# LAUGH YOUR WAY TO A MATH JOB OFFER

By Tatiana Babicheva
Illustrated by Yorushi66

## TATIANA BABICHEVA

Having experienced technical interviews from all angles—as a candidate, an interviewer, and in countless discussions—Tatiana Babicheva knows that success often extends beyond technical expertise. While numerous guides focus on "cracking the quant interview" or "acing the FAANG technical round," these often emphasise only academic knowledge, urging candidates to rigorously review university-level mathematics. However, Tatiana has found that a simple, seemingly non-academic question can sometimes make the biggest difference.

A candidate's glaring error in logical reasoning or lack of unconventional thinking can overshadow even the deepest technical expertise. In this concise guide, Tatiana introduces classic questions and ideas commonly used as warm-up exercises in technical interviews, offering readers a fresh perspective on critical thinking in high-stakes scenarios. With eleven books published in both Russian and French, Tatiana is committed to making scientific knowledge accessible. This book, her second in English, presents a unique approach to preparing for technical interviews, one that encourages you to think beyond the surface and embrace the humour in the unexpected intersections of logic and creativity.

LAUGH YOUR WAY TO A MATH JOB OFFER
Tatiana Babicheva

# Laugh Your Way to a Math Job Offer!

**Tatiana Babicheva**

**Illustrated by yorushi66**

**The author's website: babicheva.org**

# Dedication

> To crocobeavers, kittendemons, and my imaginary friends.

> To Nicolas E., who helped me to write this book.

> To my dauther Esther who inspired me to everything.

# Introduction

In my life, I have gone through technical interviews, conducted them, and discussed them with friends. There are countless books and guides on «cracking the quant interview», «how to pass a technical interview in FAANG», and similar topics. However, most of these books focus heavily on purely technical questions and emphasize the importance of thoroughly reviewing university-level mathematics.

At the same time, in my experience, a very simple question that is not necessarily «academic» can often change the course of an interview. Indeed, a glaring error in logical reasoning or an inability to think unconventionally can, unfortunately for the candidate, outweigh all their knowledge about the stock markets.

In this very simple book, I have tried to introduce you to the most classic questions and ideas that can be used as warm-up questions in technical interviews.

And I really hope that sometimes the intersection of two flat jokes is a subtle one.

And, please, write a review if you like my book:
amazon.com/author/babicheva

# Contents

# To Think or not to Think?

> A professor of logic is riding in an elevator. The elevator stops, and someone outside asks, "Is this elevator going up or down?" The professor responds, "Yes."
>
> —One joke that didn't quite land

## 1.1 Do We Really Need To Reason Logically?

In the age of information, where data is abundant and decisions often need to be made swiftly, the importance of logical reasoning cannot be overstated. Logical reasoning is the foundation upon which sound decisions are built. It allows us to analyse situations, understand relationships between concepts, and draw accurate conclusions. But do we really need to reason logically all the time?

Imagine you're in the middle of a technical interview. The interviewer asks you to solve a problem that seems simple enough. You're confident — you've got this. But then, a curveball: the problem isn't just about writing code; it's about explaining your thought process. This is where logical reasoning steps into the spotlight. Think of logical reasoning as the GPS for your brain. Without it, you might still get to your destination, but you'll probably take a lot of wrong turns, end up in a ditch, or worse — your brain might "recalculate" indefinitely.

Let's say you're asked to design a scalable web application. Without logical reasoning, you might start by throwing together a bunch of microservices, a database, and some front-end code, hoping it all works out. With logical reasoning, you think through the system's requirements, design a robust architecture, and ensure each component interacts seamlessly. Plus, you avoid creating the digital equivalent of a Rube Goldberg machine. Nobody wants to pay for a year's worth of work for just a draft realisation of the Dijkstra algorithm.

Logical reasoning is great, but sometimes, you need a bit of creativity to spice things up. Even a monkey (okay, maybe not a monkey, but ChatGPT definitely) can implement the Dijkstra algorithm, but only a human can invent a new one.

Here's an example from my life: you're tasked with optimising an algorithm. Logical reasoning helps you understand the algorithm's current performance and identify bottlenecks. But a creative spark might lead you to an innovative approach that significantly improves efficiency and makes your code run faster. For instance, if we are considering a road network, we don't even need to try starting a line of cars from Paris to New York — they are definitely in different connectivity components! Running Dijkstra for such a line would be quite long — we'd have to traverse the entire continent to prove that there is no such path.

Let's not forget that technical interviews are also about how well you play with others. You might be a logical reasoning wizard, but if you can't explain your thought process or adapt to new information, you're like a brilliant chef who can't tell anyone else how to replicate their dishes.

For example, you're working through a problem, and the interviewer asks you to pivot your approach based on a new constraint. Suppose you need to find the shortest path in a network, but you only have predictions of congestion. Logical reasoning helps you adjust your plan: for example, you might want to use the Dijkstra algorithm with some modifications. Being adaptable and clearly communicating your new strategy shows you're more than just a coding machine — you're a team player.

So, do we really need to reason logically? Absolutely. But it's not the only tool in your kit. A well-rounded candidate combines logical reasoning with creativity, adaptability, and strong communication skills.

In this chapter, we'll explore classic questions and ideas designed to sharpen your logical reasoning skills while also sprinkling in some (maybe bad!) humour and practical examples.

## 1.2 Dancing Crocobeaver

Questions in this section are inspired by Voynarovsky's Logical Thinking Test.

The Logical Thinking Test by Miroslav Voynarovsky appeared in 2005 on the author's LiveJournal and has since become widely popular on the internet among Russian speakers. The test aims to assess your ability to distinguish between correct and incorrect logical inferences. The psychologist placed all test-takers in equal conditions, achieving experimental purity.

Special mathematical, biological, or linguistic knowledge is not required for his test; you only need to rely on your understanding and answer the question clearly. The words used in the text need no interpretation. You should focus only on the action taking place, its outcome, and the possible solutions. The logical thinking test is precisely constructed, with no metaphors or hints.

In the test, you might come across unfamiliar words like ”beaverobird”. These words are intended to assess your logical thinking ability, independent of your knowledge about the world. Assume that these words can mean anything, provided that the phrase in the condition remains logically true. For example, if it is stated that ”the beaverobird runs”, this means that the beaverobird indeed knows how to run and presumably has legs or paws; it could be a person, an animal, or a walking mechanism. Sometimes, the test includes opposite terms or expressions, such as “can” and “cannot”, “big” and “small”, etc. In all such cases, intermediate options (”can, but poorly”, ”average”) are not considered.

In the questions we propose, the components are as follows:

- **Condition** — Carefully read the question. It presents a true situation that is already proven and undisputed.
- **Inferences** — This is the statement that we need to investigate. Out of the four given inferences, there is always **exactly one correct** and **exactly one clearly incorrect**. Your goal is to identify them.

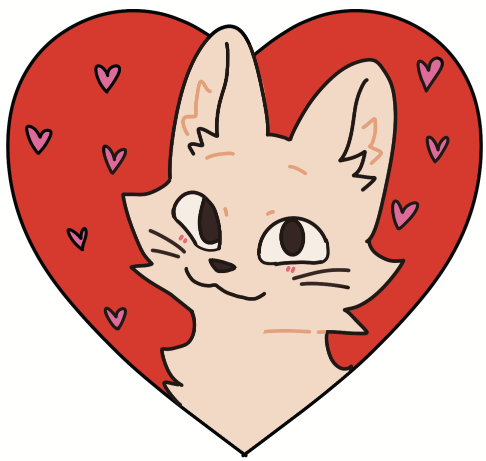
Even if you think otherwise, the questions are not related to each other.

To make this more difficult — try not to giggle even once while solving this test.

So, three, two, one… Let’s focus!

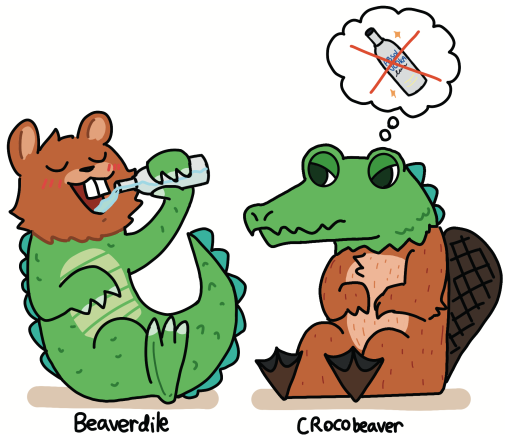
### Separate Truth and Lies

1. Not every crocobeaver can be called big.
   a) There are no small crocobeavers;
   b) Small crocobeavers exist;
   c) Every crocobeaver can be called small;
   d) Big crocobeavers will soon take over the world.
2. All crocobeavers are predators. All predators love kittens.



   a) Kittens are loved by crocobeavers;
   b) Crocobeavers hate kittens;
   c) Kittens are predators;
   d) Crocobeavers are kittens.
3. It is not true that all crocobeavers love to drink milk and dance.
   a) There is at least one crocobeaver who does not like to drink milk or does not like to dance, or does not like both;
   b) Each crocobeaver likes to drink milk and dance;
   c) Among those who love to drink milk and dance, there is at least one crocobeaver;
   d) Dancing crocobeavers love kittens.
4. There are kittenbeavers with crooked tails (all kittenbeavers have tails).
   a) Not every kittenbeaver has a straight tail;
   b) Not every kittenbeaver has a crooked tail;
   c) There are kittenbeavers with straight tails;
   d) Among those with crooked tails, there are definitely no kittenbeavers.
5. If you kiss a kittenbeaver, a storm will start. A kittenbeaver was kissed.
   a) The storm has already started;
   b) The storm will start sometime;
   c) The storm will start sometime or has already started;
   d) The storm is not correlated with the kissing of kittenbeavers.

6. If you kiss a crocobeaver, a storm will start immediately. There has been no storm in the last hour.
   a) A crocobeaver was not kissed in the last hour;
   b) A crocobeaver was kissed in the last hour;
   c) You shouldn't kiss just anyone;
   d) A crocobeaver was kissed more than an hour ago.
7. Both crocobeavers and beaverodiles live in the forest. No crocobeaver drinks vodka.




   a) There are vodka-drinking beaverodiles in the forest;
   b) There is at least one non-drinking crocobeaver in the forest;
   c) All beaverodiles in the forest drink vodka;
   d) Among those who drink vodka in this forest, there are crocobeavers.
8. It is not true that all beaverodiles are long and light-coloured.
   a) All beaverodiles belong to the set of long and light-coloured beings;
   b) There exists a beaverodile that is short or dark-coloured, or both;
   c) Every beaverodile is short or dark-coloured, but not both;
   d) A long and light-coloured beaverodile will definitely become shorter and darker.
9. If you feed a kittenbeaver, it will calm down. A calm kittenbeaver can be milked.
   a) If you don't feed the kittenbeaver, it cannot be milked;
   b) You can milk the kittenbeaver without feeding it; it will find something to eat by itself;
   c) After feeding, the kittenbeaver can be milked;
   d) The kittenbeaver cannot be milked regardless of its condition.
10. If you make a kittenbeaver happy, it will give milk. A kittenbeaver will be happy if you pull its tail.
   a) If you pull the kittenbeaver's tail, it will give milk;
   b) Everyone will be happy if you pull their tail;
   c) If you don't pull the kittenbeaver's tail, it won't give milk;
   d) Pulling the tail does not affect the kittenbeaver's condition.


11. All squirrel-parrots sometimes sing, but not always.
    a) Some squirrel-parrots never sing;
    b) Some squirrel-parrots sing very loudly;
    c) Some squirrel-parrots sometimes sing quietly;
    d) There are no squirrel-parrots that never sing.
12. The information that this book is about - funny mathematical problems - turned out to be false.
    a) The information turned out to be false;
    b) The book is not about mathematical problems;
    c) The book is not about mathematical problems, but not funny ones;
    d) There was no such information.


13. A huge blue crocobeaver scared our CEO.
    a) The CEO had a nightmare;
    b) The CEO tried some bad liquor;
    c) The CEO was scared;
    d) Huge blue crocobeavers do not exist.
14. Everyone who croaks loudly is definitely kissed. All beaverdemons constantly croak loudly.
    a) Everyone who croaks loudly is a beaverdemon;
    b) All beaverdemons are definitely kissed;
    c) Some beaverdemons are not kissed;

   d) Everyone who is kissed is a loudly croaking beaverdemon.
15. If a hencat sees an enemy, it will hide. The hencat hid.
   a) The hencat saw an enemy;
   b) The hencat is hidden now;
   c) The hencat did not see an enemy;
   d) The hencat never hides.
16. Beaverangels always search for food at night. They definitely eat something.
   a) Some beaverangels are active at night;
   b) Beaverangels never search for food during the day;
   c) Beaverangels search for food only at night;
   d) Beaverangels search for food only during the day.

17. Beaverodiles can either jump high or run fast.
   a) Some beaverodiles can jump high and run fast;
   b) No beaverodiles can jump high;
   c) If a beaverodile cannot jump high, he definitely can run fast;
   d) All beaverodiles can jump high.
18. Every hencat has at least one red spot.
   a) No hencat has blue spots;
   b) Some hencats have more than one red spot;
   c) All hencats have red spots;
   d) Some hencats have no spots at all.
19. There are problems both about crocobeavers and beaverodiles in the book.
   a) There are no problems about crocobeavers in the book;
   b) There are problems about beaverodiles in the book;
   c) There are only problems about crocobeavers and beaverodiles in the book;
   d) Beaverodiles do not exist!
20. Greomols can be either good or bad. It is not true that this greomol is not bad.
   a) This greomol is good;
   b) This greomol is mediocre;

   c) This greomol is bad;
   d) This greomol is not logical.
21. When you breathe, you always purr.
   a) If you purr, it means you are breathing;
   b) If you do not breathe, you do not purr;
   c) If you do not purr, it means you are not breathing;
   d) If you do not purr, it means you are breathing.
22. Jean stopped taking this test after answering only 22 questions.
   a) Jean got tired while taking this test;
   b) Jean got exhausted while taking this test;
   c) Jean did not finish this test;
   d) This test could have had exactly 20 questions.
23. If you hug a beaveduck, it will spoil immediately. This beaveduck is not spoiled. Now I will hug it.
   a) Do not bother the beaveduck;
   b) The beaveduck will spoil;
   c) The beaveduck will not spoil;
   d) Anyone will spoil if hugged.
24. If you hug a beaveduck, it will spoil immediately. This beaveduck has not been spoiled.
   a) The beaveduck has not been hugged;
   b) The beaveduck has been hugged;
   c) Leave the beaveduck alone;
   d) Someday this beaveduck will spoil.

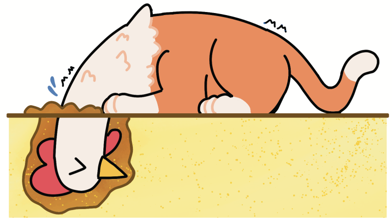
**And what were the right answers?** We hope you first tried to take the test and then check the answers or will at least honestly take it.

1. Not every crocobeaver can be called big.
   **True** Small crocobeavers exist.
   **False** There are no small crocobeavers.
2. All crocobeavers are predators. All predators love kittens.
   **True** Kittens are loved by crocobeavers.
   **False** Crocobeavers hate kittens.
3. It is not true that all crocobeavers love to drink milk and dance.
   **True** There is at least one crocobeaver who does not like to drink milk or does not like to dance, or does not like both.
   **False** Each crocobeaver likes to drink milk and dance.
4. There are kittenbeavers with crooked tails (all kittenbeavers have tails).
   **True** Not every kittenbeaver has a straight tail.
   **False** Among those with crooked tails, there are definitely no kittenbeavers.

5. If you kiss a kittenbeaver, a storm will start. A kittenbeaver was kissed.
   **True** The storm will start sometime or has already started.
   **False** The storm is not correlated with the kissing of kittenbeavers.
6. If you kiss a crocobeaver, a storm will start immediately. There has been no storm in the last hour.
   **True** A crocobeaver was not kissed in the last hour.
   **False** A crocobeaver was kissed in the last hour.
7. Both crocobeavers and beaverodiles live in the forest. No crocobeaver drinks vodka.
   **True** There is at least one non-drinking crocobeaver in the forest.
   **False** Among those who drink vodka in this forest, there are crocobeavers.
8. It is not true that all beaverodiles are long and light-coloured.
   **True** There exists a beaverodile that is short or dark-coloured, or both.
   **False** All beaverodiles belong to the set of long and light-coloured beings.

9. If you feed a kittenbeaver, it will calm down. A calm kittenbeaver can be milked.
   **True** After feeding, the kittenbeaver can be milked.
   **False** The kittenbeaver cannot be milked regardless of its condition.
10. If you make a kittenbeaver happy, it will give milk. A kittenbeaver will be happy if you pull its tail.


   **True** If you pull the kittenbeaver's tail, it will give milk.
   **False** Pulling the tail does not affect the kittenbeaver's condition.
11. All squirrel-parrots sometimes sing, but not always.
   **True** There are no squirrel-parrots that never sing.
   **False** Some squirrel-parrots never sing.
12. The information that this book is about - funny mathematical problems - turned out to be false.
   **True** The information turned out to be false.
   **False** There was no such information.
13. A huge blue crocobeaver scared our CEO.
   **True** The CEO was scared.
   **False** Huge blue crocobeavers do not exist.
14. Everyone who croaks loudly is definitely kissed. All beaverdemons constantly croak loudly.
   **True** All beaverdemons are definitely kissed.
   **False** Some beaverdemons are not kissed.



15. If a hencat sees an enemy, it will hide. The hencat hid.
    **True** The hencat is hidden now.
    **False** The hencat never hides.
16. Beaverangels always search for food at night. They definitely eat something.
    **True** Some beaverangels are active at night.
    **False** Beaverangels search for food only during the day.
17. Beaverodiles can either jump high or run fast.
    **True** If a beaverodile cannot jump high, he definitely can run fast.
    **False** Some beaverodiles can jump high and run fast.
18. Every hencat has at least one red spot.
    **True** All hencats have red spots.
    **False** Some hencats have no spots at all.


19. There are problems both about crocobeavers and beaverodiles in the book.
    **True** There are problems about beaverodiles in the book.
    **False** There are no problems about crocobeavers in the book.
20. Greomols can be either good or bad. It is not true that this greomol is not bad.
    **True** This greomol is bad.
    **False** This greomol is good.
21. When you breathe, you always purr.
    **True** If you do not purr, it means you are not breathing.
    **False** If you do not purr, it means you are breathing.
22. Jean stopped taking this test after answering only 22 questions.
    **True** Jean did not finish the test.
    **False** This test could have had exactly 20 questions.

23. If you hug a beaveduck, it will spoil immediately. This beaveduck is not spoiled. Now I will hug it.
    **True** The beaveduck will spoil.
    **False** The beaveduck will not spoil.
24. If you hug a beaveduck, it will spoil immediately. This beaveduck has not been spoiled.
    **True** The beaveduck has not been hugged.
    **False** The beaveduck has been hugged.

**What do my results mean?** Each correct answer earns you one point, so each question can give you up to 2 points. Add up your points and see where you land on the scale of brilliance:

- **45-48 points:**
  You're a logic wizard! Your reasoning skills are top-notch, and any mistakes you make are likely just due to bad luck or being tired. But remember, even geniuses can improve — if you're up for the challenge, of course.
- **35-44 points:**
  You're pretty sharp! Your logical thinking is solid, though you might trip up on the trickier puzzles. Before declaring your deductions as fact, take a moment to double-check. It's okay if someone points out an error; they're just helping you polish your skills.
- **20-34 points:**
  *Option 1:*
  You lacked the patience to complete the entire test and chose from the remaining options at random.
  *Option 2:*
  Your logical thinking is underdeveloped. If you try to reason publicly, you might be ridiculed. You will have to rely on other strengths if you want to convince someone or learn something. However, you may not be entirely hopeless if you try to improve.
- **24 points:**
  *Possible scenario:*
  You're perfectly logical but a bit absent-minded. You might have skipped marking the *definitely wrong* options.
- **8-19 points:**
  *Option 1:*
  You took the test by randomly selecting options.
  *Option 2:*
  You have no logical thinking at all. The result you achieved could be obtained by simply guessing. Do not attempt to "reason logically", especially in public. People might consider you, to put it mildly, "strange".
- **4-7 points:**
  You did not want to take the test.
- **0-3 points:**
  You're a master of logic, but you decided to have some fun and answer everything wrong on purpose. Well played!

## 1.3 Brain Teasers and Gridlock

Some problems can be grouped into a common category known as "logic puzzles". The hardest task is to explain the solution to the problem in such detail that even a first-year student can understand it ("I couldn't do it. I couldn't reduce it to the freshman level. That means we don't really understand it", R. Feynman).

One classic method of solving logic problems is by creating tables. Let's consider the following problem.

**Problem 1.1.** Three friends — Jean, Alex, and Serge — teach mathematics, physics, and literature in schools in New York, Los Angeles, and Paris. Jean doesn't work in Los Angeles, Alex doesn't work in New York, a New York resident teaches literature, a Los Angeles resident doesn't teach physics, and Alex doesn't teach mathematics. What subject does each of them teach and in which city?

*Solution.* Let's construct a table of pairwise correspondences between names, cities, and subjects. We will combine them into one table below.

| | J | A | S | NY | LA | P |
|---|---|---|---|---|---|---|
| Ma | | | | | | |
| Ph | | | | | | |
| L | | | | | | |
| NY | | | | | | |
| LA | | | | | | |
| P | | | | | | |

We'll fill in the table with facts from the problem statement: for this, we'll mark minuses in the cells where the statement is false and pluses in those where the statement is true. Our statements are mutually exclusive, as in such problems, a person cannot live (or work) in different cities simultaneously. Thus, when we put a plus in a cell, we must put minuses in the remaining cells in the $3 \times 3$ sub-table in the same row and column.


After marking a plus based on the statement that the literature teacher lives in New York, we'll fill the empty cells in the third row and the first column of the right sub-table with minuses. As soon as we put a minus after the statement "the Los Angeles resident doesn't teach physics", we realise that there is only one empty cell left in the Los Angeles column of the right sub-table, so we put a plus there. After initially filling the table out, it should look like Figure a).

| | J | A | S | NY | LA | P |
|---|---|---|---|---|---|---|
| Ma | | − | | − | + | − |
| Ph | | | | − | − | + |
| L | | | | + | − | − |
| NY | | − | | | | |
| LA | − | | | | | |
| P | | | | | | |

a) After initially filling it out

| | J | A | S | NY | LA | P |
|---|---|---|---|---|---|---|
| Ma | − | − | + | − | + | − |
| Ph | − | + | − | − | − | + |
| L | + | − | − | + | − | − |
| NY | + | − | − | | | |
| LA | − | − | + | | | |
| P | − | + | − | | | |

b) Solved problem

We already have a lot of information: the literature teacher lives in New York, the mathematics teacher lives in Los Angeles, and the physics teacher lives in Paris.

Let's reason further. Alex doesn't live in New York, Alex isn't a mathematician, the mathematician lives in Los Angeles $\rightarrow$ Alex doesn't live in Los Angeles $\rightarrow$ Alex lives in Paris. We put a plus. Now we know that Jean lives in New York and Serge lives in Los Angeles. Combining our knowledge, we get the final table from as in Figure b). $\square$

### Around so-called Einstein's Puzzle

The following puzzle is often credited to Albert Einstein on many websites. He allegedly concocted it in the early 1900s, claiming that only 2% of the population could solve it. (Is this figure still true today? Was it ever true?)

However, the puzzle made its first public appearance in Life International magazine on December 17, 1962, seven years after Einstein's passing.

This puzzle, known in English-speaking literature as the Zebra Puzzle, is sometimes linked to Lewis Carroll as well. Yet, there's no solid proof that either Einstein or Carroll created it. Plus, the Life International version references cigarette brands that weren't around during Carroll's lifetime or Einstein's youth.

What is the best-known problem setting?

The following version of the puzzle appeared in *Life International* in 1962:

**Problem 1.2.** In a posh Madison Avenue bar in New York, one stranger accosts another with a mimeographed sheet of paper and the question, "Have you seen this?" In university dormitories, the problem is tacked to the doors. In suburban households in Westchester, Long Island and Connecticut, the ring of the telephone is likely to herald a voice that asks: "Is it the Norwegian?" The cause of the excitement is the brain teaser reproduced on this page, with an illustration provided by Steve Cook. The facts essential to solving the problem, which can indeed be solved by combining deduction, analysis and sheer persistence, are as follows:

1. There are five houses.
2. The Englishman lives in the red house.
3. The Spaniard owns the dog.
4. Coffee is drunk in the green house.
5. The Ukrainian drinks tea.
6. The green house is immediately to the right of the ivory house.
7. The Old Gold smoker owns snails.
8. Kools are smoked in the yellow house.
9. Milk is drunk in the middle house.
10. The Norwegian lives in the first house.
11. The man who smokes Chesterfields lives in the house next to the man with the fox.
12. Kools are smoked in the house next to the house where the horse is kept.
13. The Lucky Strike smoker drinks orange juice.
14. The Japanese smokes Parliaments.
15. The Norwegian lives next to the blue house.

Now, who drinks water? Who owns the zebra?

In the interest of clarity, it must be added that each of the five houses is painted a different colour, and their inhabitants are of different nationalities, own different pets, drink different beverages and smoke different brands of American cigarettes. One other thing: In Statement 6, *right* means *your right.*

Life International will be glad to receive answers from its readers and will publish one or more of those which best combine, in the editors' judgment, the proper solution with brevity and clarity in expounding the logic by which the solution was reached. No intuitive answers, please.

— *Life International*, December 17, 1962

As a note, like Vassberg and Vassberg remarked in their article "Is Einstein's Puzzle Over-Specified?" from 2009, in the originally published form, hint 12 is poorly written. It states, "Kools are smoked in the house next to *the* house where the horse is kept." It should read, "… smoked in *a* house next…" Since the word "the" implies in-the-singular, Kools would have to be smoked in either the second or the fourth house in the row, and this leads to a contradiction.

### Other logic grid puzzles

One can construct an infinite number of similar problems, but with lesser (or greater?) complexity, by reducing the number of parameters to consider. There are even online generators for such logic puzzles, which are called "intégrammes" in French or "logic grid puzzles" in English.

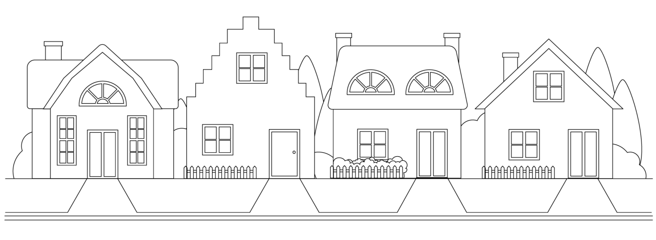
Here is an example of a simplified version of the Zebra puzzle, which seems better suited for a job interview:

**Problem 1.3.** On a street, there are four neighbouring houses of four different colours. In each house lives a person of a different nationality; each of the four owners has a different profession. The following clues are provided:

1. There are two houses between the grey house and the house of the computer scientist.
2. The Swede lives in the black house.
3. The Englishman lives in house number four.
4. The Dane lives to the right of the banker.
5. The Belgian lives immediately to the right of the Swede.
6. The astronomer lives in the blue house.

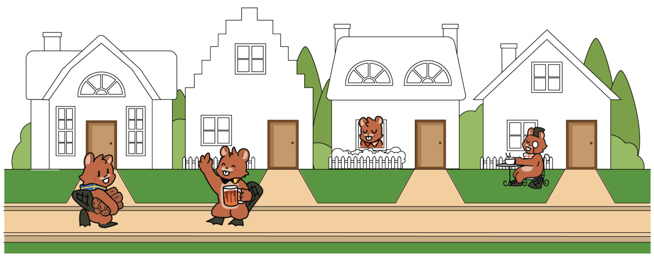
Where is the white house?

*Solution.* How can we solve this problem without constructing huge tables? Let's start by arranging the four houses:



The Englishman lives in the fourth house.

The Dane lives to the right of the banker, and the Belgian lives to the right of the Swede. Neither of these people can be in the leftmost house, so the Swede lives in the leftmost house. The Belgian lives immediately to the right of the Swede, so the Dane lives in the third house. Let's note this:



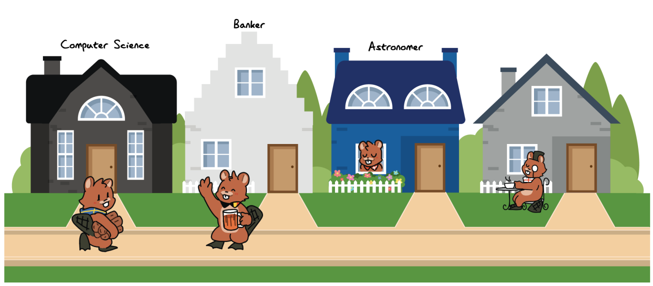
To avoid confusion, let's only keep the conditions that we haven't fully used yet:

1. There are two houses between the grey house and the house of the computer scientist.
2. The Swede lives in the black house.
3. The Dane lives to the right of the banker.
4. The astronomer lives in the blue house.

Note that the Swede lives in the black house. From the first condition, the grey house must be either house number 1 or 4. But house number 1 is black, so the grey house is house number 4. Hence, the computer scientist lives in the first house.

The Dane lives to the right of the banker. Since the computer scientist lives in the first house, the banker lives in the second house.

The astronomer lives in the blue house, which must be the third house. Therefore, the white house is house number 2.



□

However, let's get back to our zebras. No one can rule out the possibility that during an interview, you might be given a sheet of paper with a problem where you instantly recognise the Zebra puzzle. Excitedly, you might say, "The zebra belongs to the Japanese!" And that's when the interviewer will realise that you didn't even bother to read the modified clues of the puzzle. Awkward, isn't it?

And yes, it's unlikely that you will actually have to solve the entire puzzle. Usually, it's enough to start reasoning logically for the interviewer to move on to the next question.

— *Can you really calculate faster than a computer?*
— *Yes.*
— *What is* $25$ *times* $25$*?*
— *Nine!*
— *That's incorrect...*
— *But it's fast!*

## 1.4 Knights and Knaves

Another important puzzle topic involves knights (truth-tellers), knaves (liars), and spies (who can tell both truths and lies, possibly with certain restrictions). Let's consider a problem on this topic.

**Problem 1.4.** Out of three people, $A$, $B$, and $C$, one is a knight, another is a knave, and the third is a spy. $A$ said: "I am a spy". $B$ said: "$A$ and $C$ sometimes tell the truth". $C$ said: "$B$ is a spy". Who is the knave, who is the knight, and who is the spy?

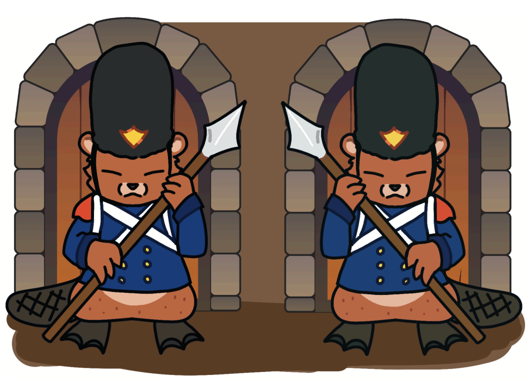
*Solution.* $A$ says that they are a spy, so they cannot be a knight because, in that case, they would lie.

Let's assume $A$ is a knave. Then they never tell the truth. Therefore, $B$ lied when they claimed that $A$ and $C$ sometimes tell the truth because if they meant that both sometimes tell the truth, they were lying. As such, $B$ is a spy. Thus, $C$ is a knight, and $C$ tells the truth. This option fits. But we must check all possible options.

Let's assume $A$ is a spy. Then $C$ definitely lies because $B$ cannot also be a spy. So, $C$ is a knave because we already have a spy. Then $B$ is a knight. But $C$ never tells the truth, so $B$ lied, hence they are not a knight. It's a contradiction.

Thus, we have obtained the only answer: $A$ is a knave, $B$ is a spy, $C$ is a knight. □

Sometimes, it is up to us to decide which questions to ask.



**Problem 1.5.** You are faced with two doors. One door leads to your dream job offer, and the other is the Moon Door. In front of each door is a guard. One guard always tells the truth. The other always lies. You can ask one question to determine which door is the correct one (you don't like flying!). What will you ask?

*Solution.* To determine which door leads to your job offer, you can ask either guard the following question:

"If I were to ask the other guard which door leads to the job offer, which door would they point to?"

Here's the reasoning behind the question.

If you ask the truth-telling guard this question, they will truthfully tell you what the lying guard would say. Since the lying guard would point to the wrong door, the truth-teller would point to the wrong door as well.

If you ask the lying guard this question, they will lie about what the truth-telling guard would say. Since the truth-teller would point to the correct door, the liar would point to the wrong door.

