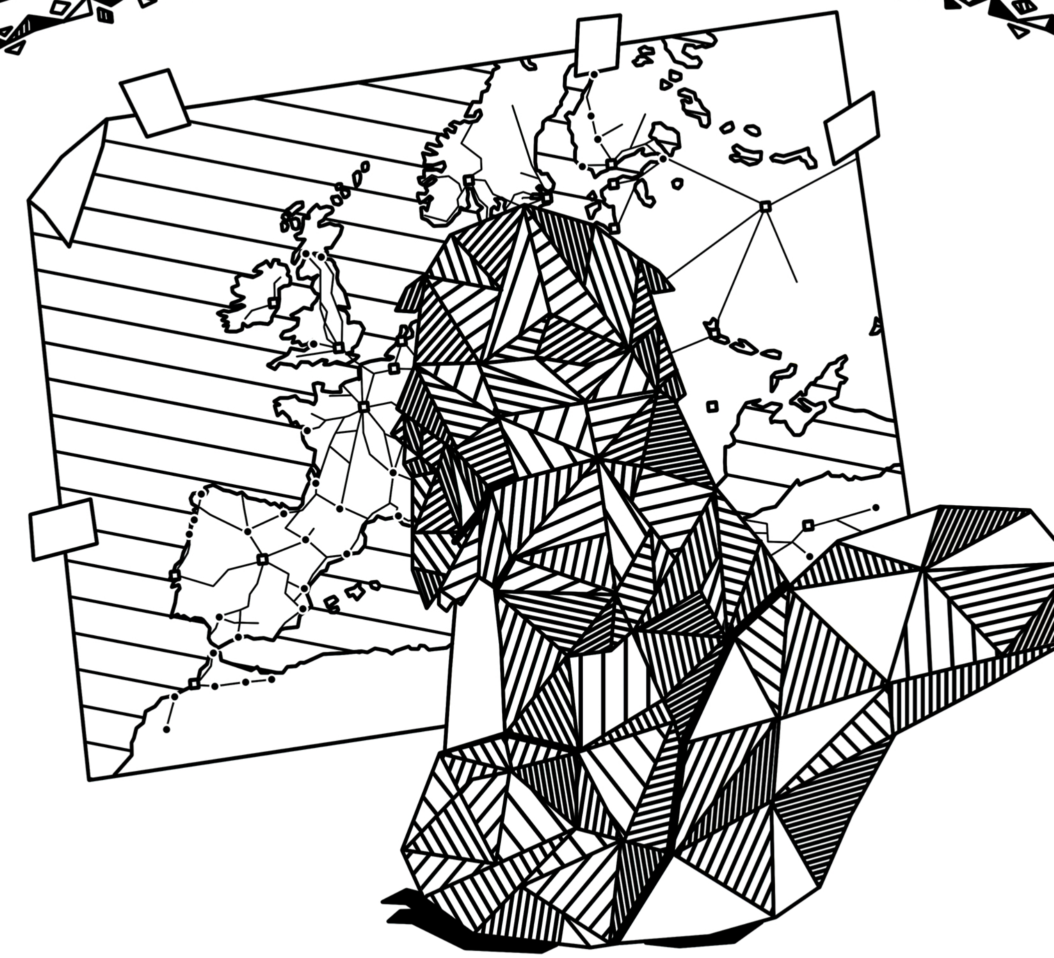
Tatiana Babicheva is an inspiring teacher who believes that teaching is not just about imparting knowledge but also about being a lifelong learner. This passion for growth and discovery led her to earn two PhDs: one in applied mathematics and its applications in Russia, and another in computer science from Université Paris-Saclay in France. She has published 10 books in Russian and French in the field of popular science, including mathematics at the competition level.

Dmitry Babichev is a gold medalist of the 2008 International Mathematical Olympiad. After completing his Master's degree, he pursued a PhD in Machine Learning and Parametric Statistics under the supervision of Francis Bach at DI/ENS. His thesis, titled On Efficient Methods for High-Dimensional Statistical Estimation, focuses on innovative solutions for complex statistical challenges. Currently, Dmitry works in the field of machine learning. He believes in the importance of explaining complex ideas in simple terms and is passionate about popularizing mathematics.

