005/2006.7.1): Max colored several cells in a 6×6 square. Afterward, it turned out that in every 2×2 square, the same number of cells was colored, and the same was true for every 1×3 strip. Prove that diligent Max colored all the cells.

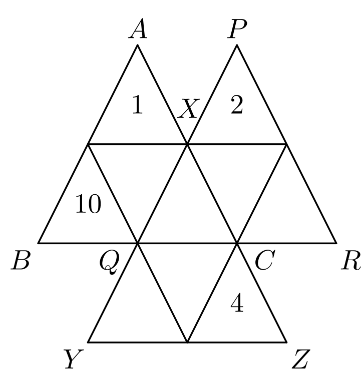
Problem 12.6. (MF — 1996.7.1) The digits $1, 2, 3, \ldots, 9$ are arranged in a circle in an arbitrary order. Every three digits, arranged clockwise, form a three-digit number. Find the sum of all nine such numbers. Does it depend on the order in which the digits are arranged?

Problem 12.7. (MMO — 1992.8.3): Each participant of a two-day Olympiad solved the same number of problems on the first day as the sum of the numbers solved by all the other participants on the second day. Prove that all participants solved the same number of problems.

Problem 12.8. (Mos2ARSO — 2008.7.1): Is it possible to place 4 ones, 3 twos, and 3 threes in the circles of a five-pointed star (as shown below) such that the sums of the four numbers on each of the five straight lines are equal?



Problem 12.9. (JMO) The diagram shows three triangles: ABC, PQR, XYZ, each of which is divided up into four smaller triangles. The diagram is to be completed so that the positive integers from 1 to 10 inclusive are placed, one per small triangle, in the ten small triangles. The totals of the numbers in the three triangles ABC, PQR, and XYZ are the same. Numbers 1,2,4 and 10 have already been placed. In how many different ways can the diagram be completed?



Problem 12.10. (JMO) A jar contains red and white marbles in the ratio $1 : 4$. When

Jenny replaces 2 of the white marbles with 7 red marbles, the ratio becomes $2 : 3$. What is the ratio of the total number of marbles in the jar now to the total number in the jar before?

Problem 12.11. (Nederlandse Wiskunde Olympiade) On each of the $10{,}000$ squares of a $100 \times 100 -$ chess board a number is written. Along the top row, the numbers 0 to 99 are written from left to right. In the left column, the numbers 0 to 99 are written from top to bottom. The sum of the four numbers in a $2 \times 2 -$ block always equals 20. Which number is written in the bottom right square of the board?

Skill Assessment Problems

Skill Assessment Problem 12.1. (COM — 2010.7.9) The network of bus routes in the suburbs of Beaverville is arranged so that:

a) Each route has exactly three stops, including both terminals;

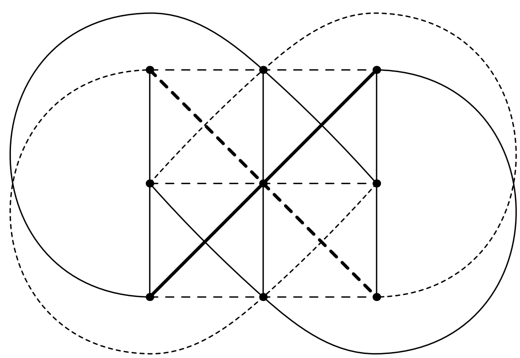
b) Any two routes either have no common stops at all or have only one common stop.

What is the maximum number of routes that can exist in this suburb if there are a total of 9 stops?

Skill Assessment Problem 12.2. Given 25 numbers. The sum of any four of them is positive. Prove that the sum of all of them is also positive.

Solutions to Skill Assessment Problems

Solution to Problem 12.1: Suppose there is a stop through which at least 5 routes pass. According to the problem setting, apart from this stop, no two routes out of these 5 can have another common stop, and since each route has 2 more stops besides the given one, there must be at least $5 \cdot 2 = 10$ stops other than this one. This contradiction proves that no more than 4 routes pass through each stop. Since there are only 9 stops and each can have at most 4 routes passing through it, all routes pass through all stops no more than $9 \cdot 4 = 36$ times, i.e., there are no more than $36 \div 3 = 12$ routes in total. An example with 12 routes is shown in the diagram below.