In both cases, the answer you get will be the door that does not lead to the job offer. Therefore, you should choose the other door. □

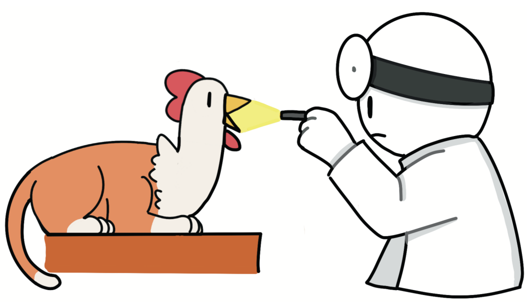
## 1.5 Prove Me Wrong

In the previous section, we used the idea "let's assume that we have a contradiction." This approach requires the ability to construct a proper negation, which we have practised before.

The name of this method ("Proof by Contradiction") essentially speaks for itself. It is a particular case of a method known as "reductio ad absurdum" (Latin), or "reduction to absurdity".

If we need to prove a certain statement $A$, we will assume that it is false, meaning the negation of $A$ is true. After that, we need to arrive at a contradiction, which would mean that our assumption was incorrect and the problem is solved.

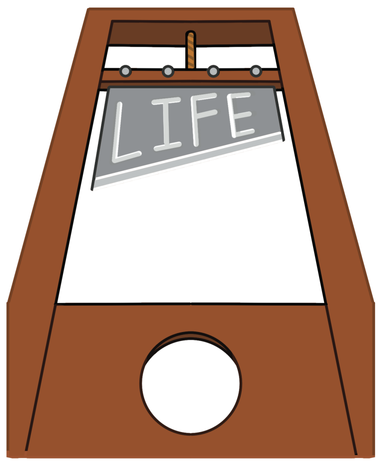
This method of proof is based on the law of double negation in classical logic. The law of double negation is a principle underlying classical logic, according to which "if it is not the case that not $A$, then $A$ is true". This principle is also called the law of double negation elimination.



We often encounter reasoning based on proof by contradiction in real life. Suppose, at some point, you didn't want to go to an annual work gala and pretended to be sick. You told your doctor that your throat hurts. He looked at your throat and said, "If your throat really hurt, it would be red. It's not red, so it doesn't hurt". Okay, this example might not be perfect, as doctors assume adults won't lie for such a silly reason, but I hope you get what I mean.

This principle can be very useful in solving mathematical problems because it allows us to establish the truth of a statement by demonstrating that its negation leads to a contradiction.

One classic example involves a cruel king who wanted to execute his minister without up-setting the kingdom's people. He announced that he, being a kind king, would forgive the minister and trust in their ancient gods. He wrote "death" and "life" on two identical-looking pieces of paper, and the minister would choose one. Depending on what was written on it, his fate would be decided. The minister, being wise, realised that both papers said "death", yet he managed to survive by swallowing the chosen paper. To determine his fate, the people looked at the remaining paper, which read "death". This proved that he was fortunate to have picked the paper with "life" written on it.




**Problem 1.6.** Integer points on a line are coloured red and blue. Prove that there is a segment with both ends and the midpoint coloured in the same colour.

*Solution.* Let's assume the opposite, i.e., let's assume that such a segment does not exist. Consider all points with even coordinates except $0$. Either among them, there is a point of the same colour as $0$, or they are all of a different colour than $0$. In the second case, we immediately come to a contradiction by considering the points $2$, $4$ and $6$. So, we have points of the same colour as $0$ and $2x$, where $x \in \mathbb{Z}$. Then $-2x$, $x$, and $4x$ must be of a different colour. But $x$ is the midpoint of the segment with ends $-2x$ and $4x$, leading to a contradiction. □

**Problem 1.7.** Among any ten of sixty animals, there are at least three of the same species. Prove that among all of them, there are $15$ animals of the same species.

*Solution.* Let's assume the opposite, i.e., let's assume that the first part of the problem setting is met even if there are no more than $14$ representatives of each species. Let's call animals that represent their species alone "singles", in contrast to "groups", where there will be at least 2 animals. Since the first part of the problem setting must be met, there can be no more than 4 "groups"; otherwise, we could select a group of 10 animals with 2 of each species. On the other hand, if there are no more than 3 groups, then there will be at least $60-3\cdot 14 = 18$ singles, and by choosing 10 singles, we immediately arrive at a contradiction. Thus, there are exactly 4 groups, in which case there are at least $60-4\cdot 14 = 4$ singles, and by choosing 2 animals from each group and 2 singles, we arrive at a contradiction. Contradictions obtained in all possible distributions of animals complete the solution to the problem. □

**Problem 1.8.** Ten friends sent each other holiday postcards, so that each sent 5 postcards. Prove that there are two friends who sent postcards to each other.

*Solution.* Let's assume the opposite, i.e., that such people do not exist. This would mean that in any pair of people, only one card was sent or none at all. There are a total of

$$10 \cdot 9 \cdot \frac{1}{2} = 45$$

pairs, which means that no more than 45 cards were sent. However, there were exactly $10 \cdot 5 = 50$ cards sent in total — a contradiction. □

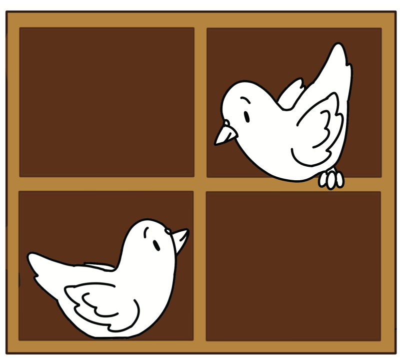
*One day, you'll ask me who I love more — you or mathematics. I'll answer: "Let's assume it's mathematics..." And you'll leave, never knowing that it was proof by contradiction.*

## 1.6 Stuffing Pigeons in Cubbies

Although the pigeonhole principle first appeared as early as $1624$ in a book attributed to Jean Leurechon, it is commonly referred to as Dirichlet's box principle, Dirichlet's drawer principle, or just the Dirichlet principle.

The solutions to problems using the pigeonhole principle usually rely on the method of proof by contradiction.

Typically, students engaged in mathematical olympiads encounter the pigeonhole principle in the early years of secondary school. It can be stated as follows: "If there are $N$ pigeonholes and at least $N + 1$ pigeons, then there must be at least two pigeons in one of the pigeonholes". Let's use the method of proof by contradiction – suppose "this is not the case", meaning there are **less than** two pigeons in each pigeonhole, i.e., $1$ or $0$ pigeons. Then, in $N$ pigeonholes, the maximum number of pigeons would be $N \cdot 1 = N$, which is less than $N + 1$.



A natural generalisation is the following statement: "If there are $N$ pigeonholes and at least $kN + 1$ pigeons, then there must be at least $k + 1$ pigeons in one of the pigeonholes."

In reality, you are unlikely to encounter a problem where you actually have to place pigeons in pigeonholes. In each specific problem, you need to understand what plays the role of pigeons and what plays the role of pigeonholes.

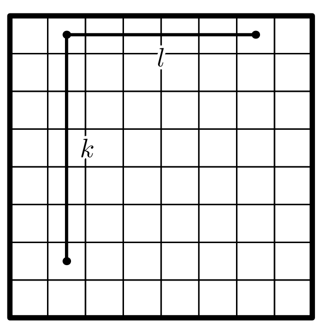
**Problem 1.9.** Prove that in any company, there will be two people with the same number of acquaintances from within that company.

*Solution.* Let's find the "pigeons" and "pigeonholes". It's not hard to guess that people will be the "pigeons" and the number of acquaintances will be the "pigeonholes". Each person can have from $0$ to $n-1$ acquaintances, where $n$ is the number of people in the company. So, the pigeonholes will have numbers from $0$ to $n-1$. It seems like there are equal numbers of "pigeons" and "pigeonholes", so why doesn't the pigeonhole principle work here? It does but with a slight modification. Notice that in pigeonholes $0$ and $n-1$, pigeons cannot sit simultaneously. Indeed, then there would be a person who is not acquainted with anyone and a person acquainted with everyone, which is impossible. Therefore, we actually have $n-1$ non-empty pigeonholes. That is, the pigeonhole principle still works in this case.

This contradiction completes the solution of the problem. □

**Problem 1.10.** In an $8 \times 8$ grid, integer numbers are placed such that any two numbers in adjacent cells differ by no more than $4$. Prove that among these numbers, there are at least $2$ equal ones.

*Solution.* Suppose the opposite. Therefore, in the grid, there are no two equal numbers — all numbers are different. Let's consider the smallest and largest of them — they will differ by at least $63$. But one can reach from one of them to the other through adjacent cells in at most $k+l \leq 7+7=14$ moves (see the figure below).




Since the difference between adjacent numbers is no more than $4$, making $\leqslant 14$ moves, the difference will be no more than $4 \times 14 = 56$. This contradiction completes the proof of the problem. □


**Problem 1.11.** Your drawer contains 3 red socks, 101 yellow socks, and 13 blue socks. Being a busy and absent-minded STO of the company, you randomly grab a number of socks out of the drawer and try to find a matching pair. Assuming each sock has an equal probability of being selected, what is the minimum number of socks you need to grab in order to guarantee a pair of socks of the same colour?

*Solution.* When you have 3 colours (3 pigeonholes), by the pigeonhole principle, you will need to have $3 + 1 = 4$ socks (4 pigeons) to guarantee that at least two socks have the same colour (2 pigeons share a hole). □

## 1.7 You Need Few Little Grey Cells

Let's solve a few more problems, similar to those that my friends and I have encountered in interviews or in books for interview preparation. Of course, these problems can be formulated in completely different ways, but the ideas are always the most important.

**Problem 1.12.** A prankster drink machine offers three options — Unicorn Elixir, Dragon Brew, or Random Potion, but the machine has been bewitched so that none of the buttons dispense what they claim. If each magical drink costs you one leg, how many legs must you sacrifice to figure out which button dispenses which mythical beverage?

*Solution.* In my opinion, the best solution here would be to not spend any legs and just walk away from such a machine on your own two feet. But let's assume you have no choice, and your legs might grow back after figuring out the correct solution since these are magical drinks after all.

Let's take a Random Potion. Since it is labelled as random, it definitely is not random. Suppose we get Unicorn Elixir. Then, in the column where it says Dragon Brew, it must be random, and in the Unicorn Elixir column, only Dragon Brew remains. If we get Dragon Brew, the solution is determined similarly. We only spent one leg, hooray! We can still hop away from those who make us solve such problems. □

**Problem 1.13.** You've got four mysterious cards displaying $007$, $42$, $AI$, and $FF$. The rumor is: if there's at least one vowel on one side of a card, then there has to be an even number on the other. You can flip some cards to uncover the truth. Which cards do you flip, and why?

*Solution.* Let's recall the laws of mathematical logic. From truth follows truth, and from falsehood follows anything. So, we need to flip $007$ (in case there is a vowel, which would cause a contradiction), and we also need to flip $AI$ (in case there is an odd number). The other cards don't make a difference.

By the way, this kind of problem is classic for the beginning of the first term in colleges like Caltech. □

**Problem 1.14.** Three people counted a pile of balls of four colours. Each of them correctly distinguished two colours, and two others could be confused: Dmitry confused green and red, John confused red and violet, and Tatiana violet and blue. The results of their counts are given in the table. How many balls of each colour were there, actually?

| | green | red | violet | blue |
|---|---|---|---|---|
| some person | 2 | 5 | 7 | 9 |
| another person | 2 | 4 | 9 | 8 |
| and one more person | 4 | 2 | 8 | 9 |

*Solution.* When you solve this problem, you will understand the world I am living in. I do confuse violet and blue, and my brother Dmitry confuses green and red. I don't have friends who confuse red and violet, but let's say that it is our imaginary friend.

Only Dmitry confuses green with something else. So, two people do not confuse green with anything, and Dmitry is definitely the third person. The actual number of green balls is $2$. Dmitry will give the correct counts for violet and blue balls. Therefore, there are $8$ violet balls and $9$ blue balls.

Only John confuses blue with something else, so the actual number of blue balls is indeed $9$, and John is the second person. John will give the correct counts for green and red balls, so there are $4$ red balls. We have uncovered all the secrets. □

*An engineer, a physics professor and a mathematician travel by train. The engineer says, after seeing a black sheep:*
*— Sheep are black in Scotland.*
*The physics professor says:*
*— You can only state that some sheep are black in Scotland.*
*And the mathematician says:*
*— You can only state that there is at least a sheep that has at least one black side.*

# Are you enjoying the book so far?

Your quick, honest review helps us immensely and only takes a minute!

Use your phone to scan the QR code and go directly to the Amazon review page.

SCAN ME

Thank you for taking the time—it truly means the world to us!

# To do or not to do?

> The logician's wife sends him to the store and says, "Get some bread, and while you're there, pick up some eggs." The logician never came back.
>
> —One joke that didn't quite land

## 2.1 When Do We Need to Construct Something?

In maths interviews, you might be asked to come up with an algorithm or some complex construction. Don't worry; it's not because the interviewers are trying to relive their traumatic school days. More likely, they just want to see if you can think logically and handle tasks you might encounter on the job.

Sometimes, it seems like these problems have nothing to do with real work. That's true if you're asked to pour water from one container to another or make sure a goat doesn't eat the cabbage while you ferry a wolf across a river; it's just an exercise in logic and strategy. This logical and strategic thinking is what sets you apart from "soulless machines" that are already pretty good at handling many tasks.

These problems show how you approach solving issues, how you break them down, and how you find the best solutions. It's like playing a strategy game where every move affects the outcome. Future employers value the ability to foresee moves and plan actions.

For instance (yes, I work in the transportation sector), consider the problem of finding the shortest path on a road network. Of course, we could just use Dijkstra's algorithm. But now, imagine you have a website with millions of users, and the road network is that of Europe. Dijkstra's algorithm isn't very suitable here. However, we can precompute. We break our problem into two parts: precompute and then find the shortest path on the precomputed graph. In transportation, one of the most popular precomputation methods is Contraction Hierarchies (we won't go into the details), which has its own stages. You can optimise something at each stage.

It's also important to be able to provide examples and counterexamples of how things work (or don't work). For instance, at one stage, we might use the Ford-Fulkerson maximum flow algorithm, but then we must be able to give an example where this classic algorithm never converges.

In this chapter, we won't dive into the technical details of the questions. Instead, like in the first chapter, we'll solve puzzles and problems that don't require specific knowledge but do require a clever idea or just an understanding of the type of problem.

## 2.2 Is it Acheron or Styx?

Let's start our section with the most famous puzzle on this topic.

**Problem 2.1.** A man needs to ferry a wolf, a goat, and a giant cabbage across a river in a boat. The boat can carry only the man and either the wolf, the goat, or the cabbage at one time. If the wolf is left alone with the goat, the wolf will eat the goat. If the goat is left alone with the cabbage, the goat will eat the cabbage (or at least nibble on it). However, in the man's presence, no one eats anyone else. The man successfully ferries his load across the river. How does he do it?

*Solution.* We won't ask why the man needs a wolf. Let's just solve the puzzle.

Let's try taking the wolf first. But as soon as the boat leaves with the wolf, the goat, left unattended, will eat the cabbage.

If we try taking the cabbage first, the wolf will eat the goat.

Therefore, we must take the goat first. Next, we return alone to the original side. Now, we take the wolf.

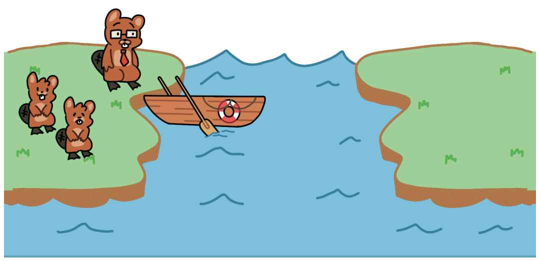
We cannot leave the goat alone with the wolf. Returning with the wolf doesn't help since we just brought it over. Therefore, we return with the goat.

Next, we take the cabbage across the river. Finally, we return to fetch the goat and take it across.

Everyone has crossed safely, and the puzzle is solved! □

These puzzles are known as **river crossing puzzles**. They come with various conditions but always involve moving people or objects from one side of something to the other.

**Problem 2.2.** A teacher approached a river that needed crossing. The bridge was damaged, and the river was deep. Suddenly, the teacher noticed two boys, Misha and Jean, in a small boat. However, the boat was so tiny that it could carry only one adult or two boys at a time! Yet, everyone managed to cross the river using that boat. How did they do it?



*Solution.* If the teacher takes the boat and crosses, the boys will be left without the boat. So, first, we ferry the two boys.

One boy, Misha, returns with the boat.

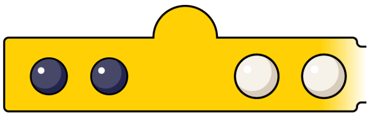
The teacher then crosses the river.

Now, Jean returns to fetch Misha, and they both cross to the right side.

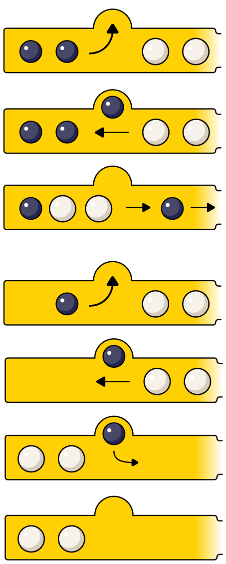
Everyone has crossed safely, and the puzzle is solved! □

Not all puzzles involve rivers.

**Problem 2.3.** In a narrow and very long channel, there are four balls: two black ones on the left and two white ones of slightly larger diameter on the right. In the middle of the channel, there is a small niche in the wall where only one ball (any) can fit. Two balls can be placed side by side across the channel only at the niche. The left end of the channel is closed, but at the right end, there is an opening through which any black ball can pass, but not a white one. How can all the black balls be rolled out of the channel without removing any balls?



*Solution.* We will provide the solution through pictures:



□

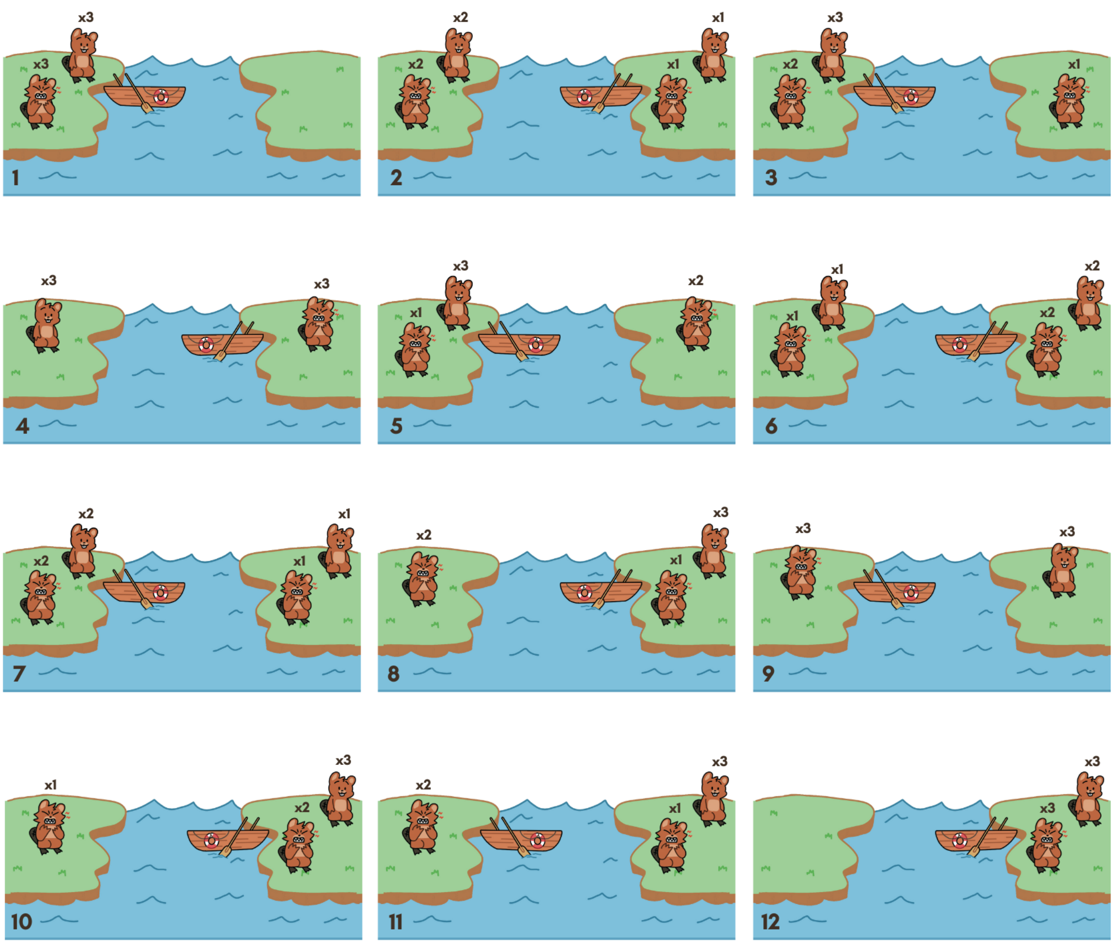
Now, let's answer a tricky question.

**Problem 2.4.** Two travellers come to a river. On the riverbank, they find a boat that can carry only one person. Nevertheless, they both manage to cross the river and continue their journey. How could this be?

*Solution.* This is possible if they approach the river from opposite banks. The first traveller takes the boat and crosses to the other side, and then the second traveller takes the boat and crosses. □

**Problem 2.5.** Three missionaries and three cannibals need to cross a river. They have one boat that can hold only two people. To avoid a tragedy, there can never be more cannibals than missionaries together. How can they cross the river?

*Solution.* Let's illustrate the solution to the problem:




□

All these problems did not consider the time needed to cross anything; we had only capacity constraints. Let's add a time dependency.

**Problem 2.6.** One night, the author's family gathered on one side of a bridge: baby Esther, cat Sophie, Tatiana herself, and her husband Alexey. They want to get to the other side, but here's the catch: the bridge is old and rickety, so only two individuals can cross at a time. And, as luck would have it, they only have one flashlight.

Each of them moves at a different speed. Esther, crawling at a turtle's pace, takes $10$ minutes to cross and cannot be carried because she will scream and attract all the surrounding werewolves. Sophie, who takes $5$ minutes, stumbles in a sinusoidal pattern. The tired Tatiana takes $2$ minutes to cross the bridge, while the cheerful Alexey runs across in just $1$ minute. If two go together, they will move at the speed of the slower person since there's only one flashlight, and it must light the way for both.

How can they all get across the bridge in $17$ minutes?

*Solution.* Just to clarify, throwing the flashlight is not allowed!

First, Alexey and Tatiana briskly cross the bridge together, which takes $2$ minutes. Then Alexey, being the fastest, returns with the flashlight, adding another $1$ minute.

Now for the toughest part: Sophie and Esther, our slowest, cross the bridge. To clarify any doubts: the cat is holding the flashlight in its mouth. This takes $10$ minutes. Tatiana, who has been resting on the other side, returns with the flashlight, taking another $2$ minutes.

Finally, Alexey and Tatiana cross the bridge together again, taking the last 2 minutes.

Thus, everyone manages to cross the bridge in 17 minutes. Well done, team! □

## 2.3 Advent it and Do it

In some problem settings, we are just asked to provide an example or construct a structure with specified properties. Presenting the solution usually does not pose any problems — it is sufficient to provide the required example and, perhaps, explain why it fits. Proving that it is the "most elegant" or the only solution is not needed.

You may be asked to provide an example related to practically anything, from arithmetic constructions to geometric or logical ones. In real-life job interviews, you can be asked to give an example or construction related to your expertise.

**Problem 2.7.** Write the number 2024 using ten twos and arithmetic operations.

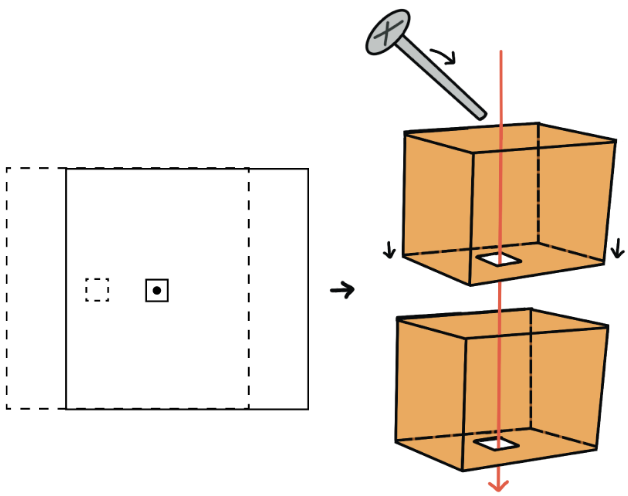
*Solution.* In such problems, it is usually acceptable not to put arithmetic operation signs between numbers, which allows us to obtain, for example, the number 2222 using four twos.

$$2024 = 2222 - 222 + 22 + 2.$$

□

**Problem 2.8.** Little Jean decided to play as a builder and cut out two identical shapes from cardboard. He placed them overlapping each other on the bottom of his rectangular box, and the bottom ended up being completely covered. For extra security, he hammered a nail into the centre of the bottom. The question is: could this nail pierce one piece of cardboard without piercing the other?

*Solution.* Imagine that the cardboard pieces initially lie on top of each other with their edges aligned. If the pieces are square, by sliding one along the other and overlapping them, it's easy to form a rectangle, but its centre will lie inside both pieces. But this can be fixed! Notice that the centre of the resulting rectangle does not coincide with the centre of the first square piece of cardboard. Therefore, if you cut a small hole in the centre of the rectangle in the first piece of cardboard, a nail hammered into this hole can pierce only one piece of cardboard. And that's how our little builder can continue his experiments with cardboard without worrying about the nails!



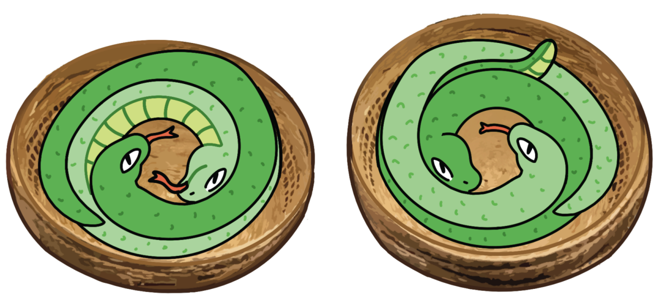
□

## 2.4 From Planning to Action

This topic is a logical continuation of the previous one. However, while we previously needed to provide a "static" example, in this topic, we are looking for a "dynamic" example — one that describes a sequence of actions that can lead us from the initial position to the desired one.

The process, from the Latin "processus", means a step or progress. Problems related to algorithms and operations involve step-by-step execution of certain actions (usually a limited number of them). The solution to such problems is reduced to initiating a particular process — an algorithm, a sequence of steps — and it is important to demonstrate that it will be finite and lead to the desired result.

**Problem 2.9.** In parliament, no one has more than three enemies. Prove that the parliament can be divided into two chambers so that each deputy in their chamber will have no more than one enemy.



*Solution.* Let's demonstrate the organisation of the division process step by step. We will assign deputies to chambers one by one. Initially, we randomly divided the parliament into two chambers. Then, we find a deputy who has at least two enemies in their chamber and move him to the other chamber. The total number of enemy pairs sitting in the same chamber decreases. The process will be finite and stop because the number of hostile pairs is no more than the total number of people in the parliament multiplied by three (in fact, even less, but this is graph theory). The termination of the process means the construction of the desired division. □

Unfortunately, erroneous reasoning sometimes occurs. For the above problem, for example, one might give the following incorrect “solution”.

*Wrong solution.* Place all deputies in the hall. Choose one deputy and send them to chamber $A$. Then, choose a deputy who has no enemies in $A$ (as long as such deputies exist) and send them to $A$. Repeat this iteration until there are no such deputies left. Then, choose a deputy from the hall who has exactly one enemy and send them to $A$. Repeat this several times until such deputies are exhausted. Then, assign the deputies who have three enemies in $A$ to chamber $B$. Since they do not quarrel with each other, this will not contradict the condition. Deputies remaining in the hall have exactly two enemies in $A$. But then in $B$, they have no more than one enemy, and they can be sent to $B$. □

The mistake in the last “solution” is as follows: by sending deputy $X$ to $A$, who has one enemy $Y$ in $A$, and sending $Z$, who also has one enemy (let it again be $Y$), we end up with $Y$ having two enemies in $A$.

*A mathematician is asked to solve a problem: “Given a gas stove, a water tap, and a kettle, the task is to boil water.”*
*“This is easy,” he replies. “First, we pour water into the kettle. Then we light the fire and place the kettle on the stove.”*
*“Okay, now a new task,” they say to him. “The task is to boil the kettle with water already poured into it.”*
*“Well, this is even easier! We pour the water out of the kettle and reduce the task to the previous one.”*

**Problem 2.10.** Three stakes are stuck into the ground. Two are bare, while the third holds disks with diameters of 3, 2, and 1 decimeters stacked from bottom to top. The following is permitted: taking the top disk from one stake and placing it onto another, with the condition that only a disk of smaller diameter can be placed on top of a disk with a larger diameter. The task is to move all the disks to another stake in the same order using these operations.

*Solution.* Represent the solution in the form of a table, where we will display the disks placed on each stake at each step (the leftmost digit denotes the top disk).

| | | | |
|---|---|---|---|
| Step 0 | 123 | | |
| Step 1 | 23 | 1 | |
| Step 2 | 3 | 1 | 2 |
| Step 3 | 3 | | 12 |
| Step 4 | | 3 | 12 |
| Step 5 | 1 | 3 | 2 |
| Step 6 | 1 | 23 | |
| Step 7 | | 123 | |

The result obtained at the last step completes the solution to the problem. □

This last problem is known as the ”Tower of Hanoi” and is widely used in different areas of life. For example, it is used in psychological research on problem-solving. In the 2011 film Rise of the Planet of the Apes, this puzzle, called in the film the ”Lucas Tower”, is used as a test to study the intelligence of apes. I hope you are able to pass it.

## 2.5 The Birth of a Counterfeiter

If you have participated in puzzle competitions before, you may have encountered weighing puzzles.

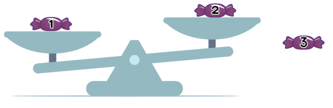
They usually involve identifying an object that differs in weight from the others within a finite number of weighings. The search is carried out by comparing individual elements as well as groups of elements, either among themselves or with weights of a certain value. Different types of scales may be used, each with its own capabilities. The simplest case is using balance scales, which allow for comparing objects or groups of objects with or without given weights.

Here's a simple problem on this topic.

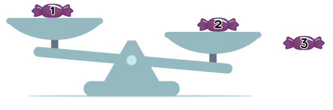
**Problem 2.11.** There are 3 externally identical candies, one of which is tasteless (lighter because it lacks filling). How can you identify the tasteless candy using balance scales without weights in just 1 weighing?

*Solution.* Let's number the candies and place candy number 1 and candy number 2 on different sides of the balance scales. There are three possible cases.

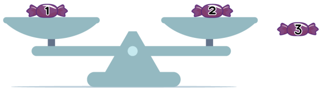
1. If the side with candy number 1 is heavier, then candy number 2 is tasteless.




2. If the side with candy number 2 is heavier, then candy number 1 is tasteless.



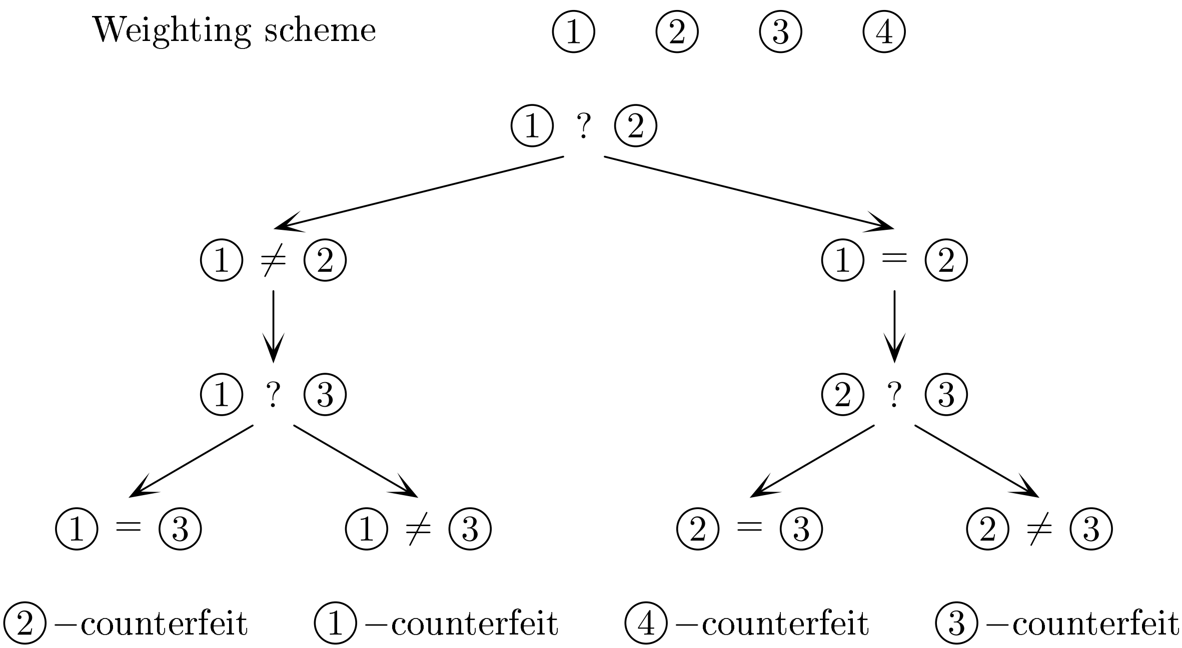

3. If the scales are balanced, then candy number 3 is tasteless.



The solution to the problem is complete after considering all possible cases. Remember to consider all possible cases; missing even the most obvious case can cost you your job offer. □

**Problem 2.12.** Among four coins, exactly one is counterfeit (and it is unknown whether this coin is lighter or heavier than the genuine ones). How can you identify it using the balance scales twice without weights?