INTRODUCTION TO DISCRETE MATHEMATICS

Mathematical Competitions. Levels A1-A2

3.Introduction to Discrete Mathematics



TATIANA BABICHEVA
Dmitry Babichev
llustrators: Kilyanova Natalia; Yuliya Chayka

Mathematical Competitions. Levels A1-A2 Book 3. Introduction to Discrete Mathematics

ISBN: 9798303874022

Cover design: *Natalia Kilyanova*
Images: *Dmitry Babichev and Yulia Chaika*

Dedication

> “Dedicated to our students, who started they shortest path math journey.

Introduction

Introduction to the series

Begin your preparation for Competition Mathematics with our carefully crafted series. These books are designed to inspire a love for problem-solving and foster critical thinking. They are ideal for both budding mathematicians and passionate enthusiasts.

Inside, you will find a wide range of challenges, puzzles, and problems. Each one is selected to enhance your mathematical abilities. Experience the challenge of solving complex equations and gain confidence by deciphering complex geometric puzzles. Every book has engaging content to stimulate your mind and expand your skills.

If you're preparing for regional competitions, national tournaments, or simply want to deepen your mathematical knowledge, this series is an invaluable resource. The books provide clear explanations, strategic insights, and numerous practice problems. They aim to build your confidence and equip you with the skills needed to tackle any mathematical challenge.

While school mathematics forms a foundation, this series goes beyond it without requiring advanced knowledge to understand the material. Our course covers a wide range of topics, reflecting the diverse nature of Olympiad problems. Solving a geometry problem may require knowledge of combinatorics, while a number theory problem might involve understanding invariants and the pigeonhole principle.

Olympiad problems are generally not restricted to specific grade levels, making these books suitable for high school students. Some of the problems included have been featured in the final stages of national math Olympiads for higher grades. The goal is to demonstrate how to solve problems using straightforward and elegant methods, avoiding unnecessary complexity.

We have categorized competition mathematics into levels similar to the international standards used for foreign language proficiency. This approach is based on the concept of the «language» of competition mathematics. Traditional grade-based divisions are often outdated, as understanding a topic might only require elementary-level math. Moreover, the topics in these books are interconnected. Without a grasp of a topic at level A1, understanding its expanded form at level A2 can be challenging.

Here's what to expect at each level:

Let's use an analogy with foreign languages:

Level A1. You understand (generally) foreign speech and can talk about family, activities, hobbies, travels, weather, and buying things. In short, the standard tourist set. Can you conjugate basic verbs and be familiar with different tenses? The question «How are you?» doesn't stump you. Congratulations! You have a good A1 level! This is enough for survival.

Similarly, in olympiad math — you can «survive» at beginner-level olympiads, understand what is required in problems, and formulate solutions. You likely won't need math knowledge beyond seventh grade to understand topics at this level. (The problem might be from an 11th-grade olympiad, but the solving method remains the same.)

At level A2, you can discuss preferences in art, cultural differences, and main social trends, etc. You form complex sentences («This is Peter, whose dad works at the bank. I've already told you about him»), can write to a friend on Facebook, describe a vacation, and understand the essence of any conversation in the language.

You can recognize and solve middle-level Olympiad problems. You will be able to avoid common mistakes and present your solutions effectively. Topics at this level typically require knowledge up to the eighth grade.

This series of books generally covers levels A1 and A2 of competition math: you will understand any problem from most competitions, formulate your solution, and even change the solution of ChatGPT to match the real competition problem. However, you are still far from being a native speaker.

What is in these Books?

This series uses a proof-based approach to problem-solving, which is usually reserved for advanced levels in countries like the USA and the UK. However, this method helps build a solid foundation in mathematics.

Each chapter is divided into four parts:

1. The first part covers the theoretical background and provides detailed solutions to typical problems.

2. The second part presents a problem set labeled by source. Olympiad problems are marked with notations like «Year.Grade/Round.Number.» For example, «ACM 2016.10A.5» is the fifth problem from the 10th-grade 10A variant of the ACM Olympiad 2016. Grade numbering may vary between countries, so adjust accordingly. Non-grade-specific Olympiads, like AIME, are marked by version (I or II) instead of grade.