□

Solution to Problem 12.2: If the sum of any four numbers is positive, then the sums of any 24 numbers are also positive. There are 25 such sums of the form $S - x_i$. Summing them up yields $25S - x_1 - \ldots - x_{25} = 25S - S = 24S$, and this is greater than zero. Therefore, the sum S of all the numbers is also greater than zero. □

Estimation + Example

> “ Question: How do you make seven an even number?
> Answer: Just remove the «s.»

—One joke that didn’t quite land

Theory and Practice

This topic stands out in Olympiad mathematics: essentially, it is not a separate topic — a problem on almost any topic can be an estimation + example.

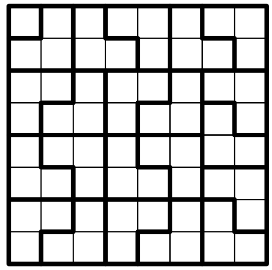
This topic can be identified by words like «maximum» or «minimum» in the problem statement. For example, a problem might be of the form: find the largest number that has a given property. What does solving such a problem imply? Often, one encounters solutions of this type: «this number works», followed by proof that it works, and that's where the solution ends. But what was required was to find the largest number, not just any number that works. The part of the solution where a specific number is presented is called an example. However, it is also necessary to provide an estimate, showing that this number is indeed the largest. Typically, less than half the points that the problem «costs» are awarded for the example, but the solver may think they have solved the problem and move on to the next one, missing out on the remaining points.

It's even more frustrating when the situation is reversed: an estimate has been proven, but the solver considers the example obvious and doesn't bother writing it down. Unfortunately, according to the grading criteria, the grader cannot award full points for such a solution.

Example 13.1. What is the maximum number of three-cell corners that can be cut out of an 8×8 grid?

Solution: **Estimation.** There are 64 cells in the square. Therefore, it is not possible to cut out 22 or more corners, because then the total number of cells in them would be at least $22 \cdot 3 = 66$. Hence, the number of corners is at most 21.

Example. We can cut out 21 corners — an example is shown in the diagram below.



Thus, the maximum possible number of corners is 21. The logic of the reasoning is clear: we showed that the number of corners does not exceed 21 (**estimation**), and sometimes equals it (**example**). Therefore, 21 is the maximum number of corners. □

Example 13.2. (Circle 57 — 2005/2006.7): What is the maximum number of squares on an 8×8 board that can be colored black, such that in every corner of three squares, there is at least one uncolored square?

Solution: **Estimation.** Divide the board into 16 squares of size 2×2. Notice that in each of these squares, we can color at most two squares (to avoid creating black corners). Therefore, the total number of colored squares does not exceed $2 \cdot 16 = 32$.

Example. Color the chessboard with stripes row by row: $4 \cdot 8 = 32$. Alternatively, the chessboard coloring itself will also contain 32 colored squares. □

Example 13.3. What is the smallest number of coins in 3 and 5 kopecks that can make up a total of 37 kopecks?

Solution: If the number of coins is no more than seven, then the sum will be no more than $7 \cdot 5 = 35$ kopecks. Therefore, seven or fewer coins will not be enough. Suppose there are eight coins. They cannot all be 5-kopeck coins ($8 \cdot 5 = 40$). Seven 5-kopeck coins and one 3-kopeck coin give a total of 38 kopecks. If there are no more than six 5-kopeck coins, then the sum does not exceed $6 \cdot 5 + 2 \cdot 3 = 36$ kopecks. Hence, using eight coins, it is also not possible to make 37 kopecks. So, there must be at least nine coins. Here is an example of a suitable set of nine coins: five 5-kopeck coins and four 3-kopeck coins

$$5 \cdot 5 + 4 \cdot 3 = 37.$$

Therefore, the smallest possible number of coins is nine. □

Note: You are not obligated to explain how you came up with the example! When writing the solution, simply present the example (and show, if it's not obvious, that it works). It's not necessary to describe the reasoning behind your example.

Example 13.4. (MechMathCircle — 2012/2013) 30 pikes were released into a pond, and they started eating each other. A pike is considered satiated if it has eaten at least three pikes. What is the maximum number of pikes that could become satiated if the eaten pikes are also counted in the tally?

Solution: **Estimate.** There are various ways to estimate it.