*Solution.*




One of the methods for presenting the solution to such problems is a diagram of possible actions. □

Furthermore, weighing problems can also be given for ordinary scales that digitally display weight.

**Problem 2.13.** The king was presented with the annual tribute – one sack of gold coins from each of the 10 provinces of his kingdom, but the secret service reported that one of the sacks contains counterfeit coins weighing 9 grams each (while all the others contain genuine coins weighing $10$ grams each). The king has scales that show the exact weight, which he can use only once. How can the king find the sack with the counterfeit coins? (There are a large number of coins in each sack.)

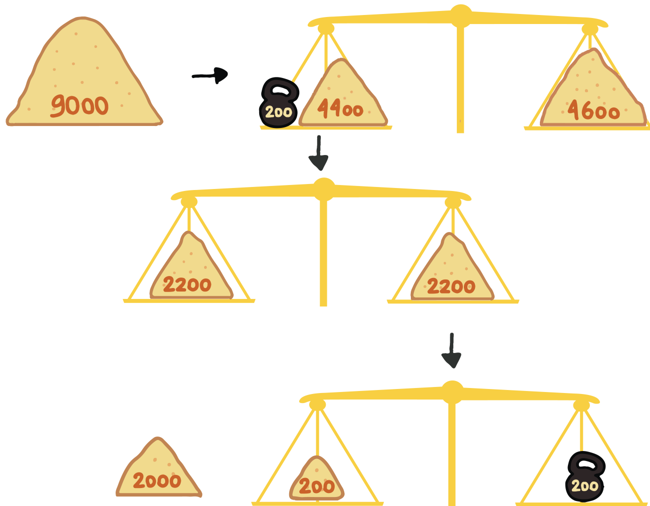
*Solution.* To solve this problem, we can rely on the contradiction between the expected weight (if all coins were genuine) and the actual weight due to the counterfeit coins. For example, we can take a different number of coins from each sack: $1$ coin from the first sack, $2$ from the second, and so on. If there were no counterfeit coins, the scales would show a weight of $550$ grams $\big((1+2+\ldots+10)\cdot 10\big)$. However, the presence of counterfeit coins will reduce it to $550-x$, where $x$ is the number of the sack with the counterfeit coins. □

If you encounter a type of problem you haven't seen before, don't worry; just give it a try!

**Problem 2.14.** There are $9$ kg of grain and balance scales with a $200$ gram weight. How can you measure exactly $2$ kg of grain using the scales only $3$ times?

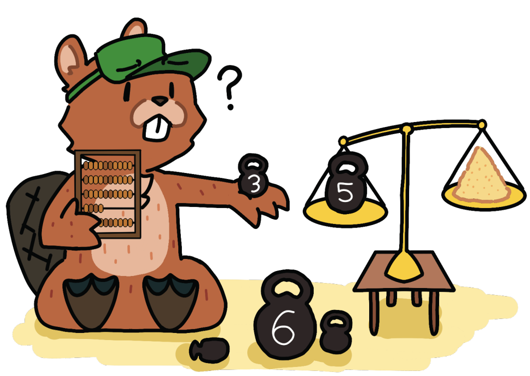
*Solution.* Take a pile of grain weighing $m$. We can obtain two piles weighing either $m/2$, $m/2$ (pouring onto two pans without using the weight), or $m/2-100$, $m/2+100$ (the same, but with the weight on one pan).

We have $9000$ g of grain. Divide the grain into piles of $4400$ and $4600$ g. Now "halve" $4400 \to 2200+2200$. The last weighing is to "subtract" $200$ g from one pile. Thus, we have used three weighings.




□

**Problem 2.15.** The grain keeper beaver has 6 weights marked 1, 2, 3, 4, 5, and 6 kg. However, he suspects that the markings on two of the weights may be swapped. He cannot determine this information by visual inspection due to the weights being made of different materials. How can he determine whether the markings are correct using the balance scales twice, where any groups of weights can be compared?




*Solution.* Be careful! In this problem, the goal is not to determine which 2 weights are swapped, but simply to determine whether the markings are correct or not.

For the first weighing, place the weights marked "6" and "1" on one side of the scales, and the weights marked "2" and "5" on the other side. If the scales are not balanced, it means the markings are definitely swapped, and the second weighing is unnecessary. If the scales are balanced, then the possible situations are:

- weights “6” and “1” are swapped;
- weights “2” and “5” are swapped;
- weights “3” and “4” are swapped.

For the second weighing, place the weights marked “6” and “2” on one side, and the weights marked “5” and “3” on the other. Reasoning similarly to the first weighing, if the scales are balanced, the possible situations are:

- weights “6” and “2” are swapped;
- weights “3” and “5” are swapped;
- weights “1” and “4” are swapped.

Thus, the possible error scenarios are inconsistent, so with these two weighings, we can guarantee whether the markings on any 2 weights are swapped or not. □

**Problem 2.16.** You have 12 candies that are identical externally. One of the candies is heavier or lighter than the rest (you don’t know which and why; it can be without the filling or with a clue inside; you don’t want to test it). Using just a balance that can only show you which side of the tray is heavier, how can you determine which candy is the bad one with no more than 3 measurements? You are asked to both identify it and determine whether it is heavy or light (is it dangerous or just not tasty?)

*Solution.* The key, as always, is to divide the original group (and any intermediate subgroups) into three sets instead of two. The reason for this is that comparing the first two groups will always provide information about the third group.

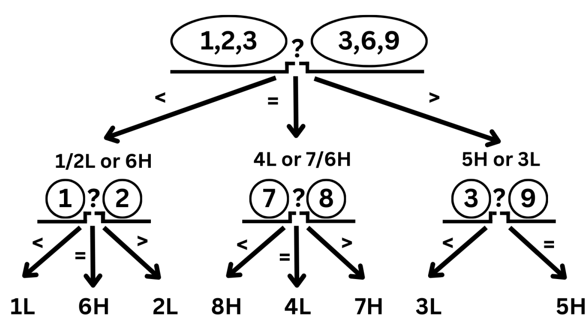
How much information can one weighing provide? One "ternary bit", as we have three possible outcomes: "$>$", "$<$", and "$=$". The information about the different candy includes $24$ possibilities, as the candy can either be lighter or heavier than the normal ones. Therefore, we need at least three weighings, because $3^2 = 9 < 24 < 27 = 3^3$. We should always divide the option space into three groups. For the first weighing, we can aim to have $8$ options for each weighing result.

Considering that the solution is wordy to explain, you can draw a tree diagram to show the approach in detail. Label the candies $1$ through $12$ and separate them into three groups with $4$ candies each.

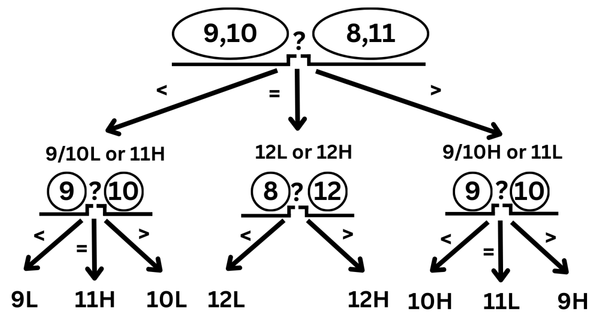
Weigh candies $1$, $2$, $3$, and $4$ against candies $5$, $6$, $7$, and $8$. Then, explore two possible scenarios: either the two groups balance, or one of the groups, without loss of generality, consisting of candies $1$, $2$, $3$, and $4$, is lighter than the group consisting of candies $5$, $6$, $7$, and $8$.

Consider the case of inequality. Either one of the candies $1$—$4$ could be lighter than the correct weight, or one of the candies $5$—$8$ could be heavier than the correct weight. What is certain is that candies $9$—$12$ are normal.

Let's demonstrate the remaining weighings:



In the case of equality, the scenario is considered similarly.



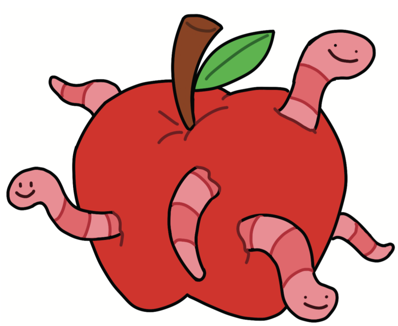

□

**Problem 2.17.** What is the smallest number of integer weights required to exactly balance every integer between $1$ and $40$. Prove it.

*Solution.* In this problem, we need to provide both an example and proof of why fewer weights won't suffice. This is an "estimate+example" type of problem, which we will discuss in more detail later.

For this problem, we can use weights of $1$, $3$, $9$, and $27$ units. For example, a weight of $20$ grams can be balanced with weights of $3$ and $27$ on one side, and weights of $1$ and $9$ on the other. Why can we represent all weights this way? Each weight can be placed in three positions: right, left, or not used. Thus, we have $3^4 = 81$ potential weights. All weights, except zero, are accounted for twice: once when placed on the right side and once on the left side. No two weights coincide. This also proves the minimal number of weights: as $3^3 = 27$ is less than the $40$ possible weight values. □

*Once Euclid was asked:*
*"What would you prefer — two whole apples or four halves?"*
*"Four halves," replied Euclid.*
*"But isn't that the same thing?"*
*"Of course not. By choosing halves, I will immediately see whether these apples are wormy or not."*



## 2.6 Moonshiners and Mages

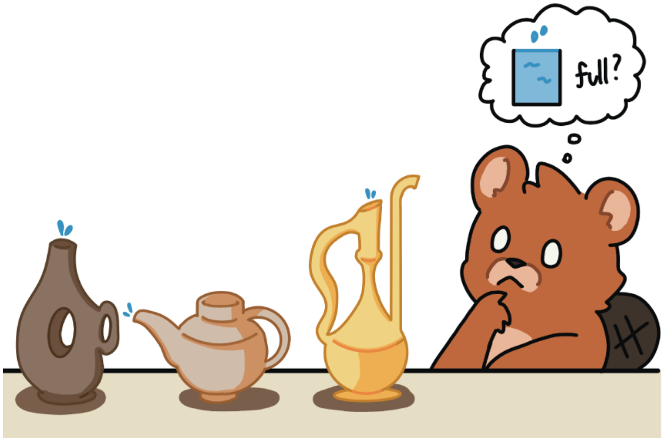
One puzzle that raises many questions for those unfamiliar with the context is the water jug puzzle. According to a popular story, the famous French mathematician, mechanic, and physicist Siméon Denis Poisson (1781—1840) solved similar problems in his youth and later claimed that it was this problem that motivated him to become a mathematician.

This topic is also referred to as decanting problems or just jug-pouring problems. Such a problem was used in the action film Die Hard: With a Vengeance (1995). The characters John McClane and Zeus Carver (played by Bruce Willis and Samuel L. Jackson) solve the variant with two jugs and water from a public fountain to try to prevent a bomb from exploding. They obtain $4$ gallons of water using only $5$-gallon and $3$-gallon jugs.

**Problem 2.18.** Jean has $12$ litres of vodka and wants to drink half of it. But he only has $8$-litre and $5$-litre jugs. How can he measure out $6$ litres?

*Solution.* A typical solution from someone unfamiliar with this topic might be: "Jean should pour half of the twelve-litre container into the eight-litre one. $12 : 2 = 6$, so we have $6$ litre."

Unfortunately, this solution is not correct. As is often the case, the problem statement does not specify what the authors consider obvious — namely, that all the jugs are opaque and of such shape that it is impossible to see how full they are.




The only actions available are pouring all the contents from one jug into another or pouring in as much liquid as the container can hold.

The standard method for presenting solutions to such problems is to construct a table showing the amount of liquid in each container at each step.

| Container Size | 12 | 8 | 5 |
|---|---|---|---|
| Step 0 | 12 | 0 | 0 |
| Step 1 | 4 | 8 | 0 |
| Step 2 | 4 | 3 | 5 |
| Step 3 | 9 | 3 | 0 |
| Step 4 | 9 | 0 | 3 |
| Step 5 | 1 | 8 | 3 |
| Step 6 | 1 | 6 | 5 |

This can be interpreted as follows.

In the first step, we completely fill the 8-litre vessel by pouring vodka from the 12-litre vessel. This leaves $12 - 8 = 4$ litres of vodka in the 12-litre vessel.

In the second step, we pour as much as possible from the 8-litre vessel into the 5-litre vessel. The 5-litre vessel can hold 5 litres, so the 8-litre vessel will have $8 - 5 = 3$ litres of vodka remaining. Etc., etc...

In the end, we pour 2 litres from the 8-litre vessel, leaving $8 - 2 = 6$ litres in the 8-litre vessel, which we can finally drink. □

Consider the following problem.

**Problem 2.19.** The evil wizard Croc-o'Beaver is brewing a potion. In the final stage, he must add exactly 4 millilitres of dead water. Unfortunately, he only brought flasks of 3 millilitres and 5 millilitres to the dead water spring. What should Croc-o'Beaver do to finish brewing his potion?

*Solution.* The solution to this problem is very similar to the previous one. The only difference is that the volume of one of the vessels — the spring — is unlimited. We can choose to include or not include this "virtual" vessel with $\infty$ capacity in our table.

| Vessel Size | $\infty$ | 3 | 5 |
|---|---|---|---|
| Step 0 | $\infty$ | 0 | 0 |
| Step 1 | $\infty$ | 0 | 5 |
| Step 2 | $\infty$ | 3 | 2 |
| Step 3 | $\infty$ | 0 | 2 |
| Step 4 | $\infty$ | 2 | 0 |
| Step 5 | $\infty$ | 2 | 5 |
| Step 6 | $\infty$ | 3 | 4 |

When solving pouring problems, it is important to consider that repeating one of the previous steps at any point clearly overcomplicates things and goes in the wrong direction. □

By Bézout's identity from the Number Theory (we will revise it a bit later), such puzzles have a solution if and only if the desired volume is a multiple of the greatest common divisor of all the integer volume capacities of jugs.

Dudeney, in the "Amusements in Mathematics" book, provides an interesting historical account of the "pouring problems" or "Tartaglia's measuring puzzles".

The first printed puzzle involving the measurement of a specific quantity of liquid by pouring between vessels of known capacities was introduced by Niccolò Fontana, famously known as "Tartaglia" (the stammerer, $1500$—$1559$). His puzzle involves dividing $24$ ounces of valuable balsam into three equal parts using only vessels with capacities of $5$, $11$, and $13$ ounces. There are numerous solutions to this puzzle, each requiring six manipulations or pourings between the vessels.

**Problem 2.20.** The evil wizard Croc-o'Beaver is brewing a potion. In the final stage, he must add exactly $3$ millilitres of dead water three times in a row. Unfortunately, this time, he only brought a flask with $9$ millilitres of dead water to his tower, and he didn't have time to run to the spring again. He found empty vials of $5$, $4$, and $2$ millilitres volume at his place. What should Croc-o'Beaver do to finish brewing his potion?

*Solution.* The table will look like this.

| Container / Container Size | Flask 9 | Test Tube 1 5 | Test Tube 2 4 | Test Tube 3 2 |
|---|---|---|---|---|
| Step 0 | 9 | 0 | 0 | 0 |
| Step 1 | 7 | 0 | 0 | 2 |
| Step 2 | 3 | 0 | 4 | 2 |
| Step 3 | 3 | 4 | 0 | 2 |
| Step 4 | 0 | 4 | 3 | 2 |
| Step 5 | 4 | 0 | 3 | 2 |
| Step 6 | 6 | 0 | 3 | 0 |
| Step 7 | 1 | 5 | 3 | 0 |
| Step 8 | 1 | 3 | 3 | 2 |
| Step 9 | 3 | 3 | 3 | 0 |

The complexity of the problem lies in the unusual formulation. ”Three times in a row” means that Croc-o’Beaver doesn’t have time to measure out 3 millilitres after he has already added a portion to the potion. So, it is necessary to distribute the potion into the test tubes so that each of them contains exactly three millilitres at the same time. A solution where three millilitres are obtained several times and then poured into the potion is incomplete. Once we understand the problem, it’s not difficult to solve. □

*A physicist, a mathematician, and a mystic were asked to name the greatest invention of all time. The physicist chose fire, which gave humanity control over matter. The mathematician chose the alphabet, which gave humanity control over symbols. The mystic chose the thermos.*
*“Why the thermos?” the others asked.*
*“Because the thermos keeps hot liquids hot in winter and cold liquids cold in summer.”*
*“Yes, and so?”*
*“Think about it,” said the mystic reverently. “How does that little bottle know?”*

## 2.7 It is not the Squid Game

It is usually quite simple to understand that a problem is about a game: it is explicitly stated in the problem statement. Typically, there are two players in the game.

By considering the moves, one understands certain actions allowed by the game's rules, and generally, the execution of a move is mandatory. Moves are made in turns. With very rare exceptions, at some point, the game ends — after a certain, pre-known number of moves or due to some position reached in the game. After that, the result of the game is determined: victory for one of the players or a draw. However, draws in games that may appear in puzzles and interview questions are not so common. The question in such problems is usually formulated as follows: what will be the result if both sides play correctly? It is said that player $A$ wins with optimal play by both sides or wins forcibly if there is a strategy that allows them to make moves in such a way that the result of the game will be a win for player $A$ in any case. It is easy to understand that in this case, there is no strategy for player $B$ to guarantee that they will not lose. It is said that with optimal play by both sides, the game ends in a draw if each player has a strategy that results in a draw or their own victory in any case.

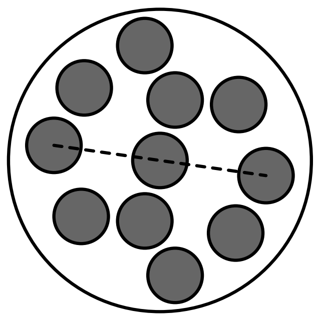
During the game or when trying to formalise the solution, the concept of a "best" move is often applied. But there is a problem: in most games, it is almost impossible to clearly define what this "best" move means without analysing the position to the very end. For example, in chess, there is the concept of a "sacrifice of a piece", which at first glance may seem like a "bad" move because the material is lost. But in perspective, this move can turn out to be quite "good" as it will lead to a position with a higher evaluation in a few moves.

Thus, after making a move, it is often impossible to immediately determine whether it is the "best" one. Therefore, this concept can only be applied in games where we can calculate the opponent's moves to the very end, and in all other cases, one should actively avoid this slippery concept.

One of the most common strategies is the **symmetric strategy** and its variations. In this case, the player with the winning strategy adheres to some symmetry: for each move of the opponent, they respond with a symmetric move.

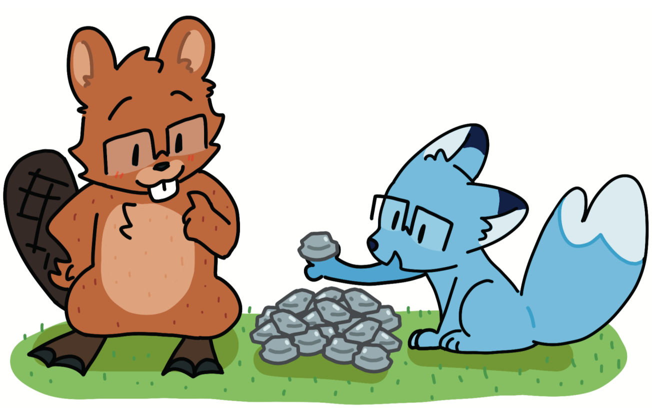
**Problem 2.21.** Suppose there is a round table and an unlimited number of identical round coins. Two players take turns putting coins on the table, and they cannot place them on top of each other. The player who has no place to put a coin loses. Who wins with optimal play?

*Solution.* Let's present a winning strategy for the first player, i.e., we will guide their moves. As the first move, we place a coin in the very centre of the table. Following any move made by the second player, we will respond with a move that is centrally symmetric, i.e., symmetric with respect to the centre of the table (shown in the figure below).



Let's prove that such a strategy will work: it is sufficient to show that the first player can always make a move. Due to the strategy, after the first player's move, the arrangement of coins on the table will always remain centrally symmetric. This means that if the second player puts a coin somewhere on the table, a centrally symmetric place remains free — and it is exactly where we will place the coin. □

Let's give an example of another problem where symmetry is less pronounced.



**Problem 2.22.** Suppose there are two piles, each containing 20 stones. On each turn, a player can take any number of stones, but only from one pile. The player who cannot make a move loses. Who wins with optimal play?

*Solution.* Let's prove that the second player has a winning strategy, i.e., we will guide their moves. Following any move made by the first player, we will take the same number of stones from the other pile. Then, after any move made by the second player, the number of stones in both piles will be equal. This means that after any move made by the first player, the number of stones in the piles will be different, and the second player will be able to make their move.

It is easy to see the solution to this problem when the number of stones in the piles is unequal — in this case, the first player wins, who by their first move equalises the number of stones, reducing the problem to the previous one.

As you can notice, the symmetry in the above problem is no longer geometric: the symmetry was in the equality of the piles. □

When speaking about each player making the "best" move or following the "correct" strategy, it's worth noting that in the latter case, the only correct move for the first player is indeed equalising the number of stones in the piles, while all other moves would be incorrect. Of course, one might argue that "what if the other player makes a mistake, I can still win even by playing incorrectly", and even provide concrete examples like the 2006 World Chess Championship match between Kramnik and Topalov, where both players made a mistake by missing mate. However, this would not be relevant to mathematics — in the mathematical model describing the game, there is no room for error and the emotional factors inherent in chess.

*A programmer and a web designer are walking down the street. Suddenly, a brick falls right in front of them, then another.*
*Tetris! — thought the programmer.*
*A drop-down menu! — thought the web designer.*

# To Happen or not to Happen?

> “Two students come to a professor’s house to take an exam in combinatorics. It’s late, and the professor lets them stay the night.
> The students share one room, the professor and his wife are in another, and the professor’s daughter is in a third room. One student wakes up, decides to visit the professor’s daughter, and sneaks into the room with one head visible. Meanwhile, the professor wakes up, decides to check on his daughter, and ends up in a room with one head. The second student wakes up and follows the same path.
> In the morning, the professor finds himself alone in the students’ room, a student with his daughter, and another student with his wife. Scratching his head, he mutters, “I’ve been teaching combinatorics for years, but I’ve never seen such a chaotic permutation!”
> Note from the Author: Technically speaking, this is not a permutation.
>
> —One joke that didn’t quite land

## 3.1 The Art of Unexpected Arrangements

In the tech industry, combinatorics and probability are crucial tools for solving complex problems. These concepts often surface in interviews, where they test your ability to handle real-world scenarios in fields such as transportation or computer science.

Imagine you're managing public transportation in Paris during the Olympic Games. You have $5$ buses, but each driver has a $0.1$ probability of deciding to go on strike. Passengers will be very unhappy if fewer than $3$ buses are running because they won't fit in the remaining ones. What's the probability that you'll have to drive a bus yourself? Here, you need to calculate the probability that $2$ or fewer buses are operational. This is a binomial probability problem.

Or picture this: you've celebrated your favourite soccer team's World Cup win a bit too enthusiastically. You're standing one step away from the Seine River. For each step, you randomly move either towards or away from the edge, with a probability of $2/3$ of stepping away and $1/3$ of stepping towards the edge. What are your chances of taking an unexpected swim in this pristine river?

These problems not only test your mathematical prowess but also your ability to apply theory to real-world situations. Throughout this chapter, we'll delve into some puzzles, enhancing your combinatorial and probabilistic thinking.

However, don't think we'll be tackling anything overly complex. We're still warming up, not trying to master all of mathematics in one book!

## 3.2 Plus, Minus, and Everything in Between

In combinatorics problems, we typically need to find the number of ways to accomplish a task. This could involve counting ways to form a number with specific properties, seating people around a table under certain conditions, and more.

Let's begin by reviewing two simple rules that lead to many other formulas:

1. Sum Rule: If object $A$ can be chosen in $n$ ways and object $B$ can be chosen in $m$ ways, then choosing object $A$ **or** $B$ can be done in $m + n$ ways.

2. Product Rule: If object $A$ can be selected in $n$ ways, and object $B$ can be selected in $m$ ways, then selecting both object $A$ **and** $B$ can be done in $m \cdot n$ ways.

If you happen to forget this topic, when solving a problem, you should verbalise to yourself each time whether you need to use "**and**" or "**or**", and multiply or add accordingly.

There's another important rule:

3. Division Rule: If each option has been counted $n$ times during the calculation, then the result should be divided by $n$.

Some combinations of objects occur most frequently and have specific names: permutations, arrangements, and combinations.

**Definition 1.** Let there be a set containing $n$ elements. Any ordered set composed of all the elements of this set is called a *permutation* of this set.

**Definition 2.** Let there be a set consisting of $n$ elements. Any ordered set composed of $k$ different elements of this set is called an *arrangement of $k$ elements from $n$*, partial permutation, or sequence without repetition.

**Problem 3.1.** In a soccer team, there are 11 players. How many ways can you choose a captain and an assistant?

*Solution.* The captain can be chosen from any of the $11$ players. The assistant can be chosen from any of the remaining $10$ players. Therefore, the captain and the assistant can be chosen in $11 \cdot 10 = 110$ ways. □

**Claim 1.** The number of arrangements of $k$ elements from $n$, denoted $A_n^k$, is equal to $\frac{n!}{(n-k)!}$.

*Proof.* The first position can be filled in $n$ ways, the second position in $(n-1)$ ways, and so on, until the $k$-th position, which can be filled in $(n-k+1)$ ways. Thus,

$$A_n^k = \underbrace{n \cdot (n-1) \cdot \dots \cdot (n-k+2) \cdot (n-k+1)}_{k}.$$

This formula can be written more compactly by multiplying and dividing the right-hand side by $(n-k)!$

$$A_n^k = \frac{n(n-1)(n-2)\dots(n-k+1)\cdot(n-k)!}{(n-k)!} = \frac{n!}{(n-k)!}$$

Recall that $n! = 1 \cdot 2 \cdot \ldots \cdot n$, and by definition, $0! = 1$. □

A permutation is a simple, special case of an arrangement, but it's important enough to warrant its own consideration.

**Problem 3.2.** How many five-note melodies can be formed from the notes `do`, `re`, `mi`, `fa`, `sol` of the first octave if the notes must not repeat?

*Proof.* For the first note, there are five choices, for the second note, there are four choices, for the third note, there are three choices, for the fourth note, there are two choices, and for the last note, there is one choice. Thus, the total number of melodies is:

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120.$$

□

*A programmer visits a friend's house to check out his new piano. He walks around it, hums, then says:*
*The keyboard is inconvenient — only* $84$ *keys, half of them functional, none labeled; but... pressing shift with your foot — that's original!*

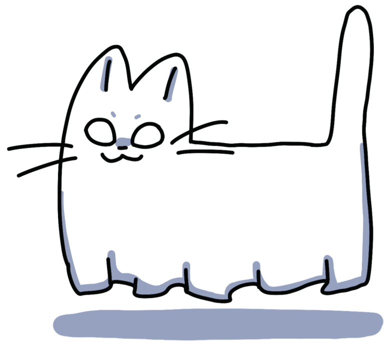
**Claim 2.** The number of permutations of $n$ elements, denoted $P_n$, is equal to $n!$.

The claim can be rephrased as finding the number of ways to arrange $n$ distinct elements in $n$ different positions.

**Problem 3.3.** How many words, not necessarily meaningful, can be formed by:

- a) Rearranging the letters in the word "ghost"?
- b) Prohibiting the substring "host" in the words formed in part a)?

*Solution.* a) All the letters in "ghost" are unique, and there are $5$ of them. Therefore, the number of words is the number of permutations of $5$ elements: $5!$



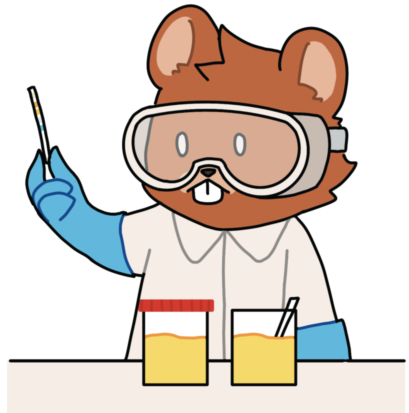
b) To solve this, first count the number of words containing the substring "host". Treat "host" as a single new letter. We then have 1 letter plus "host", giving us 2! permutations. Therefore, the number of words containing "host" is 2!. Subtract this from the total number of words to find the desired count: $5! - 2!$ □

Arrangements with repetitions are used when there are several elements (or a large number of elements of a few types), and we need to choose a certain number of them in a specific order, where the same element can be chosen multiple times. Here's a classic problem on this topic:

**Problem 3.4.** How many ways can you form a five-letter word using the 26 letters of the alphabet? Any combination of letters is considered a word.

*Solution.* The first thing to consider in such problems is whether the order in which we choose the elements matters. Clearly, the order of letters in a word matters (e.g., "cat", "tac", and "act" are three different words), so we need to allocate positions for the letters. The first position can be filled by any of the 26 letters. The second position can also be filled by any of the 26 letters, as letters can repeat. Therefore, by the multiplication rule, the number of ways to choose the first two letters is $26 \cdot 26$. The process is the same for the remaining three positions. Thus, the total number of five-letter words is $26^5$. □

Let's move on to considering combinations. Let's return to our soccer team, where we were choosing a captain and an assistant.



**Problem 3.5.** In a soccer team of $11$ players, how many ways can two players be chosen for a doping test?

*Proof.* At first glance, it seems like the situation is similar to choosing a captain and an assistant: we choose the first person in $11$ ways and the second in $10$ ways, so there are $11 \cdot 10$ ways in total. However, this is not the case here.

In fact, the pair "captain and assistant" is ordered: choosing Leo as captain and John as an assistant is not the same as choosing John as captain and Leo as an assistant. On the other hand, the pair of people sent for a doping test is unordered: sending Leo and John for the test is exactly the same as sending John and Leo for the test. Accordingly, in this problem, we are interested in the number of unordered pairs of players chosen from $11$ people.

Imagine that the unordered pair {Leo, John} is glued from two ordered pairs (Leo, John) and (John, Leo). In other words, any two ordered pairs differing only in the order of the elements give the same unordered pair. Consequently, the number of unordered pairs will be half the number of ordered pairs and is equal to

$$\frac{11 \cdot 10}{2} = 55.$$

Thus, two players can be chosen for the doping test in $55$ ways. □

**Definition 3.** Let there be a set containing $n$ elements. Any unordered set consisting of $k$ different elements of this set is called a *combination of* $n$ *choose* $k$.

In other words, a combination of $n$ elements taken $k$ at a time is simply a $k$-element subset of an $n$-element set. The number of combinations of $n$ elements taken $k$ at a time is denoted $C_n^k$ or $\binom{n}{k}$. Let's find what this number equals.

**Claim 3.** The number of combinations of $n$ choose $k$ is equal to $\frac{n!}{k!(n-k)!}$.

*Proof.* We find the number of combinations $C_n^k$ using the rules of multiplication and division. The first object can be chosen in $n$ ways. The second object can be selected in $n-1$ ways, since one object has already been chosen. The next can be selected in $n-2$ ways, and so on. The last, $k$-th object, can be chosen in $n-k+1$ ways. All these numbers need to be multiplied because we are choosing the first object **and** the second, **and** the third, and so on. Alternatively, the number of ordered sets of $k$ elements (i.e., the number of chains of length $k$) is the number of arrangements $A_n^k$.

Next, those chains that differ only in the order of the elements are glued into one unordered set. The number of such chains is equal to the number of permutations of $k$ elements, i.e., $k!$. Therefore, the required number of unordered sets of $k$ elements will be $k!$ times less than the number of chains of length $k$:

Thus, we get $C_n^k = \frac{n\cdot(n-1)\cdot\dots\cdot(n-k+1)}{k!} = \frac{A_n^k}{k!} = \frac{n!}{k!(n-k)!}$. □

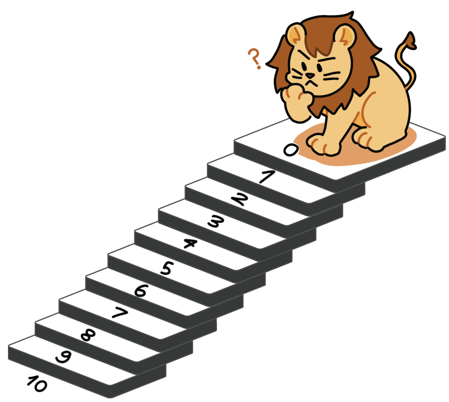
As such, $C_n^0 = 1$; $C_n^1 = n$; $C_n^2 = \frac{n(n-1)}{2}$.