You will encounter many problems from the Russian Olympiads (a country with a strong tradition in Olympiad mathematics) and various US mathematical competitions (such as AMC and AIME). We sincerely recommend not only finding the correct answer from the given AMC options but also approaching these problems from a proof-based perspective.

The problem number usually provides a sense of difficulty; generally, a higher number indicates a more challenging problem. However, this labeling doesn't always apply to some «independent» Olympiads, which can sometimes confuse genuine Olympiad participants.

3. The third part includes problems for independent solving, with some original problems introduced here.

4. Solutions are found in the fourth part.

The series consists of the following books:

1. Competitive Arithmetics
2. Ideas and Methods
3. Introduction to Discrete Mathematics
4. Introduction to Competitive Geometry
5. Competitive Number Theory
6. Competitive Geometry

This series is designed for both experienced Olympiad participants and newcomers to mathematical problem-solving. It offers a journey where theory and application meet, providing a rewarding experience. Welcome to a unique math adventure!

Introduction to this book

In this book, we will pay attention to topics that are rarely or not at all covered during school lessons.

Nevertheless, the chapters of this book will open the door to the fascinating world of incredibly beautiful and astonishingly rich science — discrete mathematics!

Typically, objects studied in discrete mathematics include integers, graphs, and logical statements.

In this book, we will focus on an introduction to graph theory and combinatorics. Number theory, a more popular topic in middle school competitions, as well as advanced aspects of graph theory and combinatorics, are covered in the next books of this series.

List of competitions used in this book

- «Математический праздник», in English mean «Mathematical festival». We note it in the book as «MF». The official site (in Russian) is https://olympiads.mccme.ru/matprazdnik/
- Городская устная математическая олимпиада для 6—7 классов, mean «City Oral Mathematical Olympiad for 6—7 grades». We note it in the book as «COM». The official site (in Russian) is https://olympiads.mccme.ru/ustn/
- Турнир городов, mean «Tournament of Towns». We note it in the book as «TOT». The official site is https://www.turgor.ru/en/
- Муниципальный этап Всероссийской олимпиады школьников, mean «second stage of All-Russian School Olympiad». We note it in the book as «2ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- American Mathematics Competitions. We note it in the book as «AMC». The official site is https://maa.org/math-competitions
- American Invitational Mathematics Examination. We note it in the book as «AIME». The official site is https://www.maa.org/math-competitions
- American High School Mathematics Examination. We note it in the book as «AHSME». The official site is https://www.maa.org/math-competitions/amc
- Московская математическая олимпиада, mean «Moscow Mathematical Olympiad». We note it in the book as «MMO». The official site (in Russian) is https://mmo.mccme.ru/
- iTest. We note it in the book as «iTest». The non-official site is https://artofproblemsolving.com/wiki/index.php/ITest
- Junior Mathematical Olympiad. We note it in the book as «JMO». The official site is https://ukmt.org.uk/junior-challenges/junior-mathematical-olympiad
- Indonesian Mathematics Olympiad. We note it in the book as «Indonesian MO». The official site is http://tomi.or.id/
- Турнир им. Ломоносова, mean «Lomonosov Tournament». We note it in the book as «LT». The official site (in Russian) https://turlom.olimpiada.ru/
- Кружок МЦНМО, mean «Circle of Moscow Center for Continuous Mathematical Education». We note it in the book as «Mccme». The official site (in Russian) is https://mccme.ru/en/math-circles/circles-mccme/20232024/
- «Покори Воробьёвы горы», mean «Conquer Vorobyovy Gory», competition

of MSU. We note it in the book as «PVG». The official site (in Russian) is https://pvg.mk.ru/

- UK Maths Trust Pink Kangaroo. We note it in the book as «Pink Kangaroo». The official site is https://ukmt.org.uk/
- Ленинградские математические кружки, means the book «Mathematical Circle: (Russian Experience)» of Genkin, Itenberg, and Fomin. We note it in the book as «Leningrad».
- Nederlandse Wiskunde Olympiade. The official site is https://www.vwo.be/vwo/
- Hamilton Mathematical Olympiad. We note it in the book as «HMO». The official site is https://ukmt.org.uk/
- Иванов, «Математический кружок», means the book «Mathematical Circle» of Ivanov. We note it in the book as «Ivanov».
- Олимпиада «Физтех», means «MIPT mathematical competition». We note it in the book as «MIPT». The official site (in Russian) is https://olymp.mipt.ru/
- Шень, «Математическая индукция», means the book «Mathematical Induction» of Shen. We note it in the book as «Shen».
- University of South Carolina High School Math Contests. We note it in the book as «University of South Carolina High School Math Contests». The official site is https://sc.edu/
- Auckland Mathematical Olympiad. The official site is https://www.auckland.ac.nz/

Contents

What is a Graph?