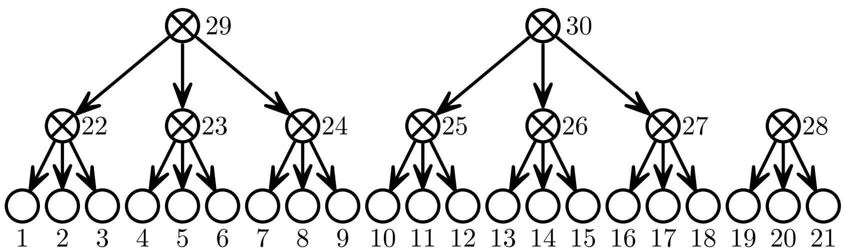
Descriptive approach to estimation. Notice that if 10 or more pikes become satiated, then 30 or more pikes will be eaten — that is, all the pikes available, which is impossible (assuming a pike cannot eat itself — which is quite obvious).

Formal-analytical approach to estimation. Let a pikes become satiated. Then, they together eat $3a$ or more pikes. Since each pike cannot be eaten twice and there will be at least one pike left in the end, the relation $3a < 30$ is evident. Hence, $a < 10$ — that is, the maximum possible number is 9.

Example. There are various ways to provide it as well.

Descriptively. Let's give an example where exactly 9 pikes become satiated. Let pikes cannibalizing each other be denoted as $a_1, a_2, \ldots, a_n$. Initially, 7 pikes ($a_1, \ldots, a_7$) become satiated, eating 21 pikes ($a_8, \ldots, a_{28}$) in total — each eating 3 pikes. There are 9 pikes left — 7 already satiated ($a_1, \ldots, a_7$) and 2 still hungry (a_{29}, a_{30}). The two remaining hungry pikes (a_{29} and a_{30}) become satiated by eating 6 satiated pikes ($a_1, \ldots, a_6$).

Graphically. The circles in the diagram below represent the satiated pikes, while the crosses represent the remaining hungry pikes.



□

Example 13.5. In a row, there are 150 flower pots, some of which have plants growing in them. Among any three consecutive flower pots, there is at least one with a plant. A zombie passed along the row and ate some of the plants, so now, among any five consecutive flower pots, there is at most one plant. What is the minimum number of plants it could have eaten?

Solution: Let's consider 2 consecutive flower pots with plants: between them, there are at most 2 empty pots, which means one of them (those with plants) will definitely be emptied. Therefore, at least half of the total number of plants will be eaten in the case where it is even, and half of the total number of plants minus one in the case where it is odd. Let's divide our 150 flower pots into 50 consecutive triples. It is clear that in each of them, there is at least one flower pot with a plant, so there are at least 50 such pots in total, resulting in at least 25 plants being eaten.

An example where exactly 25 plants could be eaten is as follows: initially, the plants are located at numbers divisible by 3, i.e., $3, 6, 9, 12, \ldots$, and then the zombie eats the plants at numbers divisible by 6. It is easy to see that this example satisfies the condition, and indeed 25 plants are eaten. □

As in-game problems, it is advisable to avoid the notion of the «best case scenario» — it is impossible to explain what this phrase means. Usually, when asked to explain what this best case is, the answer should be: «well, it's obvious.» Such an answer will not satisfy anyone.

Let's illustrate this with an example of solving the problem just provided. One of the authors of this book, being a member of the jury of an Olympiad conducted in an oral format, where this problem was given, encountered the reasoning of one of the students, starting with the following words: «The fewer plants are initially placed, the fewer plants the zombie will eat» — this is an attempt to move from the general case to the «best» one. Of course, this statement is incorrect since it does not mention that the zombie will try to eat as few plants as possible. And by moving to the «best case scenario» with 50 planted plants, the participant showed that with arrangements $1, 4, 7, 10, \ldots$; $2, 5, 8, 11, \ldots$; $3, 6, 9, 12, \ldots$ eating fewer than 25 plants seems impossible. However, since these options for placing 50 plants are not exhaustive (there are so many of them that listing them all using exhaustive enumeration would take too much time during the Olympiad), this does not prove anything even for this «best case.»

It is also worth noting that considering any «best cases» can be quite useful when searching for an example, but one should be wary of this concept only when proving an estimate.