Now, knowing what a combination is, we can immediately say that two soccer players out of eleven can be chosen for a doping test in $C_{11}^2 = \frac{11\cdot 10}{2!}$ ways.

## 3.3 Order or Anarchy? Order of Anarchy!

**Problem 3.6.** There are 7 boxes numbered from 1 to 7. How many ways can you distribute 25 identical balls into these boxes so that no box is empty?

*Solution.* Let's line up all 25 balls. To determine which balls go into which boxes, we need to place 6 dividers between the balls. This will divide the balls into 7 groups: the balls in the first group go into the first box, the second group into the second box, and so on. There are 24 gaps between the balls where the dividers can be placed. Since no two dividers can occupy the same gap (otherwise, one box would be empty), the number of ways to place the dividers is $C_{24}^{6}$. □




**Problem 3.7.** Leo noticed that his staircase has 10 steps (he starts at step 0 and needs to get to step 10). How many ways can Leo descend the staircase if:
a) He can skip any number of steps?
b) He can only step to the next step or skip one step?

*Solution.* a) Leo must get to the last step. For the remaining 9 steps, he can either step on them or skip them. Therefore, for each of the 9 steps, there are 2 choices. Thus, there are $2^9 = 512$ ways to descend the staircase.

b) Let's solve a more general problem: suppose the staircase has $n$ steps. If $n = 1$, there is only one way to descend. If $n = 2$, there are 2 ways: stepping on the intermediate step or skipping it. Let $X_n$ be the number of ways to descend a staircase with $n$ steps. We showed that $X_1 = 1$ and $X_2 = 2$.

To find $X_n$, consider Leo's first step:

1. He can step to the next step, leaving $n - 1$ steps to descend, which can be done in $X_{n-1}$ ways.

2. He can skip the next step, leaving $n - 2$ steps to descend, which can be done in $X_{n-2}$ ways.

Thus, $X_n = X_{n-1}+X_{n-2}$. From this, $X_3 = X_2+X_1 = 2+1 = 3$; $X_4 = X_3+X_2 = 3+2 = 5$, resulting in the sequence:

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

So, Leo has $89$ ways to descend the staircase. □

The numbers obtained in this problem are shifted Fibonacci numbers.

*Definition from the dictionary for mathematicians: Recursion (noun) — see recursion.*

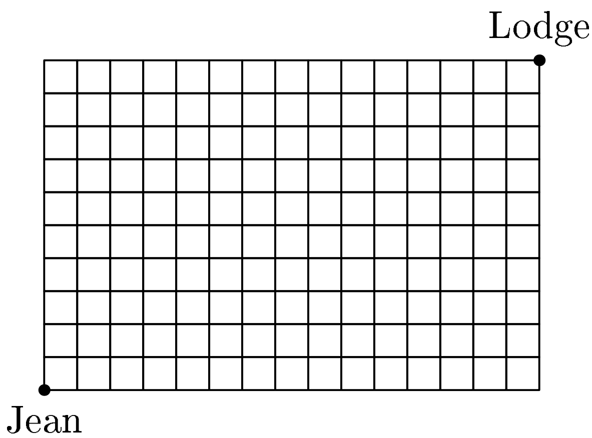
Now, let's prove one of the basic properties of binomial coefficients.

**Problem 3.8.** Prove that $C_n^k = C_n^{(n-k)}$.

*Solution.* For each $k$-element subset $A$ of an $n$-element set $M$, we can uniquely correspond its complement, which is an $(n - k)$-element subset consisting of all elements not in $A$. Therefore, the number of $k$-element subsets of the set $M$ is equal to the number of its $(n-k)$-element subsets; the former is $C_n^k$, and the latter is $C_n^{n-k}$.

In easier terms, choosing $k$ elements is the same as choosing $n - k$ elements to exclude, so the number of ways to choose the first set is equal to the number of ways to choose the second set. □

**Problem 3.9.** Consider a grid of integers. Let there be a beaver named Jean, whose lodge is located at coordinates $(16, 10)$. Jean can move either to the right or upward to an adjacent node. Now, he is at $(0, 0)$ point. How many ways can he reach his home?




*Solution.* How can Jean move? Notice that his home is 16 steps to the right and 10 steps "up" from his starting point. If Jean moves one step up first, he will have 16 and 9 steps left respectively. If he decides to move horizontally first, he will have 15 and 10 steps left. It should be obvious that the beaver must make 16 horizontal and 10 vertical moves in total, in any order. Each sequence defines a unique path. Thus, we have a sequence of 26 actions, from which we need to choose 10 to be vertical moves (or 16 to be horizontal moves). This gives us:

$$C_{26}^{10} = C_{26}^{16}.$$

□

**Problem 3.10.** Now suppose Jean wants to stop by a birch tree at coordinates $(10, 7)$ to get some materials for his lodge. How many ways can he choose his path home now?

*Solution.* In this scenario, Jean must pass through the point $(10,7)$. This breaks the task into two sub-tasks: first, moving from $(0,0)$ to $(10,7)$, and then from $(10,7)$ to $(16,10)$. Notice that there are $C_{17}^{7}$ ways to go from $(0,0)$ to $(10,7)$, and for the second part, he needs to move $6$ steps horizontally and $3$ steps vertically, giving $C_{9}^{3}$ ways. Applying the product rule, we get:

$$C_{17}^{7} \cdot C_{9}^{3}.$$

□

Combinatorics has very interesting applications in geometric problems. Typically, in these problems, we need to count the number of polygons, diagonals, intersection points, etc. Often, these problems directly reduce to counting combinations.

**Problem 3.11.** There are $10$ distinct points marked on a line. How many line segments are formed?

*Solution.* Obviously, any pair of points form a line segment. Thus, the number of line segments is $C_{10}^{2}$ since it does not matter which end is chosen first. □

**Problem 3.12.** There are $12$ distinct points marked on a circle. How many arcs are formed?

*Solution.* Unlike the previous problem, any pair of points on a circle corresponds to two arcs: one clockwise and one counterclockwise. Therefore, the total number of arcs is $2 \cdot C_{12}^{2}$. □

It is common to encounter situations where we need to eliminate duplicates caused by multiple counts of the same elements.

**Problem 3.13.** How many diagonals are there in a convex $12$-gon?

*Solution.* From each vertex, $9$ diagonals can be drawn (as the sides are not diagonals). However, each diagonal is counted twice: once from each end. Thus, the total number of diagonals is $\frac{12 \cdot 9}{2}$. □

Lines in general position are those where no three lines pass through a single point and no two lines are parallel. This configuration is convenient as any line intersects all others.

**Problem 3.14.** In a plane, given $n$ lines in general position, what is the number of triangles formed?

*Solution.* Notice that each set of three lines forms a triangle, so the problem reduces to finding the number of combinations of $n$ taken 3. The answer is $C_n^3$. □

**Problem 3.15.** In a plane, $n$ lines are drawn such that any two lines intersect, but no four lines pass through a single point. There are a total of 16 intersection points, with 6 of them having three lines pass through. Find $n$.

*Solution.* "Adjust" the configuration so that each pair of lines still intersects, but no three lines meet at a single point. If three lines originally intersected at a point $O$, there are now three pairwise intersections instead of one. Thus, the number of intersections increases by $2 \cdot 6 = 12$. Therefore, the total number of intersection points becomes $16 + 12 = 28$. Since every pair of lines intersects, the total number of intersection points is $\frac{n(n-1)}{2} = 28$, giving $n = 8$. □

## 3.4 Identify Identities

Let's return to the classic recursive function of two variables.

**Problem 3.16.** Prove that $C_n^k = C_{n-1}^{k-1} + C_{n-1}^k$.

*Solution.* By definition, $C_n^k$ is the number of ways to choose $k$ objects from $n$. These can be divided into two cases:

1. The first object is chosen. Then we need to choose $k-1$ objects from the remaining $n-1$. This can be done in $C_{n-1}^{k-1}$ ways.

2. The first object is not chosen. Then we need to choose $k$ objects from the remaining $n-1$. This can be done in $C_{n-1}^k$ ways.

Since either the first or the second case can happen, the total number of options is the sum of these two, completing the proof. □

The proven equality explains why the numbers $C_n^k$ can be arranged in the rows of Pascal's Triangle (Figure 3.1). On the sides of the triangle are ones, the numbers inside are arranged in a staggered order, and each internal number is the sum of the two numbers directly above it.

The following formula is known as the "Binomial Theorem", and for this reason, the numbers of combinations are also called binomial coefficients.

$$(a+b)^n = C_n^n a^n b^0 + C_n^{n-1} a^{n-1} b^1 + \cdots + C_n^1 a^1 b^{n-1} + C_n^0 a^0 b^n.$$

Now, let's prove one of the most well-known identities involving the sums of binomial coefficients.

**Problem 3.17.** Prove that $C_n^0 + C_n^1 + C_n^2 + C_n^3 + \ldots + C_n^n = 2^n$.

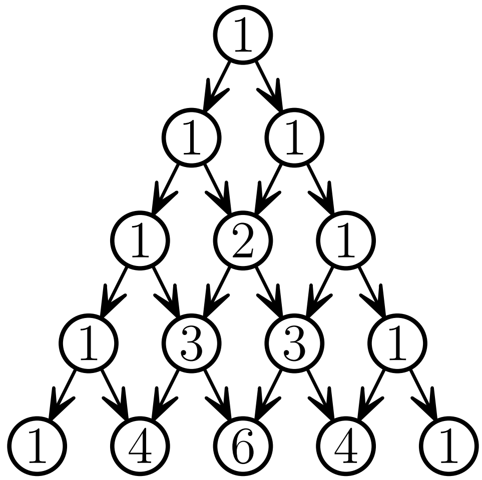



Figure 3.1: Pascal's Triangle.

*Combinatorial Solution.* Consider a sequence of length $n$ consisting of 0s and 1s. How many such sequences can we form? If there are no zeros, the count is $C_n^0 = 1$, if there is one zero, the count is $C_n^1$, and so on. Thus, the total number of such sequences is:

$$C_n^0 + C_n^1 + \ldots + C_n^n.$$

However, since each of the $n$ elements in the sequence can be either 0 or 1, we have $2^n$ such sequences. Combining both results, we obtain the desired formula. □

*Algebraic Solution.* Using the binomial theorem, we can see that when $a = b = 1$, we get:

$$(1+1)^n = C_n^0 \cdot 1^0 \cdot 1^n + C_n^1 \cdot 1^1 \cdot 1^{n-1} + \ldots = C_n^0 + C_n^1 + \ldots + C_n^n = 2^n.$$

□

While the algebraic solution in this case is elegant, do not underestimate the power of combinatorial proofs!

*The math teacher said to Jean:*
*"I warn you, if you don't behave properly, I'll tell your parents that you have talent."*

Another formula can be easily proven algebraically:

**Problem 3.18.** Find the sum $C_n^0 - C_n^1 + C_n^2 - C_n^3 + \ldots + (-1)^n C_n^n$.

*Algebraic Solution.* This sum is the expanded binomial $(1-1)^n$. Thus, the sum equals $0$. □

But let's also provide a combinatorial interpretation.

*Combinatorial Solution.* Moving all negative elements to the other side, we need to prove that the number of ways to choose an even number of elements from $n$ equals the number of ways to choose an odd number of elements:

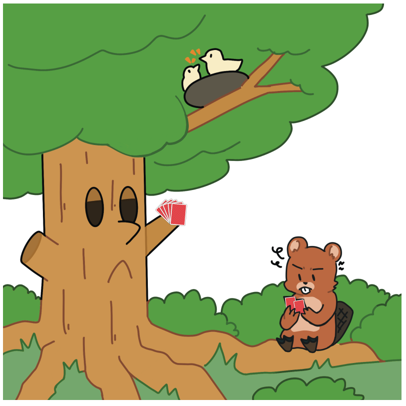
$$C_n^0 + C_n^2 + \ldots = C_n^1 + C_n^3 + \ldots$$

Consider a set of $n$ elements. Fix one element, say $a$. All possible subsets can be paired, differing only in the presence of $a$. In each pair, one subset has an even number of elements and the other has an odd number, proving their equality. □

## 3.5 I Want to Play a Game...

One of the main reasons for the emergence and development of probability theory, and even its popularity, comes from the possibility of making a lot of money in one go and without effort. For example, in the lottery or at roulette. Finding patterns to "hack the system" then becomes a powerful motivator to develop the corresponding mathematical apparatus.

It seems that if we know the laws of probability and the rules governing chance, we can win any game. These attempts, or, more accurately, the techniques aimed at securing winnings in games of chance, are called "martingales". In reality, this is a pipe dream, and we will show that below.



The main thing we need to understand is that a game is random if the player cannot influence its outcome. Thus, chess is not random. Bridge is not entirely random either while tossing a coin, roulette, and even Russian roulette are. We will call games in which chance plays a significant role — though subject to certain mathematical dependencies — "games with nature", that is, with randomness. For example, by tossing a coin, you are playing only with nature.

You can also play with someone and with nature — this is the case, for example, with "bridge". In this scenario, on the one hand, nature has dealt you the cards (or someone may cheat, but we believe in the kindness and honesty of people, and generally, we have our own deck). As for the actions of the second player, those are not random.

There are certain games where the player's action consists solely of buying a ticket, and after that, they have no further involvement. This is the case, for example, with a simple lottery. Roulette is an example of another class of games where the player has the opportunity to choose the bet and the type of game. From a mathematical point of view, a game of roulette is not fair because, in all cases, the casino wins. How do we determine if a game is fair? This requires introducing the concept of mathematical expectation, first formulated in 1670 by the Dutch mathematician Jean de Witt. He published the first modern treatise on evaluating life annuities by mathematical expectation (the present value of future payments).

What is it? Let's imagine we're playing a game. It doesn't matter to us at this point whether it's a game with nature or with another opponent. Consider a dice game with two dice. We pay \$10 for the opportunity to roll these dice. If the sum of the two rolled dice is 7, we win \$50. If any other sum comes up, we win nothing. Is this game profitable? Should you participate?

To find out, you need to calculate the probability of getting exactly 7 points on a roll of two dice. There are exactly 36 **equally probable** outcomes in total (we believe that in this game, the organisers aren't such obvious cheaters as to offer rigged dice). Among them, exactly 6 ($1+6$, $2+5$, $3+4$, $4+3$, $5+2$, $6+1$) are favorable. In other words, the probability of winning is $p_{win} = \frac{6}{36} = \frac{1}{6}$. The probability of losing is $p_{loss} = 1 - \frac{1}{6} = \frac{5}{6}$.

In our case, we know the set of possible roll values for each die: they are numbers from 1 to 6.

We don't know which number will come up on the next roll; it's ontological randomness (if the game organisers are cheaters, then for them, it would be epistemic randomness). We're interested in the average winnings per game.

**Definition 4. Mathematical Expectation** is the arithmetic mean of the possible values a random variable can take, weighted by the probability of those outcomes. In other words, if the variable $X$ takes values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$, the expectation of $X$ is defined as: $\mathbb{E}[X] = x_1p_1 + x_2p_2 + \ldots + x_np_n$.

In the considered game, if we succeed, we win \$50, so the amount of the gain is \$40 (don't forget we already spent \$10!), and if we fail, we lose \$0, so the amount of the gain is -\$10. Therefore,

$$\mathbb{E} = p_{win} \cdot S_{win} + p_{loss} \cdot S_{loss} = \frac{1}{6} \cdot 40 + \frac{5}{6} \cdot -10 = -\frac{10}{6}.$$

The mathematical expectation of a gain in this game is negative, meaning the game is not profitable for us. The more we play, the larger the mathematical expectation of gain (in absolute value) will be, and consequently, the more we will lose.

**Definition 5.** If the mathematical expectation of gain for a game is equal to zero, the game is considered to be **fair**.

How can we tell if a game is fair? Sometimes, our intuition fails. And that's why similar questions are frequently asked in job interviews.

For example, I propose a game to all who want to play in a study group or at the office: If there are at least two people who have the same birthday, then they will buy you a pizza; otherwise, you buy them pizza. Is this bet fair or not for you? Let's check.

Let's calculate some probabilities. We'll derive a formula showing the probability that there is at least one pair with the same birthdays in a group of $n$ people.

Suppose there are 366 different birthdays, and they are all equally likely (in fact, those born on February 29 should be fewer on average than those born on February 28, but this is not the most important case here). The simplest way to get the required result is to calculate the probability that each person has a birthday different from the others: the opposite of what we are looking for.

We can proceed by recurrence: the first person has 366 choices, the second has 365, the third has 364, the fourth has 363, and so on. Here we will proceed by counting, meaning we will count the number of cases where $n$ people have different birthdays and divide it by the total number of possibilities. In both cases, we assume the equiprobability of birth dates. So, for two people, the probability of no match is $\frac{365}{366}$. When the third person arrives, the desired probability of no match becomes $\frac{365}{366} \times \frac{364}{366}$, since there were 364 "available" days. By reasoning in this way, we can derive a formula that determines the probability of non-coincidence of birthdays for one of the pairs in a group of $n$ people:

$$p_n = \frac{365}{366} \cdot \frac{364}{366} \cdot \ldots \cdot \frac{366-n}{366}.$$

A very simple action remains to be done. You just need to calculate this expression until the product of the fractions becomes less than $\frac{1}{2}$. It's easier to do on a computer. It turns out that already at $n = 23$, the fraction is less than $\frac{1}{2}$, so if there are 23 people in your group, then the game is already profitable for you.

Let's look at another example, this time related to another popular game with nature: French roulette.

In the modern world, almost everyone uses the Internet, and many of us have probably received emails or intrusive ads offering to discover the secret to winning at roulette.

The game of French roulette involves spinning a small ball on a wheel with 37 slots. These are numbered from $0$ to $36$ (the numbers are part of the wheel), alternately red and black, except for zero, which is green. A player bets an amount $M$ on one of the slots. If the ball lands on their chosen slot, they are paid 36 times their bet (their gain is then $35M = 36M - M$); otherwise, they lose their bet (their loss is then $-M = 0 - M$).

There are also other betting options in the game, for example, betting on red. So, when the ball lands on a red-numbered slot, the bet is doubled. The probability of landing on a red slot is then $\frac{18}{37}$, on a black slot is also $\frac{18}{37}$, and on zero is $\frac{1}{37}$.

The expected value of a gain with a bet of one dollar is then:

$$E = \frac{18}{37} \cdot 1 + \frac{19}{37} \cdot -1 = -\frac{1}{37}.$$

As we can see, on average, we will win a negative amount, which means this game is unfair. But maybe there are other options? You can bet on a specific slot, and you will be paid 36 times your bet. Let's calculate the expected gain:

$$E = \frac{1}{37} \cdot 35 + \frac{36}{37} \cdot -1 = -\frac{1}{37}.$$

The casino cannot be tricked; the rules are designed in such a way that the expected value of a gain for each possible variant of roulette games is always exactly $-\frac{1}{37}$ of the bet.

American roulette differs from French or European roulette in its wheel layout, which has an additional number: double zero. Consequently, the rules there are even stricter.

Let's try to trick the casino using the winning strategy offered on the Internet. To simplify, let's assume the roulette game is fair, meaning the mathematical expectation of a win is zero.

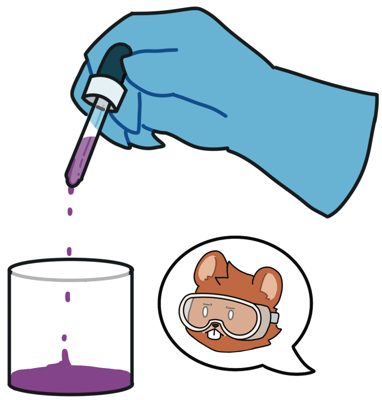
My first bet is 1 dollar on red. If I win, I put the winnings in a bank account and stop playing. If I lose, I double my bet and bet again on red. I believe it doesn't matter how many times the series lands on red. With this strategy, by constantly doubling bets after each loss and taking money when I win, I'll become invincible. Am I right?

Let's see what the probability is of getting black 20 times in a row. It will be equal to $\frac{1}{2^{20}} \approx 10^{-6}$, or about one millionth. How much money do I need to bet again in this case? It's $2^{20}$, which is about one million dollars. Do I have that much money on me? And if so, what about when black lands 21 times in a row? That's roughly two million. And 30 times in a row? It's unlikely, but the loss in that case is astronomical! Also, for this strategy, there's another problem: in real casinos, the maximum bet size is limited. Let's say you can't bet more than one million dollars.

We just watched in horror as the ball landed on black 19 times in a row. To recover the loss (well, it can't possibly land on black for the 20th time, we believe), we must bet 1,048,576 dollars, except the rules limit our bet to 1,000,000 dollars...

Even if I get lucky (and players believe they surely will), then I will be paid 1,000,000 dollars, and I've already bet 1,048,576 dollars. My net loss was 48,576 dollars, and to recover it, I must successfully play 48,576 times. Now remember, the probability of winning is not $\frac{1}{2}$ but $\frac{18}{37} \cdots$

Calculations show that even with a fair game (which is impossible), during 20 consecutive and consistent years of playing, the probability of winning is 99%. Then you need to have at least $2^{18} = 262{,}144$ dollars of pocket money to win one dollar every day. Isn't it easier to put money in the bank at a minimal interest rate? Or even to find a job? Ah, yes, that's exactly what you are trying to do....



Attempting to succeed in such a game is, after all, quite anecdotal:

*Visitor to the bartender: How much does a drop of cognac cost?*
*Bartender: Nothing.*
*Visitor: Then, pour me a glass of drops.*

The law of large numbers is at play here. We can conclude that in a fair game (and even more so in an unfair game), no strategy can lead to a guaranteed win.

## 3.6 How You Play The Cards You're Dealt Is All That Matters

Let's explore some classic interview problems with a casino theme. All decks in this casino are standard 52-card decks. Each card has a value and belongs to one of four suits.

**Problem 3.19.** A casino offers a card game using a standard deck. The rule is that you turn over two cards each time. For each pair, if both are black, they go to the dealer's pile; if both are red, they go to your pile; if one is black and one is red, they are discarded. The process is repeated until all 52 cards are used. If you have more cards in your pile, you win \$100; otherwise (including ties), you get nothing. The casino allows you to negotiate the price you want to pay for the game. Would you be willing to pay to play this game?

*Solution.* This surely is an insidious casino. No matter how the cards are arranged, you and the dealer will always have the same number of cards in your piles. Why? Because each pair of discarded cards has one black card and one red card, so an equal number of red and black cards are discarded. As a result, the number of red cards left for you and the number of black cards left for the dealer are always the same. The dealer always wins! So, we should not pay anything to play the game. □

**Problem 3.20.** Draw poker is a card game in which each player gets a hand of 5 cards.

What is the probability of getting a hand with a full house (three cards of one value and two cards of another value)?

*Solution.* The number of different hands of a five-card draw is the number of 5-element subsets of a 52-element set; thus, $C_{52}^5 = 2598960$.

We need to choose the value of the triple, which has 13 choices, and the suits of the triple, which has $C_4^3 = 4$ choices. The value of the pair can be chosen in 12 ways, and the suits of the pair have $C_4^2 = 6$ choices. So, the number of hands with a full house is $13 \cdot 4 \cdot 12 \cdot 6 = 3{,}744$. This gives us the probability of $\frac{3744}{2598960} \approx 0.144\%$. Not very high, is it? □

Try to find the probabilities for other poker hands yourself. What if we play Texas Hold'em?

**Problem 3.21.** A casino offers a simple card game. You pick up a card from the deck, and the dealer picks another one without replacement. If you have a larger value, you win; if the values are equal or yours is smaller, the house wins. What is your probability of winning?

*Solution.* Here, let us use a slightly less straightforward approach.

We have 3 different outcomes:

- $P_1$: Your card has a value larger than the dealer's;
- $P_2$: Your card has a value equal to the dealer's;
- $P_3$: Your card has a value lower than the dealer's.

By symmetry, $P_1 = P_3$. Let us find $P_2$. Suppose you have randomly selected a card. Among the remaining 51 cards, only 3 cards have the same value as yours. Thus, $P_2 = \frac{3}{51} = \frac{1}{17}$. Since $P_1 + P_2 + P_3 = 1$, we find $P_1 = P_3 = \frac{8}{17}$. Not so bad for a casino. □

**Problem 3.22.** You and three friends visit a casino. You are each dealt an equal number of cards from a standard deck. You win if each of you receives exactly one queen. What is the probability of winning this game?

*Solution.* Let's take the first queen. It is definitely dealt to someone. For the second queen, we want it to go to a different player than the first. There are 39 spots left out of 51. For the third queen, there are 26 spots left out of 50, and for the fourth, there are 13 spots left out of 49. Therefore, the probability is $1 \cdot \frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49}$. □

**Problem 3.23.** There are two decks of cards. One deck has 52 cards, the other has 104. You pick two cards separately from the same pack. If both of the cards are red, you win. Which pack will you choose?

*Solution.* Let's compare the probabilities. In a 52-card deck, there are 26 red cards. The probability of drawing two red cards is $\frac{26}{52} \cdot \frac{25}{51} = \frac{25}{102}$. In a 104-card deck, there are 52 red cards. The probability of drawing two red cards is $\frac{52}{104} \cdot \frac{51}{103} = \frac{51}{206}$. The second number is bigger. Thus, you should choose the 104-card deck. □

**Problem 3.24.** Given 100 coin flips, what is the probability that you get an even number of heads?

*Solution.* Flip the coin 99 times. No matter the outcome of the first 99 flips, the parity of the number of heads is determined by the final flip. If the number of heads is odd, you need a head to make it even. If the number of heads is even, you need a tail to keep it even. Thus, in all cases, the probability of the desired outcome is 0.5, so the overall probability is 0.5. Elegant, isn't it? We used this logic when proving the corresponding identity. □

**Problem 3.25.** You have $2025$ coins, and I have $2024$ coins; we flip all coins at the same time. If you have more heads, then you win; if we have the same number of heads or if you have less, then I win. What's your probability of winning?

*Solution.* Let's forget about your last coin. If you have $2024$ coins, what can happen?

We have $3$ different outcomes:

- $P_1$: You have more heads than me;
- $P_2$: We have an equal number of heads;
- $P_3$: I have more heads than you.

We already did something similar in one of the previous problems. We know that $P_1 = P_3$, and thus $2P_1 + P_2 = 1$. In the first case, you definitely win. In the second case, you win if you get heads. In all other cases, I win. So the probability of winning is $P_1 + 0.5P_2 = 0.5$. Very fair! □

**Problem 3.26.** Let's play a game with $2024$ socks: $1012$ white, $1012$ black and two sacks. You can arrange the socks within the two sacks in any way you want. I then come into the room and pick a sock from one of the sacks. If I pick a black, I win; a white, you win. How do you arrange the socks such that you have the highest chance of winning?

*Solution.* To maximise your chances of winning, you should place one black sock in one sack and the remaining $1011$ black socks and $1012$ white socks in the other sack. This gives you a probability of $\frac{1}{2}$ of picking the black sock from the first sack plus $\frac{1}{2} \cdot \frac{1011}{2023}$ from the second sack, which in total is close to $75\%$.

Why is this the maximum? You can try to explain it! □

*— What's the difference between a PhD in mathematics and a large pizza?*
*— A large pizza can feed a family of four!*

## 3.7 Drunk Crazy Old Lady in a Flying Cinema

*Two mathematicians are flying in an airplane. One says to the other:*
*"Lately, I've been really afraid of flying because I calculated that the probability of there being a bomb on board is even higher than the probability of dying in a car crash."*
*The second mathematician responds:*
*"Yes, I did the same calculations and came to the same conclusion. But I went further. The probability of there being two bombs on the plane is negligible, so now I always carry one with me."*

From book to book, the problem of the "drunk passenger and the airplane" or "tickets in the cinema" travels. This problem has a fascinating and intricate history that spans decades and continents. In January 2024, Konstantin Knop, with Anton Petrunin and Alexey Ustinov, traced the evolution of the problem from its origins to its current form in the mathematical literature. They documented this fascinating history while preparing the translation of Mathematical Puzzles: A Connoisseur's Collection by Peter Winkler.

The possible roots of this puzzle lie in the everyday experiences of I.B. Alekseev-Astafiev, the young author of the original problem, who frequently travelled from St. Petersburg to Pskov. Often, he would purchase a ticket for the general carriage of the train, with the seat number indicated on the ticket. However, upon boarding, it was common to find that those who arrived first had taken the best seats, claiming that the seat numbers on the tickets were merely for statistics. This situation led to a chain reaction of passengers being displaced. The original puzzle was framed as follows: "In the general carriage of the Leningrad-Pskov train, there are $N$ seats. The $N$th passenger, arriving last, finds that only the worst seat — next to the toilet — is available."

However, the editors at Kvant journal, from June 1985, decided that writing about disorder in a train and the toilet was inappropriate, so they revised the setting to a cinema. The published problem setting was as follows:

**Problem 3.27.** In a cinema with $N + 1$ seats, initially, $N$ people, including Igor, sit in any of the $N$ seats without looking at their tickets. The $(N + 1)$th spectator, arriving last, wants to take their assigned seat; if it is occupied, they displace the person sitting there, who then does the same, and so on, until a seat needed by a displaced person is found to be free. What is the probability that Igor will have to move? (In other words, what fraction of all possible seating arrangements are unfavourable for Igor?)

Around the same year, an almost identical problem circulated in academic circles. Andrey Schetnikov recalled:

"I heard of this problem around 1985. Igor Kotelnikov, who told me it, had heard it from Ryutov, who was both his and my scientific supervisor in graduate school. We referred to it as 'Ryutov's problem'. We discussed it at the summer school on physics and mathematics at Novosibirsk State University with Slava Muchnik, and I remember it well. Additionally, that summer, I roomed with Dmitry Fon-Der-Flaass. It is highly likely that he learned the problem from me; that would be the shortest chain. Whether Ryutov was the author or learned it from someone else, I cannot say. Why not? If there are no earlier witnesses, this remains the earliest testimony we can reconstruct."

The problem setting that Andrey remembers is as follows:

**Problem 3.28.** A 100-seat airplane is boarding. A hundred passengers are in line. The first one is a crazy old lady. She sits in a randomly chosen seat. The rest of the passengers are normal people: each sits in their assigned seat if it is free or in any free seat otherwise. What is the mathematical expectancy of the amount of people that sit in the wrong seat?

The problem resurfaced prominently in 1996 during the All-Russian Mathematics Olympiad in Ryazan. Vadim Bugaenko shared his memory of this event:

"The famous problem about the 'crazy old lady'. I remember well when I first heard it. It was in 1996 at the All-Russian Olympiad in Ryazan when the jury was returning from the closing ceremony. We talked about various things, including problems. Dmitry Fon-Der-Flaass told us about this problem, and it was new to everyone, creating quite a stir. I don't recall where he said he heard it. Apparently, he is no longer with us, so we can't ask him. But the entire jury heard about it for the first time in that conversation (I specifically remember Igor Rubanov and Konstantin Knop). After that, the problem became very well known."

Igor Rubanov played a crucial role in popularising the problem beyond the confines of the Olympiad. On April 27, 1997, he posted it in the Usenet group relcom.rec.puzzles:

"Hello, friends! I encountered this problem at the just-concluded All-Russian Mathematics Olympiad, where I learned it from my colleague on the jury, Dmitry Fon-Der-Flaass. I liked it and hope you will too.

A 100-seat airplane is boarding. A hundred passengers are in line. The first one is a crazy old lady. She sits in a randomly chosen seat. The rest of the passengers are normal people: each sits in their assigned seat if it is free, or in any free seat otherwise. What is the probability that the last passenger will sit in their assigned seat? Good luck!

Prof. S.M.School & Igor S. Rubanov"

This post was quickly translated and shared in the global mathematics group:

"On Sun, April 29, 1997, Prof. S.M.School wrote in relcom.rec.puzzles:

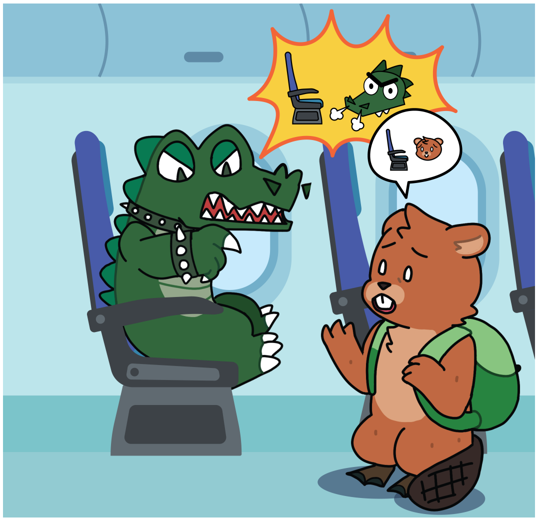
100 passengers are about to enter the airplane with 100 seats. The crazy old lady is the first in line. She occupies the wrong place. The rest of the passengers are just normal people. If their place (the one written down on the ticket) is free, they occupy it, and if it is occupied, they occupy an arbitrary one. What is the probability that the last passenger will have to occupy the wrong seat?"