> “It’s such a class I’ve got! I explain the theorem to them — they don’t understand. Another time I explain — they don’t understand. The third time I explain — I understood it myself, but they still don’t understand.
>
> —Popular joke

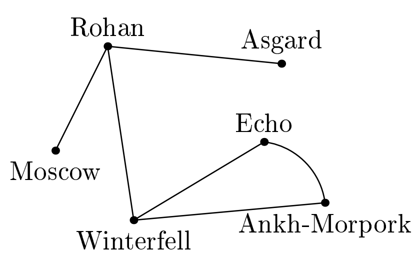
Theory and Practice

Graphs are probably one of the least favorite topics in mathematical competitions. They are rarely taught in schools and are studied sporadically, if at all. However, Olympiad problems exist on this topic, and we will be required to solve them. A novice who encounters such a problem at an Olympiad will likely limit themselves to reading its statement and then move on to another problem. However, problems involving graphs can sometimes be straightforward, and solving them may require only knowledge of terminology, a bit of theory, and common sense.

So, what is a graph?

Definition 1. *A graph* is a finite set of points, some of which are connected by lines. These points are called *vertices* of the graph, and the connecting lines are called *edges*.

This can be visually illustrated as follows: let the vertices represent some cities, and the edges of the graph represent roads connecting them (see the figure below). In a graph, the arrangement of vertices on the diagram or the fact that edges intersect is irrelevant — the same graph can be depicted in various ways.



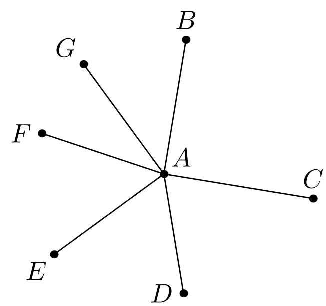
Below is another important definition.

Definition 2. Graphs are called *equal* if they have the same number of vertices, and these vertices can be numbered such that vertices are connected in the first graph if and only if vertices with the same numbers are connected in the second graph.

Graphs help to visualize the problem more clearly. Consider the following problem.

Example 1.1. (MF — 1994.6.6) A pedestrian walked along six streets of a city, passing each one exactly twice, but could not walk along each one only once. Could this have happened?

Solution: For instance, the figure below illustrates six streets that radiate from the center of the city in different directions: six segments AB, AC, AD, AE, AF, AG, with a common starting point A and no other common points.



The pedestrian can exit the center and walk back and forth along each street. However, it is evident that it is impossible to walk down each street exactly once.

Thus, the answer is: it could have happened! □

Definition 3. *A complete graph* is a graph in which every vertex is connected to every other vertex.

To determine the number of edges in a complete graph with n vertices; according to the definition, each of the n vertices is connected to the other $n-1$ vertices, so we have: $n \cdot (n-1)$, however, in this way, each edge will be counted twice. Therefore, ultimately, the number of edges in a complete graph with n vertices is equal to

$$\frac{n \cdot (n-1)}{2}.$$

Example 1.2. In Beaverland, there are 10 houses connected by 20 neat cobblestone roads, each of which connects two houses. Thrifty foxes who value their time are interested in constructing a few additional roads to ensure that each house can be reached from any other house without the need for additional roads (each pair of houses should be connected by exactly one straight road). What is the total number of additional roads that need construction?

Solution: The total number of roads can be calculated as:

$$\frac{10 \cdot (10 - 1)}{2} = 45.$$

Since 20 roads have already been constructed, we need to construct 25 more. □

Problem Set

Problem 1.1. (MMCircle — 2013/2014.6.1): In a village named Igumenka, there are 9 houses. It is known that Peter's neighbors are Ivan and Anton, Maxim's neighbors are Ivan and Sergey, Victor's neighbors are Dima and Nikita, and also the neighbor pairs are Yevgeny with Nikita, Ivan with Sergey, Yevgeny with Dima, and Sergey with Anton, and there are no more neighbors in the indicated village (neighbors are considered to be yards that share a common fence). Can Peter get through the gardens to Nikita for apples?