Example 13.6. (1ARSO — 2015.8.1): All natural numbers whose sum of digits is divisible by 5 are listed in ascending order: $5, 14, 19, 23, 28, 32, \ldots$. What is the smallest difference between consecutive numbers in this sequence?

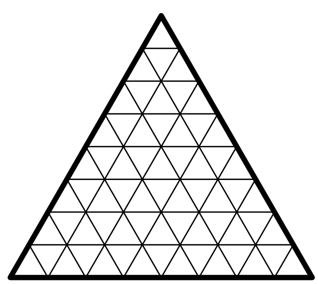
Solution: If there is no transition between two consecutive numbers in this sequence, the difference will simply be 5. When transitioning, 9 turns into 0, and the next digit increases by 1. Therefore, the sum of digits decreases by 8. When transitioning between the last two digits, it decreases by 17, and so on. By examining further, we can notice that when transitioning between the last 4 digits, the sum decreases by 35. Thus, we can find two consecutive numbers where the sum of digits is divisible by 5, for example, 49999 and 50000. This is an example where the difference is 1. Clearly, the difference cannot be smaller — this is an exceptional problem where finding the example was much more difficult than estimating. □

Problem Set

Problem 13.1. (MF — 2008.6.2): Mrs. Rabbit bought seven drums of different sizes and seven pairs of sticks of different lengths for her seven bunnies. If a bunny sees that it has both a larger drum and longer sticks than any of its siblings, it starts drumming loudly. What is the maximum number of bunnies that can start drumming?

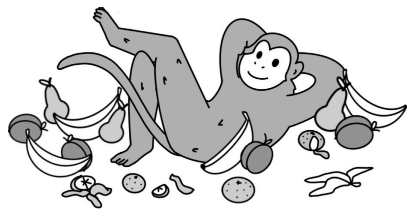
Problem 13.2. (MF — 1997.7.2): In Mexico, ecologists have achieved the adoption of a law requiring each car to be off the road for at least one day a week (the owner informs the police of the car's number and its «day off» of the week). In a certain family, all adults wish to drive daily (each for their own purposes!). How many cars (at least) should the family have if there are a) 5 adults; b) 8 adults?

Problem 13.3. (MF — 2016.6.3): A equilateral triangle with a side length of 8 is divided into equilateral triangles with a side length of 1 (as shown in the figure below). What is the smallest number of triangles that need to be colored to ensure that all intersection points (including those at the edges) are vertices of at least one colored triangle?



Problem 13.4. (MF — 1990.5.3): Forty-eight blacksmiths have to shoe 60 horses. What is the minimum time they will spend on the job if each blacksmith takes five minutes to shoe one horse? (A horse cannot stand on two legs.)

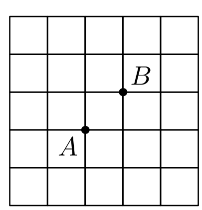
Problem 13.5. (COM — 2005.7.4): A cube frame with edges of length 1 is covered in honey. A beetle is at the top vertex of the cube. What is the minimum path it must crawl to eat all the honey?



Problem 13.6. (MF — 2015.6.5): A monkey becomes happy when it eats three different fruits. What is the maximum number of monkeys that can be made happy if there are 20 pears, 30 bananas, 40 peaches, and 50 mandarins? Justify your answer.

Problem 13.7. (MF — 2006.6.5): Grandpa called his grandson to the village: «Look at what an extraordinary garden I've planted! I have four pear trees there, and there are also apple trees, and they are planted so that exactly two pear trees grow 10 meters from each apple tree.» «Well, what's so interesting about that,» the grandson replied. «You only have two apple trees.» «Ah, but you guessed wrong,» Grandpa smiled. «I have more apple trees in my garden than pears.» Draw how the apple and pear trees could have grown in Grandpa's garden. Try to place as many apple trees as possible on the drawing without violating the conditions. If you think you have placed the maximum possible number of apple trees, try to explain why.

Problem 13.8. (MF — 2009.6.5): A curious tourist wants to take a walk through the streets of the Old Town from the train station (point A on the map) to his hotel (point B). The tourist wants his route to be as long as possible, but he doesn't want to pass the same intersection twice, so he avoids doing so. Draw the longest possible route on the map and prove that there is no longer one.