By 2001, the problem appeared in a published book with slight variations. The authors posed it as follows:

**Problem 3.29.** The $n$ passengers for a Bell-Air flight in an airplane with $n$ seats have been told their seat numbers. They get on the plane one by one. The first person sits in the wrong seat. Subsequent passengers sit in their assigned seats whenever they find them available or otherwise in a randomly chosen empty seat. What is the probability that the last passenger finds his or her assigned seat to be free?

The origin of the name Bell-Air was vaguely explained in a footnote, stating that the authors had heard the problem from David Bell.

The puzzle in its now-popular form was included in Peter Winkler's book, "Mathematical Puzzles: A Connoisseur's Collection," published in 2004. Winkler attributed the problem to an oral communication from Ander Holroyd. The problem stated:



**Problem 3.30.** "A hundred people line up to board an airplane. The first passenger loses their boarding pass and takes a random seat. Every subsequent passenger takes their assigned seat if it is available or a random free seat if it is not. What is the probability that the last passenger will find their seat taken?"

In modern publications of this problem, the first passenger is mostly said to be drunk, and that's why they take the random seat. This is the most classical problem setting in interview preparation guides. Let's discuss the solution to this problem.

*Solution.* Let's assume that, in a certain seating arrangement, the last passenger did not sit in their assigned seat (we'll call this an unsuccessful arrangement). In this case, before the last passenger arrived, their seat was taken by passenger $A$ ($A$ could even be the crazy old lady).

When passenger $A$ arrived, they had a choice of seat. In this arrangement, they took the last passenger's seat. But they could have just as likely taken the old lady's seat, allowing all remaining passengers, including the last one, to sit in their assigned seats. (Of course, we need to explain why the old lady's seat is still free when passenger $A$ arrives. However, it's not difficult to see that while the old lady's seat is free, there is exactly one passenger among those who haven't boarded yet whose seat has already been taken. Once the next passenger takes the old lady's seat, the remaining passengers will only take their own seats.) Thus, for each unsuccessful arrangement, there is a corresponding successful one that is equally likely. This implies that exactly half of the arrangements will be unsuccessful. $\square$

# To Divide or not to Divide?

> Forty days pass, the flood recedes, and Noah and his family are settling in. Noah's wife notices that all the animals are starting to reproduce, except for a pair of snakes.
>
> She asks Noah about it, and he says he'll take care of it. A week later, Noah brings his wife out to the workshop and shows her the snakes in their basket on top of the picnic table he just built. "How is this going to get them to reproduce?" she asks. "Trust me," he replies.
>
> A few days after that, she notices there are eggs in the basket. She is delighted and asks Noah how the picnic table could have possibly helped. He says, "My dear, even adders can multiply on a log table."

—One joke that didn't quite land

## 4.1 Divide et Impera

Number theory is often underestimated. It might seem like a purely academic topic or an olympiad subject — technical and uninteresting. Why would one even need to recall it?

However, in the realm of technical interviews, number theory is a silent giant. While it may seem esoteric, it forms the backbone of many problems you'll face in coding interviews. Mastery of number theory not only showcases your mathematical prowess but also your ability to think logically and solve complex problems efficiently.

Generating and working with prime numbers is a common interview topic. Algorithms like the Sieve of Eratosthenes can efficiently find all primes up to a given number and are a fundamental part of algorithmic problem-solving.

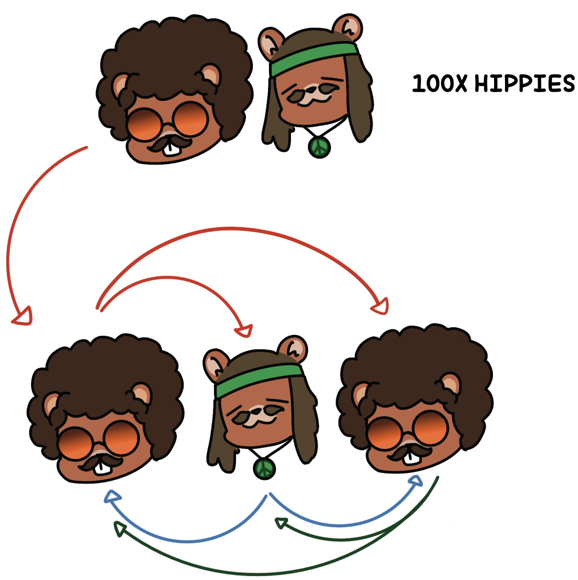
Modern cryptography relies heavily on number theory. Understanding the RSA algorithm, which is based on the difficulty of factoring large numbers, can set you apart in security-related fields.

We've recalled why this might be needed for coding interviews, but our book isn't solely about that. We're here to revisit the basics, which can also be used for warm-ups, and to solve some unusual or funny problems.

## 4.2 Two of Every Sort

*Question: How do you make seven an even number?*
*Answer: Just remove the "s".*

**Problem 4.1.** One day, the leadership of the KGB decided to provoke a group of Soviet hippies into participating in a protest against the Vietnam War near the US Embassy in Moscow. A group of 100 hippies was selected for this purpose, with plans to persuade them to join the protest. However, before that, they needed to get acquainted with each other. On the one hand, it was dangerous to gather too many strangers at once; on the other, pairing off individuals would take too much time. The KGB agent assigned to the task decided to proceed as follows: introduce people in groups of three, ensuring that people who were already acquainted could not meet again at the second meeting. Will he be able to acquaint all the hippies with each other using this method?




This type of problem is widely disliked. It is formulated in many sentences when it could be written much shorter. Some such tasks in this book teach you to filter through only what is most important in a large block of text.

*Solution.* Suppose the agent was able to meet the required conditions. Consider any arbitrary hippie. During each meeting, they get acquainted with exactly $2$ other hippies. However, they need to get acquainted with $99$ hippies in total, and since $99$ is an odd number, it is impossible to achieve this under the agent's conditions. □

**Problem 4.2.** On the board, $2025$ integers from $1$ to $2025$ are written. Each time Jean erases two numbers from the board, he writes the absolute value of their difference on the board. Can the last remaining number on the board be zero?

*Solution.* Calculate the sum of all the numbers written on the board:

$$1 + 2 + \ldots + 2025.$$

Rewrite the numbers in a different order:

$$\underbrace{1 + 2025} + \underbrace{2 + 2024} + \ldots + 1013.$$

The sum of each pair of numbers is $2026$ — an even number. The number $1013$ is odd; therefore, the overall sum is also odd.

By considering tables of addition and subtraction of even and odd numbers, we observe that they match. Consequently, if we erase two numbers and write the absolute value of their difference, the parity of the sum of all the numbers written on the board does not change. Since $0$ is an even number, it cannot be the last remaining number after a series of steps. □

## 4.3 The Mod Squad

Let us think back to a concept that is really useful in number theory.

**Definition 6.** Numbers $a$ and $b$ are said to be congruent modulo $x$ if their difference is divisible by $x$. This is denoted as $a \equiv b \pmod{x}$ or $a = b \pmod{x}$.

Negative numbers can also be congruent modulo.

For example, $13 \equiv 7 \pmod{3}$ or $11 \equiv -3 \pmod{7}$.

If $a \equiv 0 \pmod{x}$, then $a$ is divisible by $x$.

With modular congruences, you can perform operations similar to regular equalities: you can add, subtract, multiply, and exponentiate them.

If $a \equiv b \pmod{x}$ and $c \equiv d \pmod{x}$, then

- $a + c \equiv b + d \pmod{x}$;
- $a \cdot c \equiv b \cdot d \pmod{x}$;
- $a^n \equiv b^n \pmod{x}$ for any natural $n$.

Sometimes, using modular congruences allows us to significantly reduce the complexity of calculations. Let's consider an example.

**Problem 4.3.** Find the remainder of the division $2024^3$ by 7.

*Solution.* Find the remainder of the division of 2024 by 7. Of course, this can be done using long division, but let's recall a beautiful fact: $1001 = 7 \cdot 11 \cdot 13$, from which it follows that the number 2002 is divisible by 7, i.e., $2002 \equiv 0 \pmod{7}$.

Therefore, $2024 \equiv 2024 - 2002 = 22 \equiv 1 \pmod{7}$

Thus, $2024^3 \equiv 1^3 = 1 \pmod{7}$. □

In some cases, it is more convenient to switch to negative numbers. For example, $8^{100} \equiv (-1)^{100} = 1 \pmod 9$.

When we use modular congruences, many well-known divisibility tests reveal themselves from a new perspective and can even provide more information. For example, the divisibility tests for 9 and 3 can be expressed as follows:

$$\overline{a_n a_{n-1} \dots a_2 a_1 a_0} \equiv a_n + a_{n-1} + \dots + a_2 + a_1 + a_0 \pmod 9$$

and

$$\overline{a_n a_{n-1} \dots a_2 a_1 a_0} \equiv a_n + a_{n-1} + \dots + a_2 + a_1 + a_0 \pmod 3.$$

One method for solving divisibility problems is to write down all possible remainders. In some problems, finding the "good modulus" is straightforward, while in others, it needs to be guessed.

**Problem 4.4.** Prove that for any integer $n$, the number $n^2 + 3n + 4$ is not divisible by 9.

*Proof.* An integer can produce remainders $0, 1, 2, \dots, 8$ when divided by 9. Let's go through all these cases. For convenience, let's write down the remainders in the form of a table.

| $n \pmod 9$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $n^2 \pmod 9$ | 0 | 1 | 4 | 0 | 7 | 7 | 0 | 4 | 1 |
| $3n \pmod 9$ | 0 | 3 | 6 | 0 | 3 | 6 | 0 | 3 | 6 |
| $n^2 + 3n + 4 \pmod 9$ | 4 | 8 | 5 | 4 | 5 | 8 | 4 | 2 | 2 |

Consider, for example, the case when $n$ gives a remainder of 5 when divided by 9. Then $n \equiv 5 \pmod 9$, from which

$$n^2 + 3n = 4 \equiv 5^2 + 3 \cdot 5 + 4 = 25 + 15 + 4 \equiv$$

$$\equiv 7 + 6 + 4 = 17 \equiv 8 \pmod 9.$$

Other cases are handled similarly. We see that in no case did we get a remainder of 0, which means that the expression is not divisible by 9 for any integer $n$. □

**Problem 4.5.** Can a number formed by 13 twos, 13 threes, 13 fours, and 13 fives be a perfect square?

*Solution.* We will prove that this is impossible. Since the problem involves a number with known digits but an unknown order, we will use the divisibility criterion for 3. Let $N$ denote the number.

Then the sum of the digits of the number $N$ is equal to $13 \cdot (2 + 3 + 4 + 5)$. We get: $N \equiv 13 \cdot 14 \equiv 1 \cdot 2 = 2 \pmod 3$.

Let's see if the square of a natural number can yield a remainder of 2 modulo 3, as in the previous problem. Consider the cases:

| | | | |
|---|---|---|---|
| $x \pmod 3$ | 0 | 1 | 2 |
| $x^2 \pmod 3$ | 0 | 1 | 1 |

We conclude that the square of a natural number cannot yield a remainder of 2 modulo 3, which means that it is impossible to form a square from all the given digits.

□

*What a class I have! I explained the theorem to them — they didn't understand. I explained it again — they didn't understand. The third time I explained it — I understood it myself, but they still didn't understand.*

## 4.4 GCD, LCM, OMG!

Let's recall the concepts of $\gcd$ (greatest common divisor) and $\text{lcm}$ (least common multiple).

Suppose, with the help of the fundamental theorem of arithmetic, the prime factorisation of numbers $M$ and $N$ looks as follows:

$$N = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n} \text{ and } M = p_1^{\beta_1} p_2^{\beta_2} \ldots p_n^{\beta_n}.$$

Their greatest common divisor is given by

$$\gcd(M, N) = p_1^{\min(\alpha_1,\beta_1)} p_2^{\min(\alpha_2,\beta_2)} \ldots p_n^{\min(\alpha_n,\beta_n)}.$$

The least common multiple of $M$ and $N$ is given by

$$\text{lcm}(M, N) = p_1^{\max(\alpha_1,\beta_1)} p_2^{\max(\alpha_2,\beta_2)} \ldots p_n^{\max(\alpha_n,\beta_n)}.$$

The minimum and maximum values of two variables are connected by a very useful relationship: $\min(x,y) + \max(x,y) = x + y$. Indeed, if the numbers are equal, then the minimum and maximum values coincide with the numbers, and if they are not equal, then one of the numbers will be the maximum value, the other will be the minimum value, and the sum remains unchanged regardless of the order of the terms.

Hence, in particular, the formula holds: $\text{lcm}(M, N) \cdot \gcd(M, N) = M \cdot N$.

Numbers are called *relatively prime* or *coprime* if their greatest common divisor is $1$.

The concept of Bézout's identity connects relatively prime numbers and $\gcd$. Let $a$ and $b$ be integers, both not equal to zero; then, there exist integers $u$ and $v$ such that $\gcd(a, b) = a \cdot u + b \cdot v$ holds.

Let's also recall the *Euclidean algorithm* for finding the $\gcd$ of two numbers: $\gcd(a, b) = \gcd(a - b, b)$ if $a > b$ (at each step, the larger number is replaced by its difference from the smaller one). Its natural improvement is the *extended Euclidean algorithm*, where numbers are not subtracted but divided with a remainder. For example, $\gcd(315, 100) = \gcd(100, 15) = \gcd(15, 10) = \gcd(5, 10) = \gcd(0, 5) = 5$.

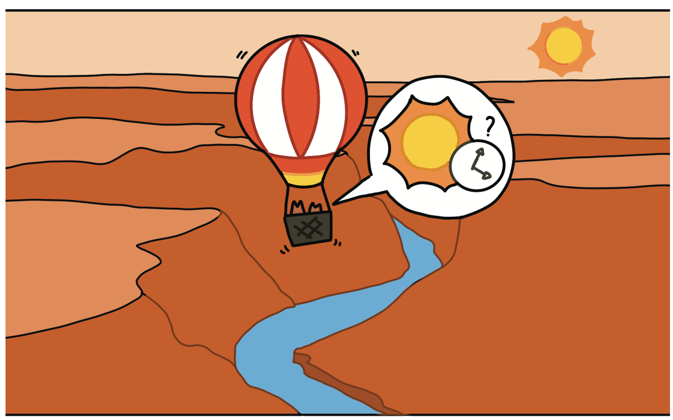
**Problem 4.6.** Find the greatest common divisor of the numbers $2^n - 1$ and $2^m - 1$.

*Solution.* Let, without loss of generality, $n > m$. Let's use the modified Euclidean algorithm:

$$\gcd(2^n - 1, 2^m - 1) = \gcd(2^n - 2^m, 2^m - 1) = \gcd(2^{n-m} - 1, 2^m - 1).$$

Here we used the fact that the second obtained number is odd, so the first number can be divided by $2^m$ without changing the $\gcd$.

One step of such a modified Euclidean algorithm performs the same operations with powers as the regular Euclidean algorithm — with numbers. Therefore, as a result of the work of such an algorithm, the number $2^{\gcd(n,m)} - 1$ will be obtained, which is what we needed to find. □



*An engineer and a physicist are lost in a hot air balloon drifting down a canyon somewhere. While the physicist is trying to use the angle of the sun to figure out how long they have to find help until night falls, the engineer shouts, "Hey! Where are we?" A few hours later, they hear a voice, "You're in a hot air balloon." The physicist then remarks, "That must have been a mathematician," "Why?" "Because the answer was both completely correct yet entirely useless."*

## 4.5 Prime Time Fun

This section combines mainly problems devoted to examples and counterexamples in number theory.

**Problem 4.7.** If we substitute numbers $n = 1, 2, 3, 4, 5$ into the expression $n^2 + n + 41$, the prime numbers $43, 47, 53, 61, 71$ are obtained. Is it true that substituting any natural number $n$ into this expression will result in a prime number?

*Solution.* Encountering such a problem in a job interview, you might be horrified because there are no explicit signs that a number is prime (while proving that a number is not prime is sometimes very straightforward). So, this problem statement is an obvious hint — you don't need to prove that all numbers of this form are prime, just provide an example of a non-prime number.

Notice that the first two terms are divisible by $n$, and the third term is $41$. If we take $n$ divisible by $41$, the sum of the numbers will be divisible by $41$, forming a counterexample, which completes the solution to the problem. □

**Problem 4.8.** Find all prime numbers $p$ such that among the numbers from $1$ to $p$ inclusive, the number of composite numbers is exactly twice the number of primes.

*Solution.* Write down all prime numbers not exceeding $59$. There are $17$ of them:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59.$$

We can see that the only suitable $p$ is 37.

Consider residues modulo $6$ for any six consecutive numbers, the first of which is at least $4$. These residues will be $0, 1, 2, 3, 4, 5$. Numbers that give residues $0, 2, 3$, and $4$ when divided by $6$ cannot be prime, since they are divisible by $2$ or $3$. Thus, among any $6$ consecutive numbers, the first of which is at least $4$, there are at most $2$ primes.

Consider some number $p_1 > 59$. There will be $17$ prime numbers from $1$ to $59$ inclusive. From $60$ to $p_1$ inclusive, there will be $p_1 - 60 + 1$ numbers, that is, at most $\frac{p_1-60+1}{6}$ blocks of six numbers plus the remaining numbers after the last block (in which a number can be prime only if it leaves a residue of $1$ when divided by $6$, since if it is a residue of $5$ when divided by $6$, then it is in a complete block). Thus, there are at most $2 \cdot \frac{p_1-60+1}{6} + 1$ prime numbers. Therefore, from $1$ to $p_1$ inclusive, there will be at most

$$17 + 2 \cdot \frac{p_1 - 60 + 1}{6} + 1 = \frac{p_1 - 5}{3}$$

prime numbers. Then, there will be at least $p_1 - 1 - \frac{p_1-5}{3} = \frac{2p_1+2}{3}$ composite numbers (since $1$ is neither prime nor composite). Thus, the number of composite numbers will be more than twice the number of primes. Hence, besides the found $p = 37$, there are no other solutions. □

The solution to this problem is thanks to the observation that prime numbers are initially quite frequent, and then they become less and less frequent, which raises a reasonable question — could it be that at some point they just end? In other words, is the set of prime numbers infinite, or is there a largest prime number? It turns out that answering this question is not that difficult. Suppose the set of prime numbers is finite. Then consider a number equal to the product of all prime numbers, plus one: $p_1 p_2 \ldots p_n + 1$. It cannot be prime, since it is greater than any of them. However, this number gives a remainder of $1$ when divided by any prime number, so it must be prime. This contradiction proves the infinity of the set of prime numbers.

**Problem 4.9.** Euclid's proof of the infinitude of the set of prime numbers suggests defining the Euclidean numbers recursively:

$$e_1 = 2, e_n = e_1 e_2 \ldots e_{n-1} + 1 \ (n \geqslant 2).$$

Are all numbers $e_n$ prime?

*Solution.* No, because $e_5 = 1807 = 13 \cdot 139$. □

*The math teacher glanced over the student's notebook and was shocked by complex calculations: "One of us has gone crazy, Leo!"*
*The next day, Leo put an envelope on the table.*
*"What's in it?" the teacher asked.*
*"A certificate stating that I am not insane."*

## 4.6 Base-ics of Numerals

*Why do mathematicians confuse Halloween and Christmas?*
*Because Oct 31 = Dec 25*

In the modern world, we use the decimal number system for counting. This means that we can break down a number into "tens", "hundreds", "thousands", and so on, i.e., powers of ten. For example, $2024 = 2 \cdot 10^3 + 0 \cdot 10^2 + 2 \cdot 10^1 + 4 \cdot 10^0$.

In the decimal numeral system, each digit can be from $0$ to $9$, meaning any non-negative integer less than the base raised to the power of the position.

And what about, for example, the binary number system? This is the numbering system in which a computer "thinks". It does so because it is most straightforward to physically represent numbers in the binary system. In this system, there are only $2$ possible digits. The digit $1$ can correspond to the presence of current in a certain segment of a circuit (or a lit bulb in a vacuum tube computer), and the digit $0$ represents its absence. The numerical representation of a binary number is interpreted as $\overline{a_n \ldots a_1 a_0}_2 = a_n 2^n + \ldots + a_1 2^1 + a_0 2^0$.

A generalisation for a numeral system with a base of $s$ is given by the formula: $\overline{a_n \ldots a_1 a_0}_s = a_n s^n + \ldots + a_1 s^1 + a_0 s^0$, where digits range from $0$ to $s - 1$. If the base of the numbering system is greater than $10$, Latin letters $A, B, \ldots$ are usually used for digits like "$10$", "$11$", and so on.

**Definition 7.** A *positional numeral system*, also known as *place-value notation*, is a numeral system in which the value of each numerical sign (digit) in the representation of a number depends on its position (order).

The first mentions of positional systems date back to Sumerian and Babylonian works. The sexagesimal system, invented by the Sumerians in the $3$rd millennium BCE, is a positional system with a base of $60$. This system became widespread due to ancient and medieval astronomers who used it primarily to represent fractions. Therefore, medieval scholars often referred to sexagesimal fractions as "astronomical". These fractions were used to record astronomical coordinates — angles, and this tradition has survived to this day. There are $60$ minutes in one degree and $60$ seconds in one minute.

The emergence of the decimal system is associated with counting on fingers. In medieval Europe, it spread through Italian merchants, who borrowed it from the inhabitants of Central Asia.

However, there are also non-positional numeral systems, such as the Roman numeral system. We constantly encounter this system in life, as the numbers of centuries, millennia, etc., are always written as Roman numerals.

Let's remind ourselves of the structure.

The meanings of the "digits" are as follows:

**I** $1$,
**V** $5$,
**X** $10$,
**L** $50$,
**C** $100$,
**D** $500$,
**M** $1000$.

For example, the number $1917 = 1000 + 900 + 10 + 7$ is written as MCMXVII.

Let's return to positional number systems. In general, it is clear that the above definition, through decomposition into digits, can convert numbers from the decimal system to another and vice versa. This is done as follows: suppose we have the number $4231$ that we want to convert to the ternary number system. First, write down all the powers of three that do not exceed this number: $3^0 = 1$, $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$, $3^6 = 729$, $3^7 = 2187$. Then start subtracting these powers from the original number, starting with the larger ones, as many times as possible: $4231 - 2187 = 2044$, $2044 - 729 \cdot 2 = 586$, $586 - 243 \cdot 2 = 100$, $100 - 81 = 19$, $19 - 9 \cdot 2 = 1$. Therefore,

$$4231 = 2187 + 729 \cdot 2 + 243 \cdot 2 + 81 + 9 \cdot 2 + 1 =$$
$$= 1 \cdot 3^7 + 2 \cdot 3^6 + 2 \cdot 3^5 + 1 \cdot 3^4 + 0 \cdot 3^3 + 2 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0 = 12210201_3.$$

But there is also another way, based on division with the remainder: the number is successively divided by the base power until it becomes $0$, and then all the obtained remainders, written in reverse order, form the number in the required number system. Let's illustrate this method using the same number:

$$\begin{aligned}
4231 : 3 &= 1410 \text{ (remainder } 1),\\
1410 : 3 &= 470 \text{ (remainder } 0),\\
470 : 3 &= 156 \text{ (remainder } 2),\\
156 : 3 &= 52 \text{ (remainder } 0),\\
52 : 3 &= 17 \text{ (remainder } 1),\\
17 : 3 &= 5 \text{ (remainder } 2),\\
5 : 3 &= 1 \text{ (remainder } 2),\\
1 : 3 &= 0 \text{ (remainder } 1),
\end{aligned}$$

Hence, $4231 = 12210201_3$.

To convert from a non-decimal number system to a decimal one, you can use these two methods applied in reverse. The first one requires little explanation, and the second one is done as follows for a system with base $s$:

$$\overline{a_n \ldots a_1 a_0}_s = \Big( \ldots (a_n \cdot s + a_{n-1}) \cdot s + \ldots + a_1 \Big) \cdot s + a_0.$$

For our example:

$$12210201_3 = \Big((((((1 \cdot 3 + 2) \cdot 3 + 2) \cdot 3 + 1) \cdot 3 + 0) \cdot 3 + 2) \cdot 3 + 0\Big) \cdot 3 + 1.$$

This gives $4231$.

It is also worth noting the possibility of quickly converting from a system with base $s$ to a system with base $s^k$, done as follows. First of all, the original number is divided into groups of $k$ digits. Starting from the right, each group is converted to the new number system, and the result should consist of 1 digit. The resulting digits are written in the order in which the groups of digits of the original number were written. Conversion from a base $s^k$ system to a base $s$ system is done in a similar way, in reverse order. Let's provide an example.

**Problem 4.10.** Convert the number $37ba7af8_{16}$ from hexadecimal to octal.

*Solution.* It could take a lot of time and effort to solve this problem by first converting it to the decimal system and then to the octal system. Therefore, let's use the method described above. First, convert the given number to the binary system by representing each digit in binary, adding leading zeros to make each group have $4$ digits:

$$3_{16} = 0011_2,\ 7_{16} = 0111_2,\ b_{16} = 1011_2,\ a_{16} = 1010_2,$$
$$7_{16} = 0111_2,\ a_{16} = 1010_2,\ f_{16} = 1111_2,\ 8_{16} = 1000_2.$$

Therefore, $37ba7af8_{16} = 110111101110100111101011111000_2$.

Now, convert the resulting binary number to the octal system by splitting it, starting from the end, into groups of 3 digits and converting them to octal digits:

$$110_2 = 6_8,\ 111_2 = 7_8,\ 101_2 = 5_8,\ 110_2 = 6_8,\ 100_2 = 4_8,$$
$$111_2 = 7_8,\ 101_2 = 5_8,\ 011_2 = 3_8,\ 111_2 = 7_8,\ 000_2 = 0_8.$$

Therefore, the number is $6756475370_8$, which is the answer. □

**Problem 4.11.** Consider the numeral system with base $n$. When will a number in this system be divisible by $n-1$?

*Solution.* Let's first understand what we need to prove. In the decimal system, the divisibility criterion for $9$ was based on the sum of the digits of a number. Maybe it will be the same here?

It is the same! A $k+1$-digit number in the base-$n$ numeral system looks like:

$\overline{a_k \ldots a_1 a_0}_n = a_k n^k + \ldots + a_1 n^1 + a_0 n^0.$

We notice that $n \equiv 1(\text{mod}(n-1))$, which implies $\forall i \geqslant 0 \quad n^i \equiv 1(\text{mod}(n-1))$.

Then, $a_k n^k + \ldots + a_1 n^1 + a_0 n^0 \equiv a_k + \ldots + a_1 + a_0(\text{mod}(n-1))$. Thus, we have even proven a stronger statement: in the base-$n$ numeral system, a number will have the same remainder when divided by $n-1$ as the sum of its digits. □

Fractions, of course, can also exist in different numeral systems.

**Problem 4.12.** Express the number $13/16$ in the base-6 numeral system.

*Solution.* Let

$$\frac{13}{16} = \frac{a_1}{6} + \frac{a_2}{6^2} + \ldots$$

Multiplying both sides by 6, we get:

$$\frac{39}{8} = a_1 + \frac{a_2}{6} + \ldots.$$

$\frac{39}{8} = 4 + \frac{7}{8}$, thus on one side there will be $4$ plus a number less than $1$, and on the other side, there will be $a_1$ plus a number less than $1$.

Therefore, $a_1 = 4$. Substituting this into the original expression, we get

$$\frac{7}{48} = \frac{a_2}{6^2} + \frac{a_3}{6^3} + \ldots,$$

which, after multiplying by $6^2$, is reduced to $a_2 = 5$.

Following similar reasoning, we see that $13/16 = 0.4513_6$ □

**Problem 4.13.** The increasing sequence $1, 3, 4, 9, 10, 12, 13, \ldots$ consists of integers that are either powers of three or the sum of different powers of three. Find the $100$th term of this sequence.

*Solution.* If we write the terms of this sequence in the base-3 number system, we get a sequence of integers that do not contain the digit $2$. Then, these numbers represent $1, 10, 11, 100, 101, 110, 111, \ldots$. Imagining that these numbers were in the binary numeral system, we get $1, 2, 3, \ldots$. To get the $100$th term of the sequence, it is necessary to write the number $100$ in the binary numeral system $100 = 1100100_2$, and then, imagining that this number were still written in the base-3 numeral system, convert it to the decimal system: $1100100_3 = 3^6 + 3^5 + 3^2 = 981$. □

## 4.7 The Weight of Nothing: Trailing Zeros

Prime numbers are the building blocks of number theory, often considered the "bricks" of mathematics due to their foundational role in constructing all other numbers. So, what acts as the "cement" that binds these bricks together?

The Fundamental Theorem of Arithmetic states that every natural number greater than one is uniquely decomposed into a product of prime numbers raised to certain powers:

$$M = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n},$$

where $p_1 < p_2 < \dots < p_n$ are prime numbers, and $0 < \alpha_1, \alpha_2, \dots, \alpha_n$ are the exponents representing the frequency of these primes.

The number of all divisors of a number can be determined combinatorially. Let's include 1 and the number itself as divisors. The task reduces to an elementary combinatorial problem: there are $n$ piles of items, and the size of the $i$-th pile is $\alpha_i$. From each pile, you can take any number of items or none at all. How many different combinations can you get? The answer is simple: each pile gives us $\alpha_i + 1$ independent selection options. Therefore, the total number of options will be $(\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_n + 1)$.

**Problem 4.14.** How many zeros are at the end of $2024!$?

*Solution.* Let's solve an auxiliary problem: how many zeros are at the end of the number $x = 2^{\alpha_2} 3^{\alpha_3} 5^{\alpha_5} 7^{\alpha_7} \dots$?

From the Fundamental Theorem of Arithmetic, it follows that for $x = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$ to be divisible by $y = p_1^{\beta_1} p_2^{\beta_2} \dots p_n^{\beta_n}$, all exponents of the corresponding factors $p_i$ in $x$ must be at least as large as in $y$.

Recall that if a number ends with $k$ zeros, it is divisible by $10^k = 2^k 5^k$. The problem reduces to finding the largest number $k$ such that $x$ is divisible by $2^k 5^k$. For our problem, this means $k = \min(\alpha_2, \alpha_5)$, i.e., the number ends with as many zeros as the smaller of the powers of 2 and 5 in its prime factorisation.

Let's compute the power of 2 in the factorisation of $n!$.

Let's write the factorisation terms, for example, for $10!$:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10$$

Under each even number, place a plus; under each odd number, place a minus.

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ - & + & - & + & - & + & - & + & - & + \end{array}$$

Did we account for all the twos? No, $4 = 2 \cdot 2$, and there is only one plus under $4$. Let's add another plus under all numbers divisible by $4$:

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ - & + & - & + & - & + & - & + & - & + \\ - & - & - & + & - & - & - & + & - & - \end{array}$$

What about eight and numbers divisible by eight?

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ - & + & - & + & - & + & - & + & - & + \\ - & - & - & + & - & - & - & + & - & - \\ - & - & - & - & - & - & - & + & - & - \end{array}$$

Under each number, we have as many pluses as the power of two in its prime factorisation. Other prime factors' powers are not considered here. Count the number of pluses in each row:

$$\alpha_2 = \left\lfloor \frac{n}{2^1} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \left\lfloor \frac{n}{2^3} \right\rfloor + \dots,$$

where $\lfloor x \rfloor$ is the floor function.

It is easy to prove that a similar formula is valid for any prime factor, particularly for $p = 5$:

$$\alpha_5 = \left\lfloor \frac{n}{5^1} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \dots$$

Let's calculate $\alpha_5$ for $n = 2024$.

$$\alpha_5 = \left\lfloor \frac{2024}{5^1} \right\rfloor + \left\lfloor \frac{2024}{5^2} \right\rfloor + \left\lfloor \frac{2024}{5^3} \right\rfloor + \left\lfloor \frac{2024}{5^4} \right\rfloor = 404 + 80 + 16 + 3 = 503,$$

while $\alpha_2$, is obviously larger. Thus, the number ends with $503$ zeros. □

*A professor is testing a very weak applicant. Eventually, it turns out that the applicant cannot even solve the simplest inequalities.*
*"Tell me," he asks, losing all hope, "what is greater:* $-1$ *or* $0$*?"*
*"Of course, minus one!" the applicant confidently replies.*
*"But why?!" the professor exclaims, grasping his head.*
*"Well,* $-1$ *is something at least, while* $0$ *is absolutely nothing."*

# To Prove or not to Prove?

"

A group of mathematicians are at a team building seminar. During the night, a fire breaks out in one of the mathematician's rooms. He quickly tears pages out of his notebook, lighting them on fire one by one. He then runs down the hall, sliding sheets of burning paper under other mathematician's doors.
After the building burns to the ground, the fire marshal asks the mathematicians how the fire spread so fast.
He responds. "I thought distributing the problem would lead to finding a solution faster."