Problem 1.2. (MF — 1992.6.3): How can you draw six line segments without removing the pencil from the paper in order to cross out 16 points that are located at the vertices of a 3×3 square grid?

Problem 1.3. (MF — 1995.6.6): In a 6×6 square, several cells are marked in such a way that from any marked cell, one can pass to any other marked cell, only crossing common sides of marked cells. A marked cell is referred to as a *terminal* if it borders exactly one marked cell on the side. Mark several cells to obtain: a) 10; b) 11; c) 12 terminal cells.

Problem 1.4. (MF — 1991.6.6): The metro of the city of Foxville consists of three lines and has at least two terminal stations and two interchange hubs, where no terminal station is an interchange hub. From each line, you can switch to each of the others in at least two places.

Draw an example of such a metro scheme if it is known that this can be done without lifting the pencil from the paper and without drawing the same segment twice.

Problem 1.5. In the school, there are 20 children, and any two of them share a common grandmother. We call a grandmother «multigrandmother» if she has at least 13 grandchildren. The school announced a «grandmother apocalypse», which is mandatory for all grandmothers to attend. Prove that at least one multigrandmother will attend the apocalypse.

Problem 1.6. In a company, every two people have exactly five mutual acquaintances. Prove that the number of pairs of acquaintances is divisible by 3.

Problem 1.7. (Ivanov): a) In a group of four people speaking different languages, any three can communicate (possibly, one translates for the other two). Prove that they can be divided into pairs, each having a common language.
b) The same for a group of 100 people.
c) The same for a group of 102 people.

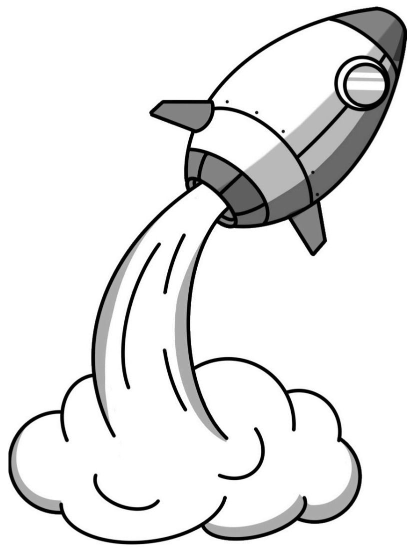
Problem 1.8. (Ivanov): Each edge of a complete graph with 17 vertices is colored in one of three colors. Prove that there exist three vertices such that all edges between them are of the same color.

Problem 1.9. (Leningrad): Each edge of a complete graph with 9 vertices is colored either blue or red. Prove that either there are four vertices with all edges between them being blue or there are three vertices with all edges between them being red.

Problem 1.10. (AHSME — 1996.10): How many line segments have both their endpoints located at the vertices of a given cube?
(A) 12 (B) 15 (C) 24 (D) 28 (E) 56

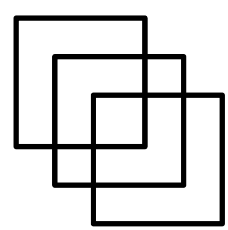
Problem 1.11. (AHSME — 1990.9): Each edge of a cube is colored either red or black. Every face of the cube has at least one black edge. The smallest number possible of black edges is
(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Skill Assessment Problems



Skill Assessment Problem 1.1. (Leningrad): A space travel system is established between the nine planets of the Solar System. Rockets fly along the following routes: Earth — Mercury, Pluto — Venus, Earth — Pluto, Pluto — Mercury, Mercury — Venus, Uranus — Neptune, Neptune — Saturn, Saturn — Jupiter, Jupiter — Mars, and Mars — Uranus. Can one reach Mars from Earth?