Problem 13.9. (COM — 2014.6.5): Max paints some cells on a 4×4 grid. Leo will win if he can cover all these cells with non-overlapping and non-extending corners

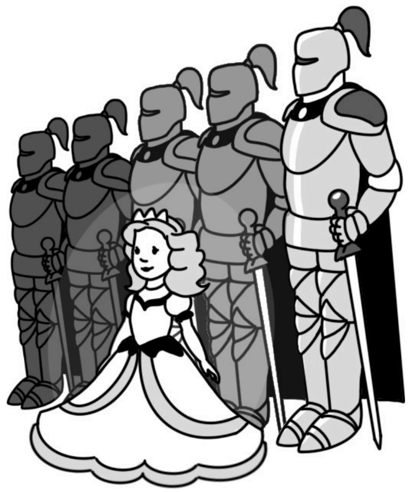
made of three cells. What is the minimum number of cells Max must paint to prevent Leo from winning?

Problem 13.10. (COM — 2013.6.6) Alice wants to cut a sheet of paper into 48 identical rectangles. What is the minimum number of cuts she needs to make if she can stack any pieces of paper but cannot fold them, and Alice is capable of cutting through multiple layers of paper at once? (Each cut is a straight line from one edge of the paper to the other.)

Problem 13.11. (COM — 2015.6.6): Using an equal number of squares with sides 1, 2, and 3, form the smallest possible square.

Problem 13.12. (COM — 2008.6.6): Find the greatest number of colors in which the edges of a cube can be painted (each edge with one color) so that for each pair of colors, there are two adjacent edges painted in those colors. (Edges sharing a vertex are considered adjacent.)

Problem 13.13. (COM — 2006.6.6) Appearing in the arena with 10 lions and 15 tigers, the animal trainer, Mrs. Owless, lost control over them, and the beasts started devouring each other. A lion will be satisfied if it eats three tigers, and a tiger will be satisfied if it eats two lions. Determine the maximum number of predators that could be satisfied and how this could happen.



Problem 13.14. (MF — 2013.6.6): Thirty-three heroes were hired to guard the princess for 240 coins. Their chief can divide the heroes into squads of arbitrary size (or put them all in one squad) and then distribute all the money among the squads. Each squad divides its coins equally among its members, and the remainder is given to the chief. What is the maximum number of coins that the chief can get if he distributes the money between the squads: a) as he pleases; b) equally?

Problem 13.15. (AT — 2012.5): In a bag, there are gold coins — doubloons, ducats, and piastres, which are indistinguishable by touch. If 10 coins are taken out of the bag, then among them, there will be at least one doubloon; if 9 coins are taken out, then among them, there will be at least one ducat; and if 8 coins are taken out, then among them, there will be at least one piastre. What is the maximum number of coins that could be in the bag?

Problem 13.16. (MF — 2003.7.5): In honor of the holiday, 1% of the soldiers in the regiment received new uniforms. The soldiers are arranged in a rectangle so that the soldiers in the new uniforms are in at least 30% of the columns and in at least 40% of the rows. What is the smallest number of soldiers that could be in the regiment?

Problem 13.17. (COM — 2013.6.7): In a row of five pots, Alice poured three kilograms of honey (not necessarily evenly distributed among them). Beatrice can take any two adjacent pots. What is the maximum amount of honey that Beatrice can guarantee to eat?

Problem 13.18. (COM — 2012.6.7): A five-digit number is called *indecomposable* if it cannot be decomposed into the product of two three-digit numbers. What is the maximum number of consecutive indecomposable five-digit numbers?

Problem 13.19. (COM — 2002.6.7): Each of 50 items needs to be painted first and then packaged. Painting takes 10 minutes, packaging takes 20 minutes. After painting, the item needs to dry for 5 minutes. How many painters and packers need to be hired to complete the work in the shortest time possible if no more than 10 people can be hired?

Problem 13.20. (COM — 2017.6.8;7.8): In each cell of a 5×5 board, there is either a cross or a zero, and no three crosses can stand in a row horizontally, vertically, or diagonally. What is the maximum number of crosses that can be on the board?

Problem 13.21. (COM — 2016.6.9): The store sells boxes of chocolates. Among them, there are at least five boxes of different prices (no two of them cost the same). Whatever two boxes Max buys, Leo can always buy two boxes, spending the same amount of money. What is the minimum number of boxes of chocolates that must be available for sale?