—One joke that didn't quite land

## 5.1 One Step Forward, Two Steps Back

In your life, you've probably had an argument or two. But even if you're a seasoned internet warrior who knows how to drop those solid proofs, proving something in a technical interview is a different beast altogether.

Often, proof techniques don't require knowledge beyond high school math. Yet, you've likely forgotten some of those elegant tricks. While no one's asking you to solve high-order differential equations, the ability to clearly and precisely prove something is valued not just in technical interviews but also on the job — though the armchair warriors might still doubt you.

Number theory and combinatorics are frequent guests in these interviews, often hiding behind seemingly simple puzzles. Whether you're estimating the runtime of an algorithm or solving a tricky combinatorial problem, solid proof techniques are indispensable.

Understanding and applying principles like the extreme principle, double counting, and invariants are essential for demonstrating your analytical prowess. These techniques not only help in solving problems but also in presenting your solutions convincingly.

Imagine being asked a brain teaser involving light switches or hats, common fare in coding interviews. It's not just about getting the right answer; it's about showing your method, proving why your solution works, and ruling out other possibilities. Your ability to construct a well-reasoned proof can be the difference between standing out and blending in.

In this chapter, we explore these fundamental proof techniques, providing you with the tools to tackle even the most challenging interview questions with confidence. So, let's dive into the world of proofs and equip ourselves with the skills to not just solve, but to prove.

And when it comes to hat problems, maybe the ability to think several steps ahead will help you survive.

## 5.2 Extreme Principle: No Half Measures

The title of the "extreme" principle speaks for itself — to solve the problem, you need to find and consider an "extreme" object. What does "extreme" mean? This term can be a little confusing, especially if it's a problem involving numbers rather than a geometric problem. "Extreme" usually means the largest or smallest number that expresses some property present in the problem. For example, it could be the greatest distance between points if a set of points is given, the smallest of the triangle areas, the smallest angle, and so on.

**Problem 5.1.** There are $11$ weights weighing $1, 2, \ldots, 11$ grams. Five of them are bronze, five are silver, and one is gold. All the bronze ones together weigh less than all the silver ones by $30$ grams. How much does the gold one weigh?

*Solution.* The difference in weight between the five heaviest weights ($11+10+9+8+7 = 45$) and the five lightest weights ($1 + 2 + 3 + 4 + 5 = 15$) is exactly $30$ grams, which means the difference of $30$ grams can be achieved in only one way, and therefore, the gold weight is the remaining one, weighing six grams. □

**Problem 5.2.** Prove that there exist $100$ consecutive integers among which exactly $7$ are prime.

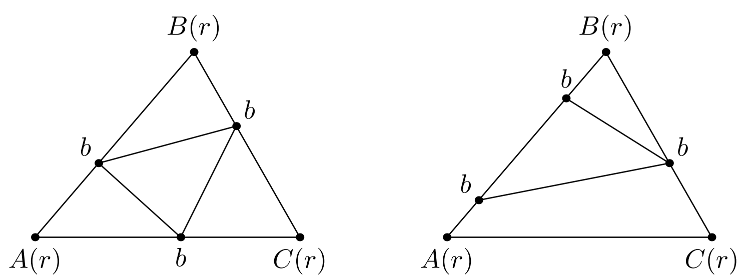
*Solution.* It is easy to understand that among the first $100$ natural numbers, there are more than $7$ primes. We have already listed them before.

It is also easy to give an example of $100$ consecutive composite numbers: $101! + 2$, $101! + 3, \ldots, 101! + 101$. The first of them is divisible by $2$, the second by $3$, ..., and the last by $101$.

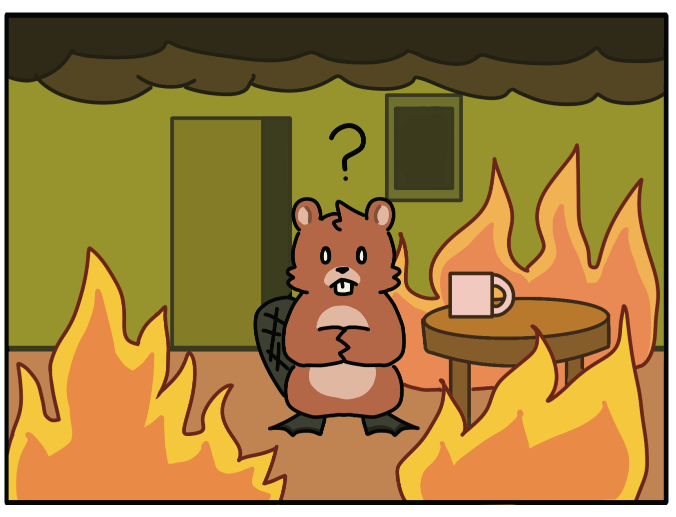
We will "shift" a row of $100$ numbers to the right along the number line: in this process, one number will be removed from the left, and one will be added to the right. During such a process, the number of composite numbers can change by at most $1$. Initially, there were more than $7$ prime numbers, and in the end, there were $0$. Therefore, due to discrete continuity, there exists a moment in time when there are exactly $7$ prime numbers, as required to prove. □

**Problem 5.3.** On the plane, some points are coloured blue and red in such a way that no three points of the same colour lie on the same line (there are at least three points of each colour). Prove that some three points of the same colour form a triangle, and there are no more than two points of the other colour on each side of this triangle.

*Solution.* Consider the triangle $ABC$ with all vertices of the same colour, which has the smallest area. Without loss of generality, we can assume that the vertices are red. Let's prove that such a triangle is the desired one. Suppose the opposite: then there are at least 3 blue points on its sides. Two cases are possible: each blue point lies on its own side, or two blue points lie on one side, and one lies on the other side. In both cases (shown in the figure below), the area of the triangle with vertices at the blue points will be smaller than the area of triangle $ABC$, which cannot be: triangle $ABC$ has the smallest area among single-colour triangles. This contradiction completes the proof of the problem.




□

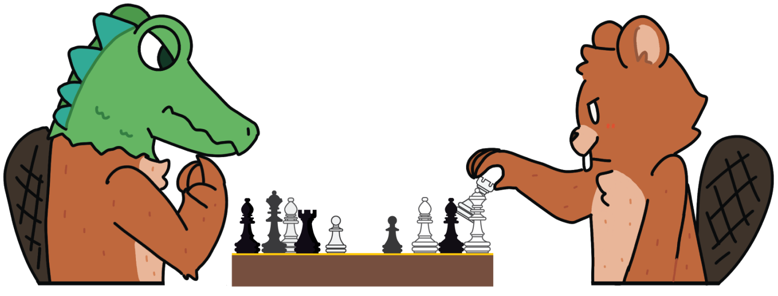


*An engineer, a physicist, and a mathematician are staying at a hotel.*
*The engineer wakes up and smells smoke. He goes out into the corridor, sees a fire, fills a waste-*

*basket with water, puts out the fire, and goes back to bed.*
*Later, the physicist wakes up and smells smoke. He opens the door, sees the fire in the corridor, calculates the flame speed, distance, water pressure, trajectory, and other parameters, and efficiently extinguishes the fire using the least amount of water and energy.*
*Finally, the mathematician wakes up and smells smoke! He enters the hall, sees the fire, then the fire hose... He thinks for a moment and exclaims: "Ah, the solution already exists!" Then he goes back to bed.*

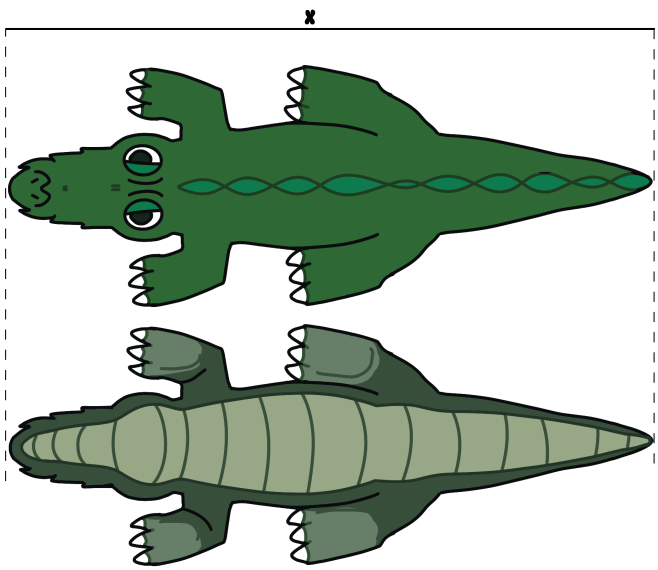
**Problem 5.4.** Four chess players from Beaverland and 7 foreign chess players played in a round-robin tournament, where each played two games with each other. The winner of the game received one point, the loser received zero, and for a draw, each received half a point. All participants scored a different number of points, and the sum of all points of Beaverland chess players turned out to be equal to the sum of all points of foreign chess players. Prove that there was at least one Beaverlander among the top three winners.

*Solution.* Count the total number of games played. There are $110$ games in total, as each player played against each other twice ($11 \cdot 10 \cdot \frac{1}{2} \cdot 2 = 110$). Thus, $110$ points were scored. Therefore, each group scored $55$ points. The best player from Beaverland scored at least $14.5$ points (since $14.5 + 14 + 13.5 + 13 = 55$). If the top three places were taken by foreign players, they scored at least $15 + 15.5 + 16 = 46.5$ points. Then, the remaining $4$ foreign players scored a total of $8.5$ points, which is less than the $12$ points they would have scored only in games against each other. This is a contradiction. ☐



## 5.3 Counting Twice: Double the Fun, Double the Trouble

In some problems, you can get the required equation if you calculate the same quantity in two ways. The difficulty is to figure out exactly what value to calculate in two ways (and which ones exactly!).



*Another wonderful illustration of applying mathematical methods to zoology.*
*Theorem: A crocodile is longer than it is wide.*
*Proof.*
*Take an arbitrary crocodile and prove two auxiliary lemmas.*
*Lemma 1: A crocodile is longer than it is green.*
*Proof. Look at the crocodile from above — it is long and green. Look at it from below — it is long but not as green (actually, it is dark grey). Therefore, lemma 1 is proven.*
*Lemma 2: A crocodile is greener than it is wide.*
*Proof. Look at the crocodile from above again. It is green and wide. Look at it from the side: it is green but not wide. This proves lemma 2.*
*The statement of the theorem obviously follows from the proven lemmas.*
*The converse theorem ("A crocodile is wider than it is long") is proven similarly.*
*At first glance, both theorems imply that the crocodile is square. However, since the inequalities in their formulations are strict, a true mathematician will draw only one correct conclusion: CROCODILES DO NOT EXIST!*

**Problem 5.5.** 30 students from the seventh and eighth grades exchanged handshakes. It turned out that each seventh grader shook hands with eight eighth graders, and each eighth grader shook hands with seven seventh graders. How many seventh graders and how many eighth graders were there?

*Solution.* Let $x$ be the number of seventh graders, and $y$ be the number of eighth graders; then $x + y = 30$. The second equation is obtained if we calculate the total number of handshakes in two different ways. On the one hand, the number of handshakes is equal to $8x$, since each seventh grader "initiates" 8 handshakes. On the other hand, the number of handshakes is equal to $7y$, since each eighth grader "initiates" 7 handshakes. Therefore, $8x = 7y$. Solving the resulting system of equations, we find $x = 14$, $y = 16$. □

The next problem is also related to the recently discussed extreme principle.

**Problem 5.6.** The cells of a $15 \times 15$ square table are coloured in red, blue, and green. Prove that there are at least 2 rows with an equal number of cells of at least one colour.

*Solution.* Suppose the opposite: this would mean that each colour has a different number of cells in each row. Thus, the minimum number of cells of any colour would be

$$0 + 1 + 2 + \ldots + 14 = 14 \cdot 15 \cdot \frac{1}{2},$$

and the total number of coloured cells would then be at least

$$14 \cdot 15 \cdot \frac{1}{2} \cdot 3 = 21 \cdot 15.$$

However, there are exactly $15 \cdot 15$ cells in total, which is less. This contradiction completes the solution to the problem. □

The following problem will also touch on the extreme principle as well as the theme of this section, and also involve an estimation plus an example.

**Problem 5.7.** Sixteen teams participated in a football championship. Each team played against each other once, with 3 points awarded for a win, 1 point for a draw, and 0 points for a loss. We call a team successful if it scores at least half of the maximum possible number of points. What is the maximum number of successful teams in the tournament?

*Solution.* Suppose that all $16$ teams could be successful. Since each team plays $15$ games in total, the maximum possible number of points that one team can score is $15 \cdot 3 = 45$. Since only integer points can be scored, for a successful performance, each team must have at least $23$ points. Then, in total, all teams must score at least $16 \cdot 23$ points. Let's try to estimate the total number of points scored by all teams differently. In one game, the teams together score either $2$ or $3$ points, so in any case, no more than $3$ points are scored. There are a total of $16 \cdot 15 \cdot \frac{1}{2}$ games, so the total number of points scored is no more than

$$16 \cdot 15 \cdot \frac{1}{2} \cdot 3 = 16 \cdot 22.5,$$

which leads to a contradiction.

An example with $15$ successful teams: Team number $16$ loses all its matches. The remaining teams play as follows: teams with even numbers win against teams with odd numbers, and vice versa. Then, each of them wins $8$ out of $15$ games and becomes successful. □

## 5.4 Guess-timation: Estimation + Example

This topic stands out in Olympiad mathematics: essentially, it is not a separate topic — a problem on almost any topic can be an estimate + example.

This topic can be identified by words like "maximum" or "minimum" in the problem statement. For example, a problem might be of the form: find the largest number with a given property. What does solving such a problem imply? Often one encounters solutions of this type: "this number works", followed by a proof that it works, and that's where the solution ends. However, the requirement was to find the largest number, not just any number that works. The part of the solution where a specific number is presented is called an example. However, it is also necessary to provide an estimate, showing that this number is indeed the largest.

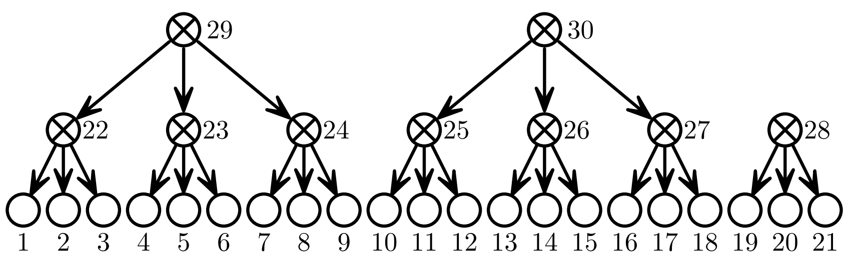
It's even more frustrating when the situation is reversed: an estimate has been proven, but the solver considers the example obvious and doesn't bother saying it.


*A physicist, a mathematician, and an engineer are standing in a field. Each is given an equal number of boards to build a fence and told to enclose the maximum number of sheep.*
*The engineer builds a small but sturdy square pen.*
*The physicist constructs a circular pen, claiming that this shape can contain more sheep.*
*The mathematician builds a tiny fence around himself, sits in the centre, and says: "Assume I'm outside."*

**Problem 5.8.** 30 pikes were released into a pond, and they started eating each other. A pike is considered satiated if it has eaten at least three pikes. What is the maximum number of pikes that could become satiated if the eaten pikes are also counted in the tally?

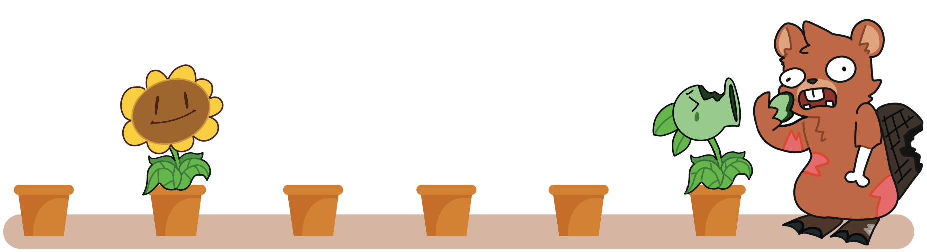
*Solution.* **Estimate.** Notice that if 10 or more pikes become satiated, then 30 or more pikes will be eaten — that is, all the pikes available, which is impossible (assuming a pike cannot eat itself — which is quite obvious).

**Example.** Graphically. The crosses in the diagram below represent the satiated pikes, while the circles represent the hungry ones.




□

**Problem 5.9.** In a row, there are 150 flower pots, some of which have plants growing in them. Among any three consecutive flower pots, there is at least one with a plant. A zombie passed along the row and ate some of the plants, so now, among any five consecutive flower pots, there is at most one plant. What is the minimum number of plants it could have eaten?



*Solution.* Let's consider 2 consecutive flower pots with plants: between them, there are at most 2 empty pots, which means one of them (those with plants) will definitely be emptied.

Therefore, at least half of the total number of plants will be eaten in the case where it is even, and half of the total number of plants minus one in the case where it is odd. Let's divide our $150$ flower pots into $50$ consecutive triples. It is clear that in each of them, there is at least one flower pot with a plant, so there are at least $50$ such pots in total, resulting in at least $25$ plants being eaten.

An example where exactly $25$ plants could be eaten is as follows: initially, the plants are located at numbers divisible by $3$, i.e., $3, 6, 9, 12, \ldots$, and then the zombie eats the plants at numbers divisible by $6$. It is easy to see that this example satisfies the condition, and indeed $25$ plants have been eaten. □

Additionally, as in game problems, it is advisable to avoid the notion of the "best case scenario" — it is impossible to explain what this phrase means. Usually, when asked to explain what the best case is, the answer should be: "Well, it's obvious." Such an answer will not satisfy anyone.

Let's illustrate this with an example by solving the problem above. I have encountered this reasoning, starting with the following words: "The fewer plants that are initially placed, the fewer plants the zombie will eat" — this is an attempt to move from the general case to the "best" one. Of course, this statement is incorrect since it does not mention that the zombie will try to eat as few plants as possible. And by moving to the "best case scenario" with $50$ planted plants, the participant showed that with arrangements $1, 4, 7, 10, \ldots$; $2, 5, 8, 11, \ldots$; $3, 6, 9, 12, \ldots$, eating fewer than $25$ plants seems impossible. However, since these options for placing $50$ plants are not exhaustive (there are so many of them that listing them all using exhaustive enumeration would take too much time), this does not prove anything, even for this "best case."

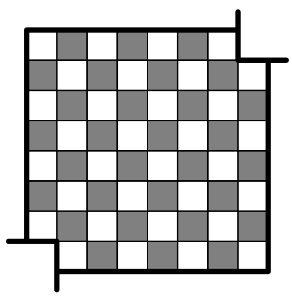
It is also worth noting that considering any "best cases" can be quite useful when searching for an example, but one should be wary of this concept only when proving an estimate.

## 5.5 Colour Wars

Well-thought-out colouring of figures can work wonders. Most of us have been familiar with the chessboard since childhood and the arrangement of black and white squares. This colouring helps solve some problems in a much more visual and understandable way.

As a warm-up, let's consider one of the most famous colouring problems in this context.

**Problem 5.10.** Imagine a board of size $8 \times 8$ cells, with the bottom-left and top-right corners removed (see the figure below). Can such a board be cut into dominoes of size $1 \times 2$?

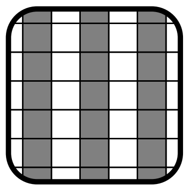


*Solution.* Trying to find an example of such a partition, we constantly fail, making us wonder — maybe it's impossible to do? But how can we prove that it's impossible? After all, there are a huge number of ways to arrange dominoes, and neither the participants of the competition nor the most powerful computer will be able to iterate through them during a competition. This is where the chessboard colouring comes to our rescue.

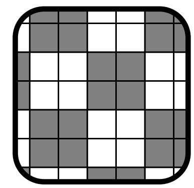
First, consider the board without the removed corners: in it, the number of white and black squares is the same — $32$ squares each. Now, notice that the bottom-left and top-right corners are of the same colour. Let's assume this colour is black (the case with the white colour is absolutely analogous). Then, the board with two removed corners has $32$ white squares and $30$ black squares. But each domino consists of one white and one black cell! Therefore, the board cut into dominoes must contain an equal number of black and white cells, which is not the case. Hence, it is impossible to cut this board into dominoes. □

This type of colouring is far from the only one used to solve problems of this type. For example, the following colourings are also quite common:

- "Striped colouring" — in this colouring, the first column is entirely black, the next one is white, then black again, and so on (see Figure a));
- "Large chessboard colouring" — almost the same as the regular chessboard colouring, but everything is divided into $2 \times 2$ squares (see Figure b)).



a) "Striped coloring"



b) "Large chessboard coloring

Colourings are usually used to prove the non-existence of the required partition. Usually, it is not immediately clear which pattern to use for the solution, so the first step is determining which colouring will help solve the problem. It may be necessary to try several different colouring methods for this. Let's consider the following problem.

**Problem 5.11.** Imagine a board of size $10$ by $10$. Can it be cut into rectangles of size $1$ by $4$?

*Solution.* Let's try using a chessboard colouring — then there will be $50$ black and white squares each. However, each rectangle occupies $2$ black and $2$ white squares, so theoretically, $25$ rectangles of size $1$ by $4$ would give $50$ black and $50$ white squares, and there is no contradiction in this case. But this does not imply that such a partition exists.

Let's use the "large" chessboard colouring. Then the board is divided into $25$ squares of $2$ by $2$, and of them, $13$ will be black and $12$ will be white. Therefore, there will be a total of $52$ black and $48$ white squares. However, each rectangle of size $1$ by $4$ occupies exactly $2$ black and exactly $2$ white squares. The resulting contradiction shows that it is impossible to cut the board into the specified rectangles. □

*The university rector reviewed the estimate brought to him by the dean of the faculty of physics and, sighing, said:*
*"Why do physicists always demand such expensive equipment? For instance, mathematicians only ask for money for paper, pencils, and erasers."*
*And, after some thought, he added:*
*"Philosophers are even better. They don't need erasers at all."*

## 5.6 The Unchangeables

Let's say there is a "process" in the problem; for example, on the board, there are numbers that can be changed based on a rule or position from which certain moves can be made. Let's assume that it is a single-player game. Such signs indicate that the problem concerns an invariant and not another topic.

**Definition 8.** *A*n invariant is a mathematical quantity or property that remains constant, i.e., does not change under a certain transformation.

*Today, I visited the profile of an old acquaintance and saw that he has written that he changed his life by* $360$ *degrees. So, guys, this is why it's important to study mathematics in school!*

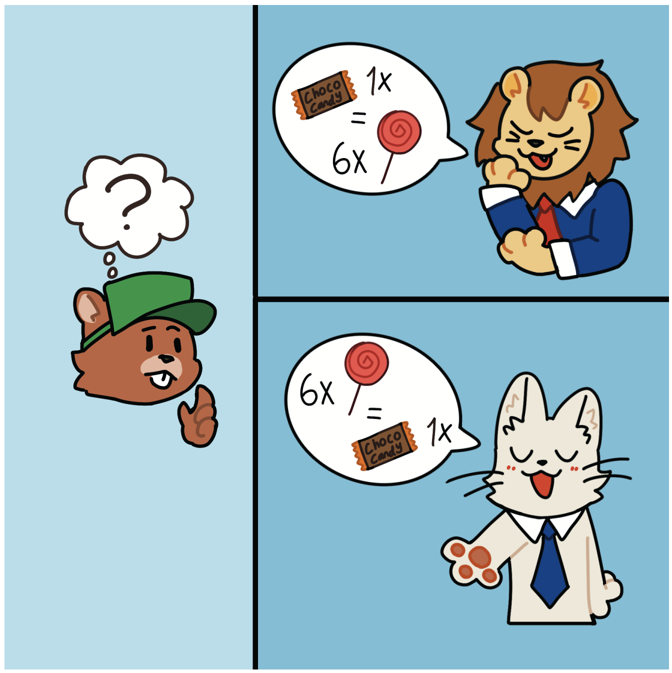
Conservation laws from physics are also invariants.

Often, the invariant is the parity or the remainder of division by some number or a combination of some numbers related to the given problem. Let's consider one of the typical problems on invariants.

**Problem 5.12.** There are $6$ numbers written on the board — $3$, $14$, $15$, $9$, $2$, $6$. In one operation, $1$ can be added to any two numbers. Can all numbers be made the same?

*Solution.* At first glance at the problem statement, the idea would be to try to increase the smaller numbers so that they reach the larger ones. However, after a bit of struggle, we can equalise five numbers, but the sixth one will be different from them. Using brute force to solve the problem won't work, as there are infinitely many cases (what if they, for example, equalise at a million). To prove that this is impossible, the principle of the invariant will help.

How to find the invariant? We need to track a certain pattern. What didn't change when one was added to two numbers? It is easy to understand that the sum of all numbers will increase by $2$, so the sum is **not** an invariant. But we know that if $2$ is added to a number, its parity does not change. Thus, the parity of the sum of all numbers always remains unchanged and equal to what it was originally. But what was it? Since among the numbers $3, 14, 15, 9, 2,$ and $6$, there is an odd number of odd numbers, then the sum of all numbers will be odd. But we need all numbers to be equal to each other. Therefore, their sum must be divisible by 6 (their quantity); that is, it must be even. Hence, it follows that these numbers cannot be made equal. □




**Problem 5.13.** In the office, a business is thriving: Leo exchanges 1 chocolate candy for 6 lollipops, and Max exchanges 1 lollipop for 6 chocolate candies. You arrive with 1 chocolate candy. You can turn to your new colleagues for an unlimited number of exchanges. Can you end up with an equal number of lollipops and chocolate candies, provided that you don't eat them?

*Solution.* Let's try to find an invariant. At each moment in time, your financial position is characterised by two numbers: the number of chocolate candies $X$ and the number of lollipops $Y$. For convenience, let's denote it by the pair $(X, Y)$. This pair could change in one move as follows: either it turns into $(X-1, Y+6)$ or into $(X+6, Y-1)$. Notice that the difference between the numbers changes by 7. This means that the remainder of the difference between the number of chocolate candies and lollipops divided by 7 does not change. Ultimately, this difference, and therefore the remainder when divided by 7, must become zero. But initially, it was equal to 1. This is impossible, so you cannot achieve this goal. □

## 5.7 The Handshake Conspiracy

Graphs are rarely mentioned in schools but you are suddenly supposed to know how to work with them when you become a college student. However, graph problems can sometimes be quite straightforward, and solving them may require only knowledge of terminology, a bit of theory, and common sense.

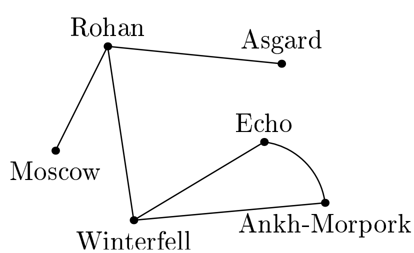
So, what is a graph?

**Definition 9.** *A graph* is a finite set of points, some of which are connected by lines. These points are called *vertices* of the graph, and the connecting lines are called *edges*.

Like many mathematical objects, graphs have their own characteristics.

**Definition 10.** The number of edges emanating from a given vertex is called the *degree* of the vertex. A vertex of a graph with an odd degree is called *odd*, and a vertex with an even degree is called *even*. The degree of a vertex $v$ is commonly denoted by $\deg v$.

For example, in the figure below, the degree of the vertex "Rohan" is $3$ since there are three roads leading from it — to Moscow, Winterfell, and Asgard.




Let's formulate and prove the following important statement.

**Lemma 1** (Handshake Lemma)**.** The number of odd vertices in any graph is even.

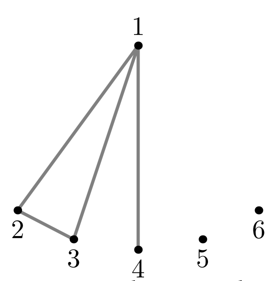
*Proof.* Consider each edge of the graph as a "thread" connecting 2 vertices. Then the sum of all vertex degrees in the graph is the total number of "threads" emanating from all vertices. But we counted each "thread" twice because it has 2 ends. Thus, the sum of the degrees of all vertices in the graph is even. Then the number of odd vertices in any graph is even, as required. □

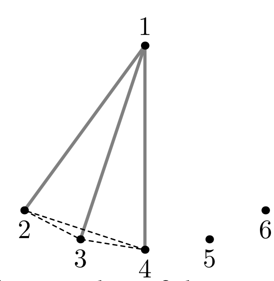
**Problem 5.14.** During his pre-election campaign, a candidate for mayor of the city of Beavers promised to install new telephone lines between houses as follows: there are a total of 25 houses in the city, and each house must be connected to exactly 5 others. Prove that he will not be able to keep his promise.

*Proof.* Let the houses be vertices, and the telephone lines be edges of the graph. Then the degree of each vertex is 5. The sum of all degrees is $5 \times 25 = 125$, which should be equal to twice the number of edges. But this is impossible, as 125 is an odd number. □

**Problem 5.15.** Each edge of a complete graph with 6 vertices is coloured with one of two colours. Prove that there are three vertices such that all edges between them are of the same color.

*Proof.* Let's consider one of the vertices of this graph. Due to the completeness of the graph, there are 5 edges emanating from it. By the pigeonhole principle, at least 3 of these edges are the same colour. Let's number the vertices of the graph. Let the considered vertex have the number 1, and the edges of the same colour extend to the vertices numbered 2, 3, and 4.




Then, if vertices 2 and 3, 3 and 4, or 2 and 4 are connected by an edge of the same colour, we have identified a monochromatic triangle. If all these edges are of a different colour, then a monochromatic triangle is highlighted on vertices 2, 3, and 4. □

## 5.8 I Put on my Robe and Wizard Hat

Hat puzzles are a favourite in the world of brain teasers. They usually hinge on the ability to think several steps ahead and deduce the situation based on limited information.

**Problem 5.16.** In a pitch-black closet, there are five hats: three blue and two red. Three clever individuals enter the closet, each picking a hat and placing it unseen upon their heads.

Outside the closet, no one can see their own hat. The first person looks at the other two and says, "I can't tell what colour my hat is". The second person hears this, looks at the other two, and says, "I can't tell what colour my hat is either". The third person is blind. The blind person says, "Well, I know what colour my hat is". What colour is their hat?

The assumption here is that everyone in the puzzle can solve it as well as we can. Just to clarify, the first two people are not blind.

*Solution.* The first person can't determine the colour of their hat. They would only know if they saw both others wearing red hats (since there are only two red hats). This isn't the case, so either both others are wearing blue hats, or one is wearing red and the other blue.

When the second person speaks, they've considered the same logic. If the third person were wearing red, the second person would know they must be wearing blue (as they can't both be wearing red). Since the second person is also unsure, the third person must be wearing a blue hat, making the second person uncertain whether they are wearing red or blue.

Thus, the third person must be wearing a blue hat. □

Things get a bit more intense when your life is on the line.

**Problem 5.17.** One hundred prisoners are given a chance to win their freedom. Each prisoner will wear either a red or blue hat, chosen at random. They can see everyone else's hat but not their own. Once the hats are on, they can't communicate. They will be called out in a random order to guess their hat colour. If correct, they're free; if wrong, they're executed.

They have one night to devise a strategy to save as many prisoners as possible. What's the best plan, and how many can they save?

*Solution.* At least 99 prisoners can be saved with a clever strategy.

The key lies with the first prisoner, who will announce their hat colour based on the parity of red hats he sees: if they see an odd number of red hats, they declare their hat red; otherwise, blue. They have a $50/50$ chance of guessing correctly.

The rest can deduce their hat colour by keeping track of the number of red hats and using the first prisoner's declaration. If the first prisoner says "red" and the second prisoner sees an even number of red hats, they know their own hat must be red, and so on. This strategy ensures that 99 out of 100 prisoners can be saved. □

I would hope not to be called first.

If you thought two colours were tricky, try three!

*Solution.* At least 99 prisoners can still be saved, but the first prisoner's survival odds drop to $1/3$. Here's the plan:

Assign scores: red = 0, green = 1, and blue = 2. The first prisoner calculates the total score for the 99 others, finds the remainder when divided by 3, and guesses accordingly:

- If $s \bmod 3 = 0,$ announce red;
- If $s \bmod 3 = 1,$ announce green;
- If $s \bmod 3 = 2,$ announce blue.

The first prisoner has a $1/3$ chance of guessing right. Each subsequent prisoner can use the announced remainder and the scores of the other $98$ hats to deduce their own hat's score.

For instance, after the first one who said "blue" was executed, the second prisoner sees $3$ red, $14$ green, and $81$ blue hats. The total score of those $32$ hats is:

$$3 \cdot 0 + 14 \cdot 1 + 15 \cdot 2 = 44 \equiv 4 + 4 = 8 \equiv 2 \bmod 3$$

If the first prisoner said the remainder is $2$ (blue), the next prisoner knows their own hat must be red. □

This clever strategy can theoretically be expanded to any number of colours, though it requires exceptional memory and calculation skills.