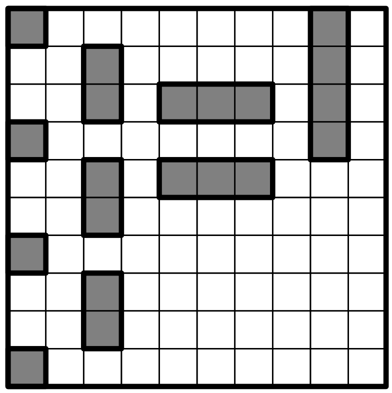
Problem 13.22. (COM — 2012.6.9): The plan of the king's palace is a square of size 6×6, divided into rooms of size 1×1. In the middle of each wall between the rooms, there is a door. The king told his architect, «Break some of the walls so that all the rooms become of size 2×1, no new doors appear, and the path between any two rooms passes through no more than N doors.» What is the smallest value of N that the king should name so that the order can possibly be executed?

Problem 13.23. (AT — 2014.6) Jean rearranged the digits in a certain number A and obtained the number B. Then he calculated the difference $A - B$, and obtained a number written using only ones (no other digits were used). What is the smallest number that he could have obtained?

Problem 13.24. (MF — 2005.7.6): On the Island of Misfortune, with a population of 96 people, the government decided to implement five reforms. Exactly half of all citizens are dissatisfied with each reform. A citizen joins a protest if they are dissatisfied with more than half of all reforms. What is the maximum number of people the government can expect to participate in the protest? (Provide an example and prove that it cannot be more.)

Problem 13.25. (MF — 2008.7.6): Leo stood at a bus stop for some time. During this time, one bus and two trams passed by. After a while, Spy arrived at the same stop. While he was there, 10 buses passed by. What is the minimum number of trams that could have passed by during this time? Both buses and trams run at equal intervals, with buses running every 1 hour.

Problem 13.26. (MF — 2010.7.6): It is easy to place a set of ships for the game «Battleship» on a 10×10 board (see the diagram). But on what is the smallest square board can this set be placed? (Recall that according to the rules, ships should not even touch by corners.)



Problem 13.27. (MF — 2013.7.6): Foxes Alice and Beatrice have grown 20 counterfeit banknotes on a tree and are now inscribing seven-digit numbers in them. Each banknote has 7 empty cells for digits. Beatrice calls out one digit at a time: «1» or «2» (she doesn't know any others), and Alice fills in the called digit in any free cell

of any banknote and shows the result to Beatrice. When all the cells are filled, Beatrice takes as many banknotes as possible with different numbers (she takes only one from several with the same number), and Alice takes the remainder. What is the maximum number of banknotes Beatrice can obtain, no matter how Alice acts?

Problem 13.28. (COM — 2014.7.9): There are 2014 points marked on a circle. A grasshopper sits in one of them, making jumps clockwise either by 57 divisions or by 10. It is known that it visited all marked points, making the least number of jumps of length 10. How many jumps did it make?

Problem 13.29. (Formula) Let us call a positive integer «ascending» if each of its digits is greater than the previous one (e.g., 7 and 3579 are ascending, but 2447 is not). What is the minimal number of ascending numbers that sum up to 2014?

Problem 13.30. (Nederlandse Wiskunde Olympiade) In a table consisting of n by n small squares, some squares are colored black and the other squares are colored white. For each pair of columns and each pair of rows, the four squares on the intersections of these rows and columns must not all be of the same color. What is the largest possible value of n?

Problem 13.31. (Grey Kangaroo) A hotel on an island in the Caribbean advertises using the slogan '350 days of sun every year!' According to this advert, what is the smallest number of days Will Burn has to stay at the hotel in 2018 to be certain of having two consecutive days of sun?

(A)17 (B)21 (C)31 (D)32 (E)35

Problem 13.32. (Nederlandse Wiskunde Olympiade) At a volleyball tournament, each team plays exactly once against the other team. Each game has a winning team, which gets one point. The losing team gets 0 points. Draws do not occur. In the final ranking, only one team turns out to have the least number of points (so there is no shared last place). Moreover, each team, except for the team having the least number of points, lost exactly one game against a team that got less points in the final ranking.

a) Prove that the number of teams cannot be equal to 6.

b) Show, by providing an example, that the number of teams could be equal to 7.

Problem 13.33. (Nederlandse Wiskunde Olympiade) Thirty students participate in a mathematical competition with sixteen questions. They have to answer each question with a number. If a student answers a question correctly within a minute, he gets 10 points for that question. If a student answers a question correctly but not within one minute, then he gets 5 points for that question. And if a student answers a question incorrectly, he gets no points at all for that question. After the competition had ended, it turned out that from all the 480 answers that were given, more than half were correct and given within a minute. The number of answers that were correct but not given within a minute turns out to be equal to the number of incorrect answers. Show that there are two students with the same total score.

Problem 13.34. (Formula) The sum of three positive integers is 100. What is the minimal possible value of the least common multiple of these numbers?

Problem 13.35. (Gauss Contest — 2015.21) The numbers 1 through 25 are arranged into 5 rows and 5 columns in the table below. What is the largest possible sum that can be made using five of these numbers such that no two numbers come from the same row and no two numbers come from the same column?

1	2	3	4	5
10	9	8	7	6
11	12	13	14	15
20	19	18	17	16
21	22	23	24	25

Problem 13.36. (mathcounts) A, B, C, D and E in the decimal representations $0.ABC$ and $0.DE$ represent the digits 1, 2, 3, 4, and 5, in some order. What is the least possible absolute difference between $0.ABC$ and $0.DE$?

Skill Assessment Problems

Skill Assessment Problem 13.1. (Mos2ARSO — 2008.7.3) A New Year's garland hanging along the school corridor consists of red and blue light bulbs. Next to each red light bulb, there must be a blue one. What is the largest number of red light bulbs that can be in the garland if there are a total of 50 light bulbs?

Skill Assessment Problem 13.2. What is the largest number of three-cell corners that can be cut out from a 5×7 rectangular grid?

Skill Assessment Problem 13.3. (MechMathCircle — 2012/2013) What is the smallest number of cells on an 8×8 board that can be colored black so that there is at least one black cell in any 2×2 square?

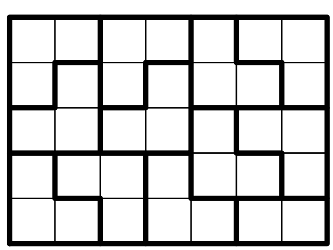
Solutions to Skill Assessment Problems

Solution to Problem 13.1: **Estimation.** Let's find the smallest number of blue light bulbs in the garland. Notice that next to each red light bulb, there must be a blue one, so three consecutive red light bulbs cannot occur. Therefore, among any three consecutive light bulbs, there must be at least one blue bulb. Then, out of the 48 blue light bulbs, there will be no fewer than $48 \div 3 = 16$. The bulbs numbered 49 and 50 cannot both be red. Thus, there must be at least 17 blue light bulbs in the garland, and consequently, there can be at most 33 red light bulbs.

Example. Let the bulbs numbered $2, 5, 8, 11, \ldots, 50$ be blue, and the rest be red. This configuration yields 33 red light bulbs in the garland. □

Solution to Problem 13.2: **Estimation.** Suppose there are 12 such corners. Then $12 \cdot 3 = 36 > 35$, so there can be at most 11 such corners.

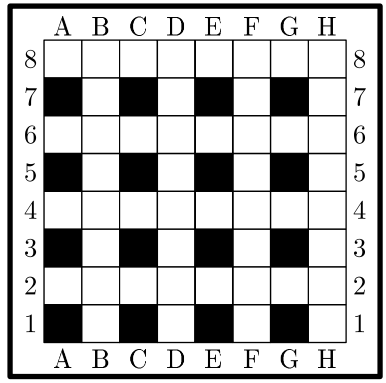
Example. The figure on the right shows an example with 11 corners.



□

Solution to Problem 13.3: **Estimation.**
Divide the original square into 16 squares of size 2×2. Each of these squares must contain at least one black cell, so there must be at least 16 black cells.

Example. Color cells $A1$, $A3$, $A5$, $A7$, $C1$, $C3$, $C5$, $C7$, $E1$, $E3$, $E5$, $E7$, $G1$, $G3$, $G5$, $G7$ black on the board (cell notation as on a chessboard). It is easy to see that 16 cells are colored black, and in any 2×2 square, there is exactly one black cell (as shown on the right). □



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