*Three mathematicians and three physicists are heading to a conference in another city. They meet at the train station. The physicists, like normal people, each buy one ticket. The mathematicians, however, buy just one ticket for all of them.*
*"How come?" the physicists ask in surprise. "Won't the ticket inspectors kick out the two of you without tickets?"*
*"Don't worry!" the mathematicians reply cheerfully. "We have a METHOD."*
*Before the train departs, the physicists take their seats but keep an eye on the mathematicians to see this mysterious "method" in action. The mathematicians all cram into one bathroom. When the inspector comes by and knocks on the door, a hand reaches out with the ticket. The inspector takes it, and the entire group travels without further issues.*
*After the conference, the same physicists and mathematicians meet again at the station. Inspired by the mathematicians' example, the physicists buy one ticket. The mathematicians, however, buy none.*
*"What will you show the inspector?" the physicists ask.*
*"Don't worry, we have a METHOD," the mathematicians reply.*
*On the train, the physicists cram into one bathroom and the mathematicians into another. Just before the train departs, one mathematician goes to the physicists' bathroom, knocks, and takes their ticket when they hand it out.*
*Thus, remember: You can't use mathematical methods without understanding their essence!*

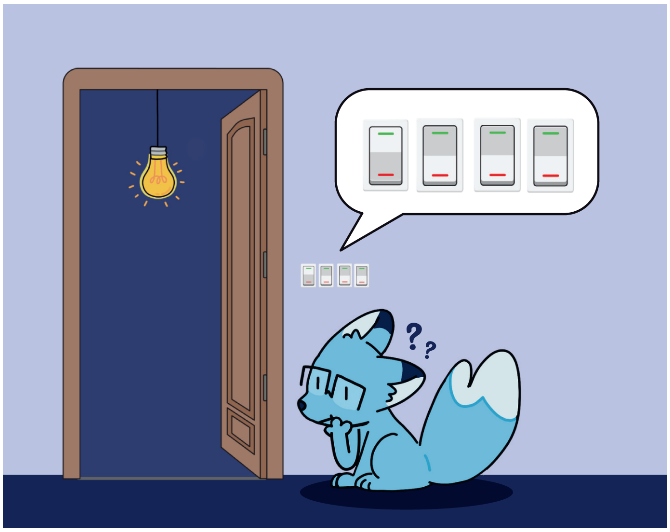
## 5.9 The Illuminati: Switch Happens

Light switch puzzles are a classic genre of brain teasers that often pop up in technical interviews, testing your logic and strategy skills. Let's dive into some of these electrifying challenges.

When it comes to light bulbs, we're not very eco-friendly: these puzzles typically assume bulbs heat up when switched on. Ok, boomer, what's the deal?

**Problem 5.18.** You're in a room with three light switches, each controlling one of three light bulbs in the next room. All lights are off, and you can't see into the other room. You can inspect the other room only once. How can you determine which switch controls which bulb? Is this possible?

*Solution.* Turn on the first switch and leave it on for a few minutes. Then turn it off and turn on the second switch. Enter the room immediately. The bulb that is on corresponds to the second switch. The bulb that is off but warm corresponds to the first switch. The bulb that is off and cold corresponds to the third switch. □



Similarly, here's another puzzle that hinges on understanding the information a light bulb can encode. Whether the light is on or off is binary, but adding the temperature factor increases the combinations.

**Problem 5.19.** There's a light bulb in a room and four switches outside. All switches are off at first, and only one controls the bulb. You can flip switches as many times as you want but can only check the bulb once. How can you figure out which switch controls the bulb?

*Solution.* Here's the plan: Turn on switches 1 and 2, then take a break and dance with a crocobeaver for a while. After a bit, turn off switch 2 and flip on switch 3. Now, rush into the room and feel the bulb.

- Light on and hot? Switch 1 is your answer.
- Light off but hot? Switch 2 did the job.
- Light on and cold? Switch 3 nailed it.
- Light off and cold? Switch 4 is the culprit.

Voilà! You've cracked the case with one trip. □

Light switch puzzles have been featured in numerous films and TV shows, often adding an element of suspense or intellectual challenge to the plot. Sometimes, it's the professor's whimsical experiments in The Mysterious Benedict Society, or it can be a life-and-death scenario like the test of the Four of Diamonds in Alice in Borderland.

If the puzzle doesn't hinge on bulb temperature, the solution frequently involves appointing someone as a "counter" to keep track of events.

**Problem 5.20.** Fifty wise men are held captive by a sultan. Each minute, one is called to flip a glass in one of two positions or do nothing. The wise men will be called randomly, possibly forever. If any wise man correctly states that all have been called at least once, they go free; if wrong, they all get an involuntary job change to eunuchs in the sultan's harem. They can only discuss a strategy before imprisonment. Nobody's eager for a new career path, so what can they do?

*Solution.* One wise man gets dubbed "The Counter". Everyone else flips the glass upside down the first time they see it bottom down. The counter flips it back to the bottom down each time and keeps a tally. Once he has flipped it 49 times, he yells, "We've all been called!" and they all walk free, avoiding an unfortunate career shift. □

Additional question: How much time will pass on average before these wise men are freed?

# To Decide or Not to Decide?

> “Three logicians walk into a bar. The bartender asks, “Do all of you want a drink?”
> The first logician says, “I don’t know.”
> The second logician says, “I don’t know.”
> The third logician says, “Yes!”

—One joke that didn’t quite land

## 6.1 Choose Wisely, Young Padawan

In your life, you've probably had to make some tough decisions. Maybe you've debated the merits of pineapple on pizza or struggled to choose between two equally appealing TV shows. In the world of technical interviews, decision theory is a bit like that — except it involves math, logic, and often, mind-bending paradoxes.

Imagine facing a problem where multiple solutions seem viable. How do you choose the best one? Decision theory provides the framework to evaluate options based on probabilities, outcomes, and the impact of each decision. It helps quantify uncertainty and assess the trade-offs, ensuring you make the most informed choice possible.

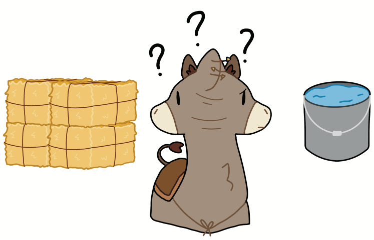
Take, for instance, the classic problem of the Monty Hall paradox. This puzzle challenges your understanding of probability and decision-making. Initially, it seems counterintuitive to switch doors, but decision theory reveals a higher probability of winning if you do. Such insights can be crucial in technical interviews, demonstrating your ability to think critically and analytically.

In this chapter, I'll explain some counterintuitive ideas in decision theory and also remind you of the most important terms so you don't feel completely lost.

So, gear up and get ready to dive into decision theory. It's not just about making choices; it's about making the right choices.

## 6.2 The Ass is Forked

There's a well-known paradox called the **Buridan's Ass Paradox** (Latin: Asinus Buridani inter duo prata). The Buridan's Ass paradox originates from the legend that an ass died of hunger between two piles of hay (or between a bucket of oats and a bucket of water) because it couldn't make a choice. Strictly speaking, this isn't a logical paradox but rather an extreme example of a dilemma taken to the absurd.




The Buridan's Ass paradox doesn't appear in any known works of Jean Buridan, although it aligns well with Buridan's theory of freedom and animals. Instead, this theme appears in Aristotle's "On the Heavens", where he ponders how a man would proceed when faced with a choice between water and food.

So, how can an ass receiving two equally tempting treats make a rational choice? Buridan addressed this by defending the position of moral determinism, arguing that when faced with a choice, one should lean towards the greater good. Buridan admitted that the choice might be complicated by evaluating the outcomes of each option.

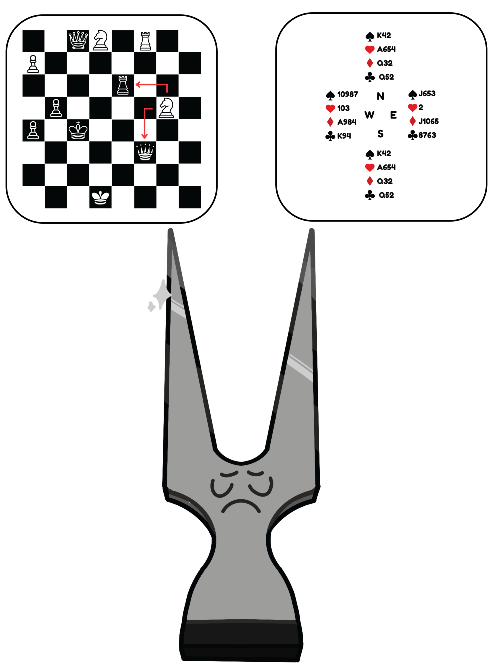
Later thinkers exaggerated this viewpoint, citing the example of an ass and two equally accessible and appetising piles of hay, arguing that the ass would surely starve to death making a decision. This version became widely known through Gottfried Wilhelm Leibniz (1646—1716), a German philosopher and scientist. He was also a member of the French Academy of Sciences.

Within the logic of the problem itself, one can show that a rational ass would never starve to death, although we can't say which pile of hay it will choose. If we consider refusing to eat as a choice and the worst outcome among three options — a pile to the left, a pile to the right, and starvation — the third option will be dominated by other strategies (we'll discuss this later), so the ass will never choose that option.

In some interpretations, a potential change in the situational context is also considered: combining the two piles into one, after which the dichotomy disappears. Although there are different types of dichotomies, this is a mind-body dichotomy, essentially a philosophical dichotomy.

We can also mention a slightly different case called **Morton's Fork**, describing a choice between two equally unpleasant alternatives, or a situation where two lines of reasoning lead to equally unpleasant conclusions. It's said that this fork originated from the rationalisation of benevolence by the 15th-century English prelate John Morton.

"Morton's Fork Coup" is a manoeuvre in the game of bridge that uses the principle of Morton's Fork. In chess, a fork is a tactical move that attacks two or more of the opponent's pieces simultaneously, aiming to gain a material advantage. Since the opponent can only protect one of the two attacked pieces, the other will be lost. We speak of a piece "making a fork" and pieces "being forked". For example, in a knight's fork, it's usually a knight that attacks two opponent's pieces at the same time. Even though it's not a pleasant situation in itself, here, generally, the principle of "choosing the lesser of two evils" works, and we then give away the less valuable piece.



## 6.3 Swipe Right or Left?

The Secretary Problem is a mathematical problem in the theory of optimal stopping in decision theory, probability theory, and statistics. The problem is also known as the Princess Problem and the Immediate Hiring Problem.

The context is as follows: someone wants to hire a secretary and observes a finite and known number of candidates. For each candidate, they must decide whether to hire them or not. If they hire a candidate, the hiring process ends without seeing the remaining candidates. Otherwise, they don't have the option to recall that candidate later. In the context of this problem, the recruiter doesn't have access to the intrinsic value of the candidates (like "this candidate is worth $7/10$"). They can only compare them (for example, "This candidate is better than the first but worse than the second").

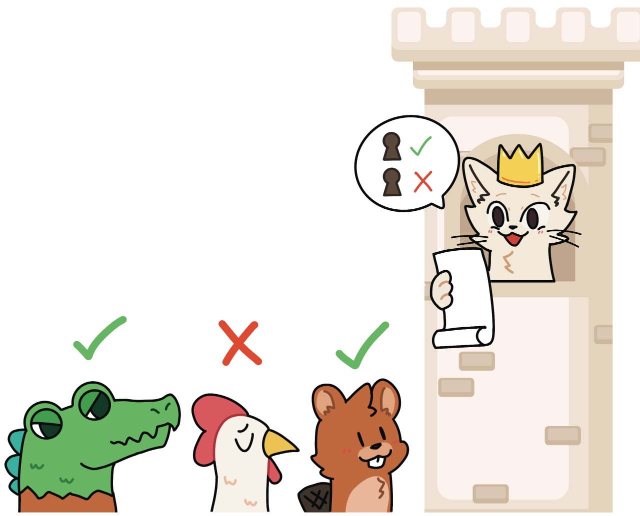
The goal is to define a strategy that maximises the probability of hiring the best candidate.

At first glance, this task seems insurmountable, even deceptive. However, this problem actually has an elegant mathematical solution. The practical wisdom derived from the problem often gets lost in the pages of books on probability theory. We think that's quite unfortunate, as there are many situations where knowing the optimal strategy for choosing among unknown alternatives can be useful. For example:

- Deciding to rent an apartment in a crowded city.
- Quickly finding the highest card while shuffling a deck.
- Searching for a low-cost store without backtracking.
- Commitments to a long-term partner.

In all these cases, you don't know which options will follow. However, you might want to make a quick but fair decision. Our goal is to explain the solution to the Secretary Problem in understandable terms and illustrate it if necessary.

Faced with total uncertainty, it's tempting to rely on luck. You might make an arbitrary decision: "Anyway, I'll choose the first option". Unsurprisingly, this random strategy doesn't work well. You're only relying on the chance that the first candidate is the best. The same holds true when you always choose the last candidate or always candidate number 2. Your odds are always the same for any pre-prepared option. The random strategy becomes less and less useful as the number of candidates increases.



You might have realised that the only variable you control is the number of options you've rejected. Your strategy can only be to decide on the number of options you want to reject before really starting to make decisions. The essence of the approach is that you want to wait long enough to have a good starting point and then choose the next candidate who is better than the options you've already looked at. In quantitative terms, this strategy is formulated as follows:

- Look at the first $X$ options and reject them. Remember the best option. Let's call it $B$.
- Continue looking at the following options until the first one with a higher score than $B$ is found. Select that option.

This strategy seems promising, but one detail needs clarification: how many options should you reject?

When the number $X$ is too large, you might set high selection criteria. But you also run the risk of saying no to the best option. When the number is too low, you have a starting point that can be really far away from the optimum. You're likely to choose a suboptimal option. What we need to do is find the optimal value for the number of rejections given the total number of candidates. To understand this, we need calculations that we'll spare you from due to their complexity.

The optimal strategy is to let $37\%$ of the candidates pass (or, more precisely, a proportion of $1/e$) and then wait for a candidate better than all those in this initial sample. This is sometimes referred to as the "37% Rule".

*"Do you love math more than you love me?"*
*"Darling! How could you even think that!"*
*"Well, then prove it!"*
*"Alright. Let* $R$ *be the set of all beloved objects."*

## 6.4 Betting on Infinity

The Saint Petersburg Paradox can be summarised by the following question: why do players refuse to risk all their money in a game where the expected value of winnings is infinite? This is not purely a mathematical problem but a paradox concerning human behaviour when faced with events of a random variable that is likely to yield a small value but has an infinite expected value. In this situation, probability theory dictates a decision that no reasonable player would make.

The player places an initial bet, taken by the bank. A coin is tossed, with the outcome being either heads or tails. As long as heads appear, the game continues. It ends when tails appear, and then the bank pays out its winnings to the player. The initial payout is one euro, doubled for each appearance of heads. Thus, the payout is $1$ if tails appear on the first toss, $2$ if tails appear on the second toss, $4$ on the third, $8$ on the fourth, and so on. Therefore, if tails appear for the first time on the $n$-th toss, the bank pays out $2^{n-1}$ dollars to the player.

What should the player's initial bet be for the game to be fair, meaning the player's initial bet equals their expected winnings? In other words, what is the player's expected average gain over the course of one game?

If heads come up on the first toss, you win \$1. The probability of this happening is $1/2$, which gives an expected gain for this case of $1/2 \times 1 = 1/2$ dollar. If heads appear for the first time on the second toss, which occurs with a probability of $1/2 \times 1/2 = 1/4$, the gain is \$2, resulting in an expected gain of $1/2$ dollar for this case. More generally, if heads appear for the first time on the $n$-th toss, occurring with a probability of $(1/2)^n$, the gain is \$$2^{n-1}$, hence an expected gain of $1/2$ dollar for that toss.

The expectation is obtained by summing the expected gains of all possible cases. The result is, therefore, an infinite sum of terms, all equal to $1/2$, making it infinite. Thus, the game is favourable to the player in all cases unless the initial bet is infinite.

The paradox lies in the fact that if only the gain mattered, it would be rational to bet one's entire fortune to play this game, given that we have just seen that it offers an infinite expected gain (which is much higher than any bet). Yet, as observed by Daniel Bernoulli, no one would actually do such a thing.

The answer to this paradox has been of three types:

- People don't do it due to an inability to understand the correct calculation and its result;
- Because the value of money is not a simple linear function: each additional dollar is assigned a different utility;
- Because risk has a cost, and a fifty-fifty chance of winning two dollars or nothing is not worth one dollar.

These three axes are not mutually exclusive; they can all be true at the same time and contribute to the decision to limit one's bet.

This paradox was formulated in 1713 by Nicolas Bernoulli (1695—1726), a Swiss mathematician. The first publication was by his brother Daniel Bernoulli, "Specimen theoriae novae de mensura sortis", in the Commentarii of the Imperial Academy of Sciences of Saint Petersburg (hence its name). However, this theory dates back to a private letter from Gabriel Cramer to Nicolas Bernoulli in an attempt to answer this paradox. For both authors, the player refuses to bet everything because they cannot risk losing all their money. In this theory of moral expectation formalised by Bernoulli, they introduce a marginal utility function. However, these two authors disagree on the utility function: natural logarithm for Bernoulli and square root for Cramer.

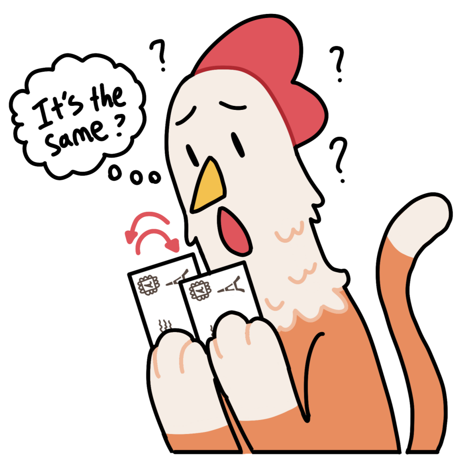
For them, it's the utility that matters to the player, not the gain, and this utility is decreasing. This means that doubling the amount won does not double in terms of utility. In the context of the paradox, the utility then takes on a finite and relatively low value, making it rational to make a limited bet. So, for players, there is a kind of inflation, or even hyperinflation, of utility.

*A mathematician organises a raffle in which the prize is an infinite amount of money paid over an infinite amount of time. Of course, with the promise of such a prize, his tickets sell like hotcakes.*
*When the winning ticket is drawn, and the jubilant winner comes to claim his prize, the mathematician explains the mode of payment: "1 dollar now, 1/2 dollar next week, 1/3 dollar the week after that..."*

## 6.5 Switcheroo

The Two Envelopes Paradox is a classic problem in decision theory that leads to a seemingly contradictory result.

There are several versions of the paradox. Most commonly, it presents the following decision situation: Two envelopes each contain a cheque. You know that one cheque contains twice the amount of the other, but you have no information on how the amounts were determined. A host offers a participant the choice of one envelope, and the amount in the chosen envelope becomes theirs.




The paradox itself lies in the following argument: before the participant opens the chosen envelope, the host advises them to switch with the following reasoning.

Let $V$ be the value of the cheque in the chosen envelope. There are two possible scenarios:

- There's a one in two chance that the other envelope contains a cheque twice the amount (thus worth $2V$);
- There's a one in two chance that the other envelope contains a cheque half the amount (thus worth $V/2$).

The expected amount obtained by switching envelopes would then be $E_{switch} = 50\% \times 2V + 50\% \times V/2 = V + V/4 = 5/4 \times V$, which is greater than $V$.

Thus, the contestant would be better off switching envelopes, which is absurd since both envelopes play the same role and the contestant, having not yet opened the first envelope, has no means to distinguish between them.

The puzzle gained popularity thanks to Martin Gardner, who described it in 1982 in his book "Aha! Gotcha". Renewed interest in the paradox came after Barry Nalebuff published an article listing several probability paradoxes in the *Journal of Economic Perspectives.* After receiving many responses to this publication, he prepared a second article directly addressing the Two Envelopes Paradox titled "The Other Person's Envelope is Always Greener".

Could the same game be "more profitable" for each of the two partners? Clearly not. Isn't this a paradox because each player mistakenly believes their chances of winning and losing are equal?

From Nalebuff's perspective, the first satisfactory explanation of his problem is given by Sandy Zabell in "Losses and Gains: The Paradox of Exchange". To paraphrase Nalebuff's explanation:

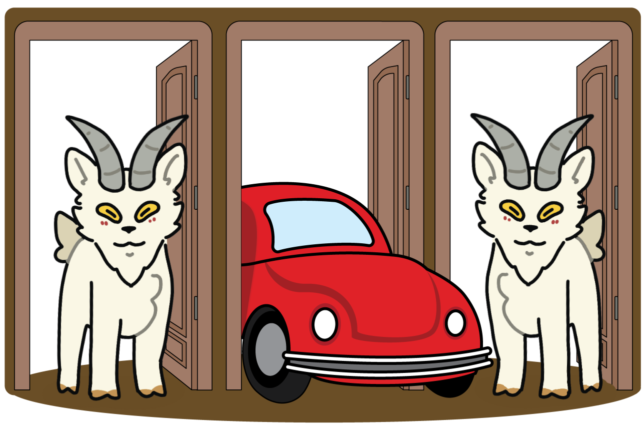
Everyone believes that the amount they see doesn't matter because there's a chance that a larger amount is in the other envelope. This means one assumes the probability of their envelope having a higher amount is $1/2$ regardless of what they see. This is only true if each value from zero to infinity is equally likely. But if all infinite possibilities are equally probable, then the probability of each value is zero. Then, each outcome has no chance, which makes no sense.

*When I needed to find a job after college, I was too lazy to write a cover letter. So, I posted a fake job ad and got about 200 applications. The best one became my cover letter.*

## 6.6 Behind Door Number Three

The so-called Monty Hall dilemma is a well-known puzzle named after the first host of the American TV show “Let’s Make a Deal”.

In this TV game, the host presented participants with a choice of three doors. Behind one door was a car, and behind the other two were goats. The car and goats were randomly placed beforehand and remained in their positions throughout the game.



After the participant made their choice, the host would always open one of the remaining two doors behind which, he knew in advance, there was no car. The contestant then had the option to stick with their initial choice or switch to the third door.

There are actually several possible strategies for Monty.

- Infernal Monty: The host suggests changing the choice if the door is correct.
- Angelic Monty: The host recommends changing the choice if the door is incorrect.
- Goaty Monty: From the start of the game, the host picks one of the goats and reveals it if the player has chosen a different door.

It is therefore preferable to base the problem on an unambiguous statement, including the constraints of the host and described by Mueser and Granberg as follows:

- There are three doors; one hides a car and the other two hide goats. The prizes are

randomly distributed.
- The host knows the distribution of the prizes.
- The player chooses one door, but nothing is revealed.
- The host opens another door, not revealing the car.
- The host suggests the contestant change their choice of door to open.

The host never opens the car door. So, if the player chooses a goat door, the host will open the only other goat door. Indeed, if the player chooses the car door, the host will randomly open one of the two goat doors (possibly pre-determined by a random draw).

The question then arises, "Does the player increase their chances of winning the car by changing their initial choice?" In other words, "Is the probability of winning by switching doors greater than the probability of winning without switching doors?"

The overwhelming majority of players and respondents refused to change their choice, even though it would have doubled their chances of winning. At the same time, people think that with the two remaining doors, the chances of winning are equal, and there's no point in changing their choice. If you think the same, don't feel embarrassed because you're not alone in this misconception.

Below is a part of a famous statement of the problem, from a letter Craig F. Whitaker published in the "Ask Marilyn" section of Marilyn vos Savant's Parade Magazine in September 1990:

> *Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?*

The publication of this article in Parade Magazine had an immediate impact on the readership and sparked countless discussions among mathematicians, both famous and unknown, and anonymous enthusiasts. Marilyn vos Savant received over 10,000 letters. As you can see, trolling thrived even in those times when it required much more time and effort than today. They even had to pay for a postal envelope and stamp.

The talented Hungarian mathematician Paul Erdős also fell into the trap and even refused to make a decision until he saw with his own eyes a computer simulation of the experiment's results. To be honest, it's hard to believe, but the rumour still spread.

| Door 1 | Door 2 | Door 3 | Result if you switch your choice | Result if you don't switch your choice |
|---|---|---|---|---|
| Car | Goat | Goat | Goat | Car |
| Goat | Car | Goat | Car | Goat |
| Goat | Goat | Car | Car | Goat |

For a winning strategy, the following is important: if you switch your door choice after the host's actions, you win if you initially chose the losing door. This will happen with a probability of $2/3$ because initially, you can pick a losing door in 2 out of 3 ways.

But often, when solving this problem, the reasoning goes something like this: the host always removes a losing door in the end, so the probability that a car appears behind the two unopened doors becomes equal, $1/2$, regardless of the initial choice. But this is not true: although there are indeed two choice options, these options (given the context) are not equally probable. This is because initially, all doors had an equal chance of winning but had different probabilities of being eliminated.

For most people, this conclusion contradicts their intuitive perception of the situation. Due to the emerging divergence between the logical conclusion and the intuitive option, the task is called the Monty Hall paradox.

The situation with the doors becomes even clearer if we imagine not 3 doors, but say, 1000, and after the player's choice, the host removes 998, leaving only 2 doors: the one the player chose and one more. It then seems more obvious that the probabilities of finding a prize behind these doors are different and not equal to $1/2$. If we switch the door, we only lose if we initially choose the prize door, with a probability of $1/1000$. We win in the case where our initial choice was wrong, with a probability of 999 out of 1000. In the case of 3 doors, the logic remains the same, but the probability of winning when the decision is changed is $2/3$ and not $999/1000$.

Did you experience cognitive dissonance trying to grasp this paradox?

*A phone rings in the dean's office of the mathematics faculty. The deputy dean, an associate professor from the calculus department, picks up the receiver.*
*"Tell me, how can I construct an angle of 50 degrees?" the question comes.*
*"Just a moment," says the deputy dean, covering the receiver with his hand, and begins to ponder aloud, "So, 50 degrees is approximately one radian..." Then he starts recalling about pi, the circumference of a circle, and so on. Seeing his struggles, another deputy dean, an associate*

*professor from the geometry department, joins the discussion. He authoritatively states that such an angle cannot be constructed with a compass and a ruler.*
*At that moment, the dean enters the room. They decide to ask him. In response, he decisively takes the receiver: "And who exactly is asking?"*
*"It's from the philology faculty," the voice says.*
*"Take a protractor," the dean cuts in and hangs up.*

## 6.7 The Test You Didn't See Coming

The Surprise Quiz Paradox was first raised by mathematics professor Lennart Ekbom in 1943. The first publication of such a paradox was O'Connor's article in 1948.

Let's present one of its settings:

A teacher announces to his students: "There will be a surprise quiz next week."

Let's clarify the terms. Three things should be understood:

- A quiz will take place during one class, either on Monday, Tuesday, Wednesday, Thursday, or Friday.
- Right before the start of the quiz, a student cannot be certain that the quiz will take place.
- Only one quiz will be given.

A clever student reasons as follows: if the quiz hasn't taken place by Thursday evening, then I will be certain it's on Friday. It won't be a surprise anymore. Therefore, the quiz can't be on Friday because it's the last possible day. But since the quiz can't be on the last day, the second-to-last day becomes, de facto, the last possible day. Thus, by recurrence, it's concluded that the quiz cannot take place.

Apparently, this seems to be just a misleading argument, similar in nature to the sorites paradoxes. It's a type of reasoning composed of a series of propositions arranged in the form: every $A$ is $B$, but every $B$ is $C$, but every $C$ is $D$, so every $A$ is $D$. The sorites is an extended syllogism.

However, the student can take the reasoning further. From the initial conclusion, the student must deduce that the teacher has necessarily lied. But in what way has he lied? If the quiz indeed takes place on Friday evening, then the lie is solely in the element of surprise. But since the teacher is a liar, there might not be a quiz at all. The initial reasoning is, therefore, no longer valid: the quiz will indeed be a surprise even if it occurs on Friday. Ultimately, the teacher will not lie if and only if he is taken as a liar. Thus, we encounter the liar paradox.

This paradox is actually inherent to the word *surprise* and the notion of *randomness*.

This paradox has multiple versions, with one of the earliest likely being Lennart Ekbom's surprise drill and the most renowned among mathematicians (courtesy of Quine and Gardner) being the unexpected hanging.

A similar paradox, named "Stevenson's Satanic Bottle Paradox", is a logical paradox described in Robert Louis Stevenson's "The Bottle Imp" (1893).

Keawe, a resident of the Hawaiian islands, buys a bottle on a trip to San Francisco. This bottle contains a little devil who grants all its owner's wishes. However, at the risk of damnation, the owner must part with it before dying. The only way to get rid of this devilish bottle is to sell it for a price lower than what was paid to acquire it. There's no other way to get rid of the bottle: if thrown away, it returns to its owner in an unknown manner. Moreover, fulfilling the wishes brings misfortune to the bottle owner's loved ones. The owner wished to become rich, and shortly after, his uncle and cousin died, leaving him a large inheritance.

The author presents a paradox in this story: what is the lowest price at which a bottle can be sold? Obviously, if you buy it at what's assumed to be the minimum price, let's say one cent, it becomes impossible to sell it at a loss. Therefore, it can't be sold for one cent because any buyer, knowing all the conditions of the transaction and its consequences, will refuse to buy it, as they won't be able to resell it. Similarly, it's impossible to sell it for two, three cents, or an approximate amount because your potential buyer will likely express doubts about the feasibility of such a transaction. Indeed, considering the possibility of a future sale, they might not find a buyer for the bottle. On the other hand, if the price of a bottle is still quite high, there's always a chance to find a buyer for it. But with each sale, the probability of finding such a buyer diminishes, and the loss of selling the bottle increases.

In the story, there are possible solutions to help the main character. For instance, the variation in exchange rates between different countries, the sacrifice of a loved one willing to buy the bottle at an extremely low price at the expense of their salvation, and finally, the indifference of this character to the consequences for their soul (because they are such a sinner that they'll burn in hell even without this curse). However, none of these solutions answer the question posed: what is the lowest price at which a bottle can be sold?

Comparing this paradox to the one described above, it becomes clear that there's no answer to the posed question. For each buyer of a bottle, except the last one, the answer to this question will depend on the case. Calculating logically your chances of selling the bottle is futile here, just like in the surprise test paradox.

*These problems are always interesting. The whole department is solving one of them right now. If they solve it, we will include it in the exam paper.*

## 6.8 Lies, Damned Lies, and Statistics

It's hard to think of an area of activity where decisions are not made from time to time. This situation and phenomenon are present always, in the past, present, and future. A person won't move a finger without making a decision about it. It's not always consciously realised, but that's how it is.

The science of decision-making has developed for a long time, one might say, one-sidedly. The classical scheme is covered by statistical theory, based on the risk function, on errors of the first and second kind.

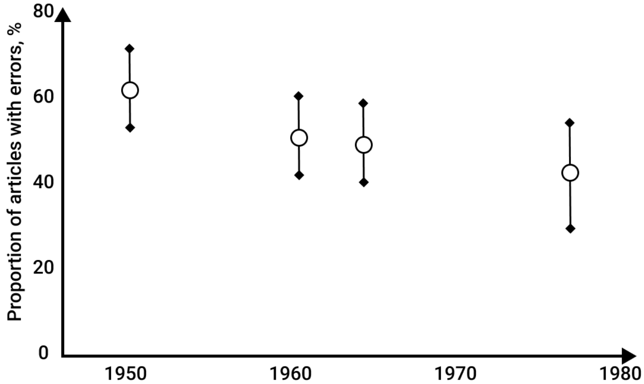
This approach to decision-making has played a positive role, and its applicability is not denied today, but it is limited by the principles of rationality. The approach is not without its shortcomings. Mark Twain wrote in a 1907 article in the North American Review: "Figures often beguile me... particularly when I have the arranging of them myself; in which case the remark attributed to Disraeli would often apply with justice and force: *There are three kinds of lies: lies, damned lies, and statistics.*"

*Three math statisticians went hunting. Suddenly, a huge wild boar runs out at them. The first statistician shoots and hits* 5 *centimetres to the left. The second one shoots and hits* 5 *centimetres to the right. The third one says with satisfaction: "Excellent, we hit it!"*

One of the fundamental properties of the human psyche is the inability to think statistically. From an evolutionary standpoint, it's necessary and crucial for us to primarily see cause-and-effect relationships: it gives us a sense of control, it allows us to survive in this world full of sabertooth tigers, crocodiles, and debt collectors, and it gives us a chance to pass our genes to the next generation. It's, so to speak, hardcoded into our DNA.

But the world is changing; we've become more complex and smarter (on average); we've invented sciences and the scientific method for a deeper and more accurate understanding of ourselves and the surrounding world. Among other things, we've invented statistics. It's a complex science — well, any science is complex, but statistics is also counterintuitive. If physics in its simple forms can be intuitively understood because its conclusions don't contradict our ability, developed at the age of two, to throw something fragile and marvel at the transformation of one large object into ten small ones, then with statistics, this trick doesn't work.

There's a very interesting study.




The figure shows the proportion of medical articles containing statistical errors. It's impossible to examine all the articles published in medical journals, so the proportion was determined from a random sample. As a result, an estimate of the true proportion of articles with errors emerges, and these estimates are shown as circles in the figure. The vertical segments represent the confidence interval, that is, the bounds within which the true proportion of articles with errors likely lies. This figure is from Glantz's book "Primer of Biostatistics". It's unknown whether anyone has checked the calculations for statistical errors in this diagram.

The brain is a powerful machine, but despite all its power, it's very lazy and doesn't like to process complex information. When decoding incoming information from the external environment, the brain, to conserve its own resources, makes errors: both type I and type II errors. For example, if you were a person from the Stone Age walking through a field and there was rustling in the tall grass around you. In most cases, the rustling is caused by the wind, but there's a small chance that a predator is hiding in the grass. A type I error would be the fear that there's a predator in the grass, and the alertness, whereas in reality, there's no predator, and you're safe. On the other hand, a type II error can be tragic if you thought it was the wind, but in reality, a predator is lurking and ready for a sudden attack.

## 6.9 Cheshire Cat's Smile

Type I error (alpha error, false positive) occurs when the true null hypothesis (for example, the absence of a relationship between phenomena or the sought-after effect) is rejected.

Type II error (beta error, false negative) occurs when the false null hypothesis is accepted.

In the USA, newborns undergo screening procedures for various congenital anomalies, including phenylketonuria and hypothyroidism. Despite the high level of type I errors, these procedures are considered appropriate because they significantly increase the likelihood of detecting these disorders at the earliest stage.

Simple blood tests used for screening potential blood donors for HIV and hepatitis have a significant level of type I errors; however, doctors have much more accurate (and consequently more expensive) tests available to verify whether a person is infected with any of these viruses.

Perhaps the most widely discussed type I errors occur in breast cancer screening procedures (mammography). In the USA, the level of type I errors in mammograms reaches 15%, the highest rate in the world. The lowest level is observed in the Netherlands — 1%.

Type II errors are a significant problem in medical testing. They create a false belief for both the patient and the doctor that the disease is absent when it actually exists. This often leads to inappropriate or inadequate treatment. A typical example is relying on the results of ergometry in detecting coronary atherosclerosis, although it is known that ergometry detects only those blood flow difficulties in the coronary artery that are caused by stenosis.

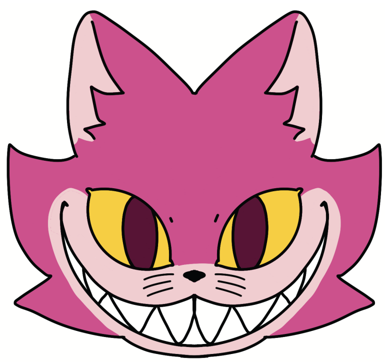
Type II errors pose serious and challenging problems, especially when the condition being sought is widespread. If a test with a 10% level of type II errors is used to screen a group where the probability of true positive cases is 70%, then many negative test results will be false.

Type I errors can also pose serious and challenging problems. This occurs when the condition being sought is rare. If the level of type I errors for a test is one case per ten thousand, but in the tested group of samples (or people), the probability of true positive cases averages one case per million, then most positive results from this test will be false.

Let's consider a theoretical case. Suppose a new disease called "bloodthirsty smile" has been discovered in the country of Cheshire Cats. Approximately one in every thousand residents is infected.

The medics in this country have proven to be very efficient: they quickly developed fairly effective tests to detect this disease. If a resident is sick, the test will show it with $99$ per cent accuracy, and there is a $1$ per cent chance of a false negative. If a resident is healthy, the test will show that they are healthy with $98$ per cent accuracy, but there is a $2$ per cent chance of a false positive.

Cheshire Cat Sophie was tested. Her test came back positive. What is the probability that she actually has the "bloodthirsty smile" disease?



It might seem that with such reliable tests, this probability should be quite high. At least, that's what our intuition tells us. Let's check if this is actually the case.

Let's consider $1{,}000{,}000$ cats. The expected number of infected cats among this thousand is $0.001 \cdot 1{,}000{,}000 = 1{,}000$ cats. Therefore, the expected number of healthy cats is $1{,}000{,}000 - 1{,}000 = 999{,}000$.

Let's consider the infected cats. The probability of identifying the disease is $0.99$, so the expected number of positive tests will be $0.99 \cdot 1{,}000 = 990$. The expected number of false negatives is $1{,}000 - 990 = 10$.

Now let's consider the healthy cats. The probability of a false positive test is $0.02$, so the expected number of false positive tests will be $0.02 \cdot 999{,}000 = 19{,}980$. The expected number of negative tests will be $999{,}000 - 19{,}980 = 979{,}020$.

| | | Is the cat sick? | |
|---|---|---|---|
| | | Sick | Healthy |
| What did the test show | Sick | 990 | 19,980 |
| | Healthy | 10 | 979,020 |

Thus, on average, the test will show that a cat is sick in $990 + 19{,}980 = 20{,}970$ cases out of a million. Of these, only $990$ correspond to the actual disease, that is, about $4.72\%$. Therefore, if Sophie received a positive test result, it is not very alarming. The probability that she is actually sick is less than five per cent, and it just means she should undergo further examination. And what is the probability of missing the disease? That is, being sick with a negative test result? It is $10/(10 + 979{,}020) \approx 0.001\%$, which is very small.

## 6.10 Causality Scene Investigation

Correlation and causality are fundamental concepts in statistics, research, and science, but they represent different relationships between variables.

**Definition 11.** **Causality** refers to a cause-and-effect relationship between two variables. If variable $A$ causes changes in variable $B$, then $A$ is said to have a causal effect on $B$. Establishing causality requires more rigorous methods than identifying correlation, typically involving experimental design and statistical analysis.

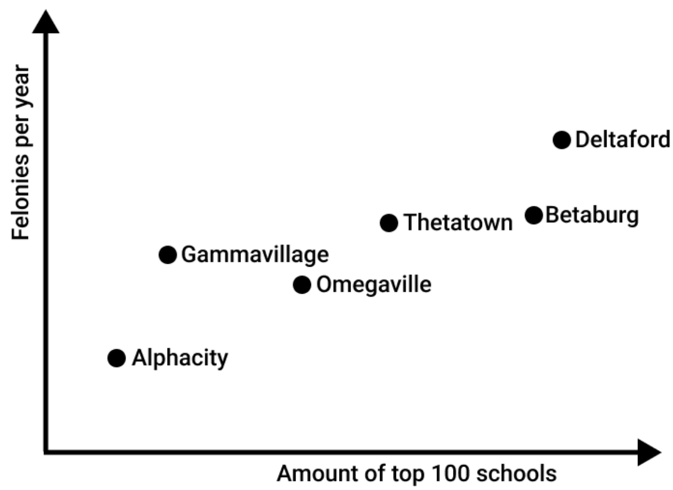
To infer causality, three main criteria must be met:

**Temporal relationship** The cause must occur before the effect.
**Correlation** A change in the cause should be associated with a change in the effect.
**No spurious relationship** There should be no confounding factors that could explain the observed relationship.

In many cases, even if two variables are correlated, it's challenging to establish a causal relationship. For example, ice cream sales and drowning deaths may be correlated because both increase during the summer, but one does not cause the other.

We can also present such statistics.

Suppose that in the Land of Oz today, there are many cities and schools. Unfortunately, along with economic and demographic growth, crime has also appeared in this country. The graph shows the number of schools from the country's top 100 in a city on the x-axis, and on the y-axis, the number of serious crimes in that city per year.




We see an obvious correlation. How can this be explained? Perhaps it is because people from high-ranking schools are so exhausted from studying that they become more aggressive? Or, conversely, do the children of criminals prefer to study harder to avoid repeating their parents' fate?

But maybe it's much simpler, and both of these parameters just depend on the size of the city?

Most scientifically-minded people understand that correlation does not imply causation.

There are numerous examples where an apparent correlation between two variables does not indicate a causal relationship:

- The number of pirates is negatively correlated with the increase in the Earth's average temperature (we understand that pirates do not affect global warming, but some economic factors led to a decrease in the number of pirates and an increase in CO2 emissions).
- The number of Nobel laureates is correlated with chocolate consumption.
- The preference for XXXL clothing sizes is correlated with the risk of a heart attack (who would have thought?!).
- The number of churches in a city may correlate with the number of bars (though it is evident that, as in the example above, both these values depend on the population size of the city).

There is a website where you can find a whole collection of interesting correlations that have nothing to do with causality:

**https://tylervigen.com/spurious-correlations**

*A physicist, a mathematician, and an engineer were each asked to establish the volume of a red rubber ball.*
*The physicist immersed the ball in a beaker full of water and measured the volume of the displaced fluid.*
*The mathematician measured the diameter and calculated a triple integral.*
*The engineer looked it up in his Red Rubber Ball Volume Table.*

The thing is, we intuitively assume that causal relationships are normal and highly probable because, otherwise, why would there be a correlation between $A$ and $B$ if there's no causal connection? It seems unlikely that there's a universal conspiracy constantly introducing factor $C$ to trigger option 3! So, when someone finds a correlation between $A$ and $B$, it's not surprising that they start saying something like:

"Sure, correlation doesn't imply causality, but... it's obvious that if you have many overweight friends, you're at risk of gaining weight, and hurricanes with female names lead to more deaths — for sexist reasons."

We desperately want to believe. It seems scientists are desperate to believe too. Because if correlation doesn't imply causality, what are they doing in most cases? If I conduct a study and get data indicating that moderate alcohol consumption is associated with a slight increase in average lifespan and should be taken into account by insurance companies, what does it matter to me if there's no causal relationship? When epidemiologists survey the entire population and triumphantly announce that there's a small association between meat consumption (but not eggs!) and average lifespan, who cares, except maybe the same insurance companies? Why are grants allocated for this, why do they spend time on this, why do they publish the results of these studies — if they don't have faith ("quasi-religious"?) that these correlations aren't just coefficients within a prediction model but reflect causality?

The world around us is the largest causal network, and it's not surprising that most of its correlations are not causal relationships.

Be careful.

# To What or What The What?

> “A biologist, a mathematician, and a physicist were asked to come up with a way to always win horse races. After a month, they were asked to share their progress.
> The biologist said:
> — In a month, I bred a breed of horses that have extraordinary speed and almost always win. To refine it, I need a couple more months.
> The mathematician said:
> — I've almost developed a theory that describes the probability of winning in each specific race. Now, I need approximately six more months, $1000, and an assistant to test it in action to reduce statistical errors.
> The physicist said:
> — To continue the work, I need $1000000, a well-equipped laboratory, a team of researchers, and another ten years. But I already have a theory of the victory of a liquid spherical horse in a vacuum.
>
> —One joke that didn't quite land

## 7.1 Laughs, Logic, and Lattes

In the last chapter of my book, you'll encounter a series of strange puzzles that go beyond the ordinary.

From figuring out salaries without spilling secrets to navigating a forest full of magically transforming beavers, these problems will tickle your brain and maybe even make you chuckle. Each puzzle is a testament to the creativity and ingenuity required to solve problems where the usual rules don't quite apply.

And remember, sometimes, the journey to the solution is just as entertaining as the solution itself. I hope you will enjoy the reading and have some fun.

In the country where I was born, they say that laughter prolongs life. I guess I will live forever, what about you?

## 7.2 Blue-eyed Tree

In logic and game theory, there's a significant difference between:

Everyone knowing something.

Everyone knowing that everyone else knows it.

Everyone knowing that everyone else knows that everyone knows it.

**Problem 7.1.** Imagine 2024 smart, logical people on an island with a quirky rule: if you deduce your eyes are blue, you must jump off a cliff at midnight. No mirrors, and no one can tell you your eye colour. One day, an outsider yells, "Hey, at least one of you has blue eyes!". What will happen?

Here's the twist: It's not just about knowing you might have blue eyes. It's about everyone knowing, knowing that everyone knows, and knowing that everyone knows that everyone knows. Only then do they start jumping!

*Solution.* Let's break this down step by step:

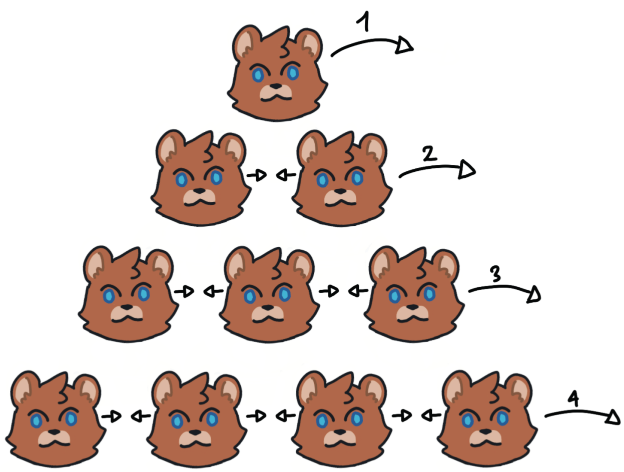
One blue-eyed person: If there were just one blue-eyed person, they'd look around, see no other blue eyes, and realise, "Wait, that must be me!" They'd tragically leave on the first night.

Two blue-eyed people: Each would see the other blue-eyed person and think, "If I don't have blue eyes, they'll leave tonight." When neither leaves the first night, they both deduce, "Oh snap, we both have blue eyes!" and both die on the second night.

Three blue-eyed people: They realise if there were only two, those two would have jumped on the second night. When no one's missing on the third morning, all three understand, "We're all blue-eyed!" and jump on the third night.

This logic scales up. Four blue-eyed folks jump on the fourth night, five on the fifth, and so on.

The crux of the matter is that the outsider's statement, "There's at least one blue-eyed person", kick-starts this chain reaction. Though everyone knew there were blue-eyed people already, the announcement synchronises their logical deductions. □




**Problem 7.2.** Imagine a magical forest with 2024 beavers and one lone tree. The forest is full of wood, but the beavers would much rather chew on that specific tree. Here are the rules:

- Whenever a beaver chews on the tree, it magically turns into a tree itself.
- All beavers are super smart, perfectly logical, and, of course, want to survive (maybe in tree form).

Question: Will the tree end up chewed down, or will it stay safe?

*Solution.* The case of two beavers and one tree is simple. If one beaver chews on the tree, it turns into a tree and gets chewed on by the remaining beaver. Thus, no beaver chews on the tree.

Three beavers and one tree: A beaver chews on the tree, turning it into a tree. Now, there are two beavers and one tree. In this situation (as shown above), the tree won't be chewed down. Result: The tree is chewed down.

Four beavers and one tree: If a beaver chews on the tree, it turns into a tree. Now, there are three beavers and one tree. In the three-beaver scenario, the tree gets chewed down. Result: No beaver chews on the tree.

Following this pattern, we find that when we have an odd number of beavers, the tree gets chewed down. When the number of beavers is even, the tree survives.

So, with $2024$ beavers (an even number) and one tree, the tree can breathe easily. □

## 7.3 Think Before You Breathe

There's a certain class of whimsical puzzles that you're unlikely to encounter in a job interview, but that might come up in a bar or a frat house.

**Problem 7.3.** You (a man) and three women decide to practice artificial respiration. One of the women has COVID-19, but it's unknown which one. You only have two masks. How can you practice on all three women without risking the spread of COVID-19? (You shouldn't get infected and shouldn't transmit the virus to any of the women.) Assume the masks provide full protection against COVID-19 when used for respiration.

*Solution.* It's actually quite simple. First, you wear two masks, one on top of the other, and perform artificial respiration on the first woman. Then, you remove the top mask and perform artificial respiration on the second woman. Afterwards, you put the removed mask back on, flip it inside out, and perform artificial respiration on the third woman. You always touch the same side of the mask, while the women always touch a clean side. □

Sometimes, the training scenario might involve a different composition of participants.

**Problem 7.4.** One day, two men and two women decided to conduct a session of artificial respiration. During the process, they realised they only had two masks. The question is: Can each person safely perform artificial respiration on all persons of different genders, completely eliminating the risk of COVID-19 transmission, or do they need to stop and get more masks?

*Solution.* Here's how it works: First, one man (let's call him Man $A$) wears both masks and performs artificial respiration on the first woman (Woman $X$). Afterwards, Man $A$ removes the outer mask and gives it to the second man (Man $B$). Man $B$ wears this mask (clean side in) and performs artificial respiration on Woman $X$. Man $A$ still has the clean inner mask on and can now perform artificial respiration on the second woman (Woman $Y$). Finally, Man $A$ removes his mask and gives it to Man $B$, who now wears both masks (clean side in) and performs artificial respiration on Woman $Y$. This ensures no one catches or spreads COVID-19. □

Feel free to try solving similar problems for different numbers of men ($m$) and women ($n$).

## 7.4 Number Nonsense

*One philosopher was shocked when Bertrand Russell told him that a false proposition implies any proposition. He said, "You mean that from the statement that two plus two equals five, it follows that you are the Pope?" Russell replied, "Yes." The philosopher asked, "Can you prove this?" Russell replied, "Certainly," and contrived the following proof on the spot:*
*(1) Suppose* $2 + 2 = 5$.
*(2) Subtracting two from both sides of the equation, we get* $2 = 3$.
*(3) Transposing, we get* $3 = 2$.
*(4) Subtracting one from both sides, we get* $2 = 1$.
*Now, the Pope and I are two. Since two equals one, then the Pope and I are one. Hence I am the Pope.*

*—Raymond Smullyan, "What Is the Name of This Book?"*

Knowledge of various mathematical concepts, even those seemingly paradoxical, like the sum of all natural numbers equaling $-1/12$, can be quite useful in interviews. It not only demonstrates your mathematical skills but also indicates your general culture and ability to think outside the box. Being able to discuss such topics shows your erudition and knowledge of geek culture, which can be appreciated in high-tech companies.

When we add numbers, increasing each subsequent addend by the same positive value, we get increasingly larger values with each new addend. For example, the sum of all natural numbers from $1$ to $10$ is $55$, from $1$ to $100$ is $5050$, and as the upper limit increases, the sum becomes larger. It logically follows that the sum of numbers from $1$ to infinity would yield an infinitely large number. However, if we perform the calculations, we get the value $-1/12$. It would be one thing if we obtained some finite positive number, although this would cause cognitive dissonance. It is entirely another matter when we obtain a fraction with a negative sign, which seems utterly absurd, as how can adding positive numbers result in a negative value? In this chapter, we will explore how mathematicians arrived at such a result.

Do you think that if you add all natural numbers, you will get infinity? The Indian mathematician Ramanujan showed at the beginning of the century that this sum equals $-1/12$. Let's dive into the depths of mathematics and figure out what is wrong with this value. Natural numbers are whole positive numbers from one to infinity. The sum of such numbers is a classic divergent series whose infinite sum should equal infinity. However, there are ways to assign a finite value to this series.

Mathematicians learned to sum divergent series back in the 19th century. For example, the method of summation by Cesàro helped find the sum of the alternating series of Grandi, which is the sequence $1 - 1 + 1 - 1 + 1 - \ldots$. This sum turned out to be $1/2$. The Abel method, developed later, allows summing more complex series like $1 - 2 + 3 - 4 + \ldots$. According to this method, the sum of such a series is $1/4$. However, none of these methods allow for the summing of all natural numbers.

So, how do we know that the sum of natural numbers is $-1/12$? It is all thanks to the work of several brilliant mathematicians, after whom the series and functions we will discuss are named. A method called regularisation of the Riemann zeta function is used to compute the sum of natural numbers. The Riemann zeta function is a function of the complex variable $s$, defined by the Dirichlet series. The value of the zeta function at $s$ is the infinite sum $\sum n^{-s}$, where the summation occurs over $n$ from 1 to infinity. If we take the value of the zeta function at $-1$, the terms of the series become equal to the natural numbers: $1^{-1} = 1$, $(1/2)^{-1} = 2$, $(1/3)^{-1} = 3\ldots$. The zeta function at $-1$ in this case equals $1 + 2 + 3\ldots$, that is, the sum of all natural numbers.

By using the relationship between the Riemann zeta function and the Dirichlet eta function, the value of the former can be quite easily computed. As a result, it turns out that the Riemann zeta function at $-1$ equals $-1/12$. This value is obtained because we transition from the real number plane to the complex plane. Thus, calculating the sum of natural numbers using this method turns out to be quite simple.

If you have read this far, then here is a pleasant bonus for you — an interesting fact. At first glance, it seems that this computation of the sum of the series of natural numbers is quite abstract and has no practical use. But in fact, the sum of this series appears in string theory and even helps describe the Casimir effect, which consists of the mutual attraction of conducting uncharged bodies in a vacuum due to quantum fluctuations. So, if school assignments like ”find the sum of all natural numbers” seemed strange to you, now you know that the results of these problems have practical applications.

There are also some numbers that are worth knowing. This has nothing to do with mathematics, but only if the interviewer mentions them or you get such a response will you understand that it was an Easter egg. Noticing it might help break the ice a bit. These include, for example, the first digits of the numbers $\pi$ and $e$, the number $42$, which is the answer to the ultimate question of life, the universe, and everything (from the movie ”The Hitchhiker's Guide to the Galaxy”), the number $28$, which is a perfect number, powers of two, and so on.

## 7.5 I Broke One and Lost the Other

*A group of scientists decided to conduct an experiment. They placed an applied mathematician, a pure mathematician, and a tester in separate, completely sealed rooms with no gaps or openings. Each person was given three steel balls to see what they would do. The next day, they checked on each participant. The applied mathematician was sitting and rolling the balls around. The pure mathematician was meditating with the balls. When they looked in on the tester, he was sitting and crying, holding only one ball. When asked where the other two balls were, he replied, "I broke one and lost the other."*

**Problem 7.5.** You have 11 laptops, each weighing a different number of kilograms from 1 to 11 kg. You know the weight of each laptop, but your customer does not. You have a very poor quality scale that breaks if you place an item weighing 11 kg or more on it. If you place something lighter than 11 kg, it shows nothing, but if it weighs at least exactly 11 kg, the scale hisses and curses. You need to sell a laptop that weighs 1 kg and prove that it is indeed the 1 kg laptop. How do you do it?

*Solution.* First, weigh the laptops weighing 1 kg, 2 kg, 3 kg, and 4 kg together. The scale does not break. The customer sees that there are four laptops on the scale, and can conclude that this must be the combination "1, 2, 3, 4".

We are not required to find the optimal solution, so let's proceed in a more straightforward manner.

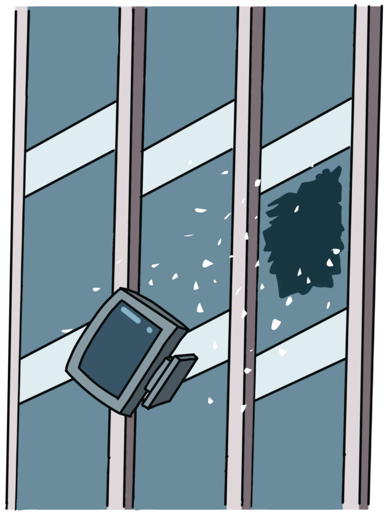
Next, weigh together laptops "$1, 2, 5$", "$1, 2, 6$", and "$1, 2, 7$". The scale does not break, confirming that two laptops in each group are the 1 kg and 2 kg laptops.

Finally, weigh "$1, 3, 5$" and "$1, 3, 6$". This will eliminate any doubts about which laptop weighs 1 kg.

Is there a more optimal solution? Think about it. □

**Problem 7.6.** You've got two computers and a 2024-story building. If you drop a computer from a window, it won't break if the floor number is less than $X$, but it will shatter if the floor number is $X$ or higher. Your mission: figure out what $X$ is. What's the game plan to keep your number of drops as low as possible, even in "the worst" case?

*Solution.* Alright, let's get strategic with these computers. Imagine we have a plan that uses $N$ drops at most. First, we'll throw the first computer from the $N$-th floor. If it breaks, we know $X$ is below that floor, and we can start dropping the second computer from the 1st floor up to $N-1$. Worst case, that's $N-1$ drops. So, in total, that's a max of $N$ drops.



Now, if the first computer doesn't break on the $N$-th floor, we have $N-1$ throws left. We'll head up $N-1$ floors higher, to floor $2N-1$, and drop again. If it breaks, the second computer checks the floors between $N$ and $2N-2$. Again, we're using up to $N$ drops.

Following this strategy, the two computers cover $N + (N-1) + (N-2) + \ldots + 1$ floors, which sums up to $N(N+1)/2$ floors. To cover all 2024 floors, we need $N(N+1)/2$ to be at least 2024. Solving for $N$, we get $N \approx 64$.

Here's the step-by-step:

- Start at floor 64. If it breaks, use the second computer from 1 to 63.
- If it doesn't break, go to floor $64 + 63 = 127$. Repeat the process:
  - If it breaks at 127, use the second computer on floors 65 to 126.
  - If not, go to $127 + 62 = 189$, and so on.

This strategy guarantees you find $X$ in at most 64 drops, ensuring you don't overdo the stress testing on your computers! □

On the topic of how crucial it is to properly test things in many fields, the author loves sharing this (possibly fictional) story:

Back in the USSR, there was a certification process for the Soviet Airbus Il 86 for the international market. One of the mandatory tests was to check the cockpit's seal integrity after a bird strike, specifically a bird hitting the pilot's windshield.

To conduct the tests according to the latest international safety requirements, they had to purchase a special pneumatic cannon from Germany. This cannon was set up at a specific distance from the cockpit and simulated a bird collision by shooting a chicken carcass of a specified weight.

They bought the cannon, set it up, checked everything, and loaded the chicken. Boom! To the horror of the factory workers, the chicken shattered the windshield frame, smashed the glass, flew through the cockpit, knocked out the pilot's door, and went about a third of the way into the cabin, damaging several seats. Catastrophic results! Emergency! They checked the cannon settings and the chicken's weight. Everything was in order. So, it was either a miscalculation of the structural strength or a defect in the manufacturing. Everyone, from designers to builders, got chewed out. They made modifications, built a new cockpit, and started new tests. Cannon calibration. Chicken weighing. Boom! The glass was broken again, and the pilot's door was seriously damaged, but the chicken didn't reach the passenger area. New calculations, reinforcing all elements, changing the glass composition, all fastenings, and connections. Calibration. Weighing. Boom! The glass shattered, chicken bits all over the cockpit, but the pilot's door stayed intact. Multiple reinforcements to the windshield frame, using the latest materials, military technologies, chicken, boom... The glass broke but stayed in place; the chicken ricocheted away, but still, there was a small gap and cabin depressurisation, which meant another test failure. The design resources were practically exhausted.

Someone finally thought, maybe the problem isn't with the plane, but with... the cannon. Maybe it's shooting too hard. They wrote a complaint to the manufacturer, attached photos of all the tests, and called in specialists for a review. The Germans arrived and asked to see the entire testing process themselves. They wanted the Russian specialists to conduct everything from start to finish to understand if they were doing everything correctly and where the possible mistake could be. The Russians set everything up strictly according to the instructions while the Germans silently observed, recording everything on video. Cannon, plane, chicken weighing, boom, the glass stayed intact, but there was still a small gap and cockpit depressurisation. The chicken ricocheted deep into the hangar, smashing everything in its path...

The Russians triumphantly looked at the Germans. The Germans, mouths agape, began to exchange glances and stare at the Russian designers. The commission members started throwing around various technical terms, proving that it was impossible to maintain cockpit integrity with such shot parameters and, accordingly, obtain the international certificate.

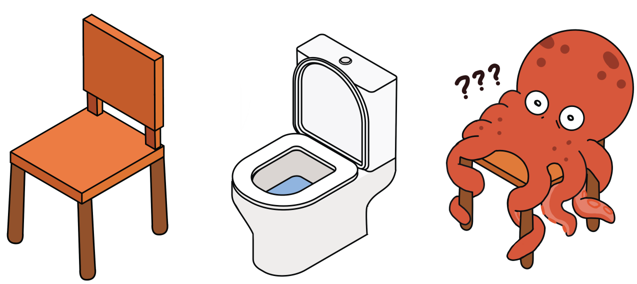
The astonished Germans were silent. When the head of the German delegation finally found his voice, he spread his arms wide and said: “Incredible that a civilian aircraft cockpit could withstand such a test, sustaining such minimal damage. It’s phenomenal! We supply our equipment to almost all aircraft manufacturers in the world, and I’m sure no other plane could withstand anything like this. Good thing you haven’t started engine crash tests with our cannon yet. By the way, the cannon is not at fault here. Hardly anything can withstand a point-blank shot with a 1.5-kilogram frozen core. You see, before loading the chicken, it needs not only to be weighed but also thawed...”

## 7.6 Go I Know Not Whither and Fetch I Know Not What

There's a popular question for analyst job candidates that's often asked during interviews. This question was presented by Artem Mitropolsky at the Analyst Days conference and goes like this:

You are standing against the wall of a huge warehouse where your friend, an alien, is currently located. You need to ask him to bring a chair from the warehouse, but:

- You can't talk to him, only pass a note.
- There is definitely a chair in the warehouse, but you don't know which one, and your friend doesn't know what a "chair" is.
- The alien can only bring you one item.
- The alien understands English but won't grasp complex associations.
- You have only 5 minutes to write the note — go!




First, give this brain teaser a shot yourself, and then read on to find out why this quirky question is such a hit.

Firstly, you've got a ticking clock — only five minutes! Secondly, it tests how well you can get your thoughts out of your head and onto paper. Thirdly, it's not one of those dreaded school Olympiad problems that nobody likes and that take forever to solve. Nope, this task is all about explaining the essence of a "chair". If you can describe it in a way that even an alien can get it, then you're golden when it comes to explaining any client requirement to a programmer.

The task comes with some built-in hints: "You don't know which chair" means you need to consider different possibilities; “The alien understands English” means you need to give a clear description.

Chairs can be all sorts of things — wooden, metal, with one leg, two legs, four legs… The more your description fits “chairs” and doesn’t fit “non-chairs”, the better.

A good answer is one where the candidate methodically explains what a chair is and what it’s used for, what it looks like, what materials it can be made of, how many legs it might have, how high its seat is from the floor, and even includes different sketches. Trust me, five minutes is enough to write all this down.

What nobody will like: only drawings, only metaphors, and ignoring the task conditions (even though it clearly says you don’t know which chair, some still describe a specific iron chair with legs shaped like griffin claws… etc.).

It’s also a red flag if your candidate can’t finish in five minutes. If, after five minutes, they’re still staring at a blank page, trying to figure out how to start, they’ll likely delay tasks, too.

You can give this task different variations. Instead of explaining to an alien, you could explain to a child, or instead of fetching a chair, it could be about going to the store for some good wine (it’s crucial not to mistake that one!).

## 7.7 Barstool Budgeting

*According to our company's rules, you're not allowed to disclose your salary to anyone.*
*Well, I wasn't planning on embarrassing myself anyway.*

**Problem 7.7.** Eight testers of silly questions from different companies gather for drinks. They're curious to know the average salary of the group. However, being cautious and humble folks, none of them want to reveal their own salary. Can you devise a strategy for these testers to calculate the average salary without anyone knowing the others' salaries?

*Solution.* This is a fun problem and has several solutions. One approach is for the first tester to choose a random number, add it to their salary, and pass the total to the second tester. The second tester adds their own salary to the total and passes it to the third tester, and so on. The eighth tester adds their salary and gives the final total back to the first tester. The first tester then subtracts the initial random number and divides the result by $8$ to get the average salary. □

You might wonder if this strategy is useful beyond being a good brain teaser. It actually has real-world applications! For instance, a third-party data provider might collect fund holding position data (securities owned by a fund and the number of shares) from all participating firms and then distribute the information back to participants. To maintain confidentiality, random numbers are added to each fund's ID before sharing. This prevents anyone from reverse-engineering the fund's holdings. This way, participants can share market information while remaining anonymous. Or we could use this method to find out the average number of hugs among students at our college, just to prepare a report for an anti-scientific conference.

# Thank You for Finishing the Book!

**We hope this book has been a valuable resource for you.**

**Loved It?**
Take a moment to leave a quick, honest review—just scan the QR code below with your phone, and it will take you directly to the review page on Amazon. It's fast, easy, and helps others discover this book!

� Scan to Review:

SCAN ME

Not Completely Satisfied?
We'd love to hear how we can improve! Share your feedback directly with us:
☒ Write to us at: tatiana@babicheva.org

**Thank you for being part of our journey to help create the next generation of math champions!**

# Goodbye, and May Your Future Choices Be as Wise as Reading This Book!

In one university, there was a professor who had the challenging task of explaining Green's theorem to his students. For those who might not recall, the proof of this theorem is quite tedious and involves numerous cases. Teaching it is far from a delightful experience.

As the professor meticulously worked through the proof, class time ended and they took a break. Upon returning, one student pointed out a minor error the professor had made three blackboards back in one of the cases. You can imagine the mixed feelings of the professor: gratitude for the correction but also the frustration of having to backtrack.

Similarly, while writing this book took considerable time and effort, I will be genuinely grateful if you inform me of any mistakes you find within its pages.

And… May your bugs be few and your offers plenty!

You know, LinkedIn is basically a reversed Tinder. Hot girls write to nerd guys and they didn't reply.

And do not hesitate to add a review if you like my book:

amazon.com/author/babicheva