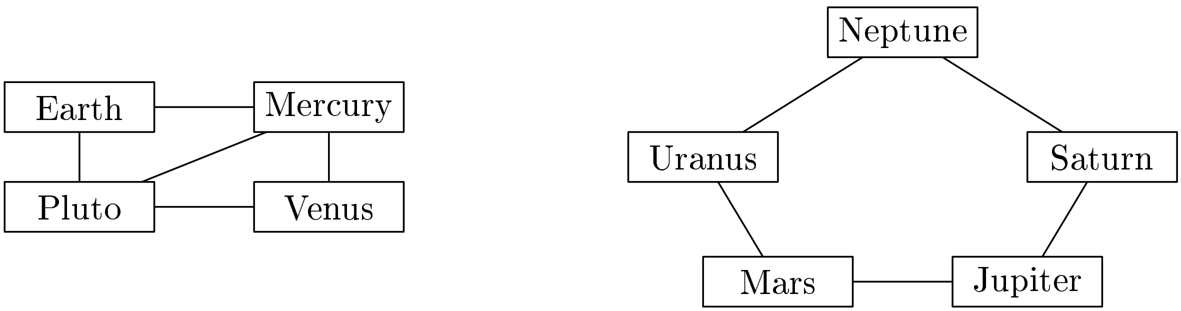
Skill Assessment Problem 1.2. (LT — 1998.5-8.11): Is it possible to draw the picture on the right without lifting the pencil from the paper and by passing through each line exactly once?



Skill Assessment Problem 1.3. (Leningrad): Each edge of a complete graph with 6 vertices is colored with one of two colors. Prove that there are three vertices such that all edges between them are of the same color.

Solutions to Skill Assessment Problems

Solution to Problem 1.1: Let's draw a diagram: the planets will correspond to points, and the routes connecting them will be non-intersecting lines.



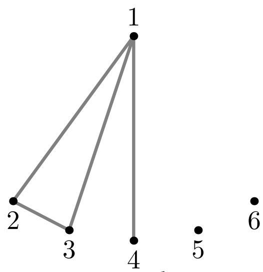
It is clear that it is impossible to reach Mars from Earth.

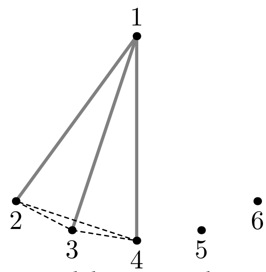
Answer: It's impossible. □

Solution to Problem 1.2: The three squares that comprise the figure are assigned a numerical value. We will begin by drawing the first square from any of its points until we reach the intersection point with the second square. Then, we stop by drawing the first square and drawing the second square until we reach the intersection point with the third square. Then we draw the third square completely, finish, then continue drawing the second square, and finally, draw the first one. Each time, we'll finish drawing the square at the same point where we started, which is the intersection point with the previous square.

Answer: Yes, it's possible. □

Solution to Problem 1.3: Consider one of the vertices of this graph. Due to the completeness of the graph, there are 5 edges emanating from it. By the Pigeonhole Principle, at least 3 of these edges are of the same color. Let's number the vertices of the graph. Let the considered vertex have number 1, and the edges of the same color extend to vertices numbered 2, 3, and 4.





Then, if vertices 2 and 3, 3 and 4, or 2 and 4 are connected by an edge of the same color, then we have identified a monochromatic triangle. If all these edges are of a different color, then a monochromatic triangle is highlighted on vertices 2, 3, and 4. □

Degree of a Vertex of the Graph

"To Leo, the math teacher said: «I warn you, if you don't behave properly, I'll tell your parents that you have talent.»

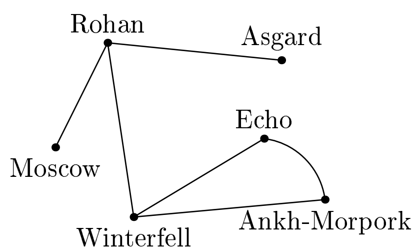
—Popular joke

Theory and Practice

In this section, we continue our discussion about graphs. Like many mathematical objects, graphs have their own characteristics, which are introduced in this chapter. First, we will ascertain the degree of a vertex in a graph.

Definition 4. The number of edges emanating from a given vertex is called the *degree* of the vertex. A vertex of a graph with an odd degree is called *odd*, and a vertex with an even degree is called *even*. The degree of a vertex v is commonly denoted by $\deg v$.

For example, as shown in the figure below, the degree of the vertex «Rohan» is 3, since there are three roads leading from it — to Moscow, Winterfell, and Asgard.



It is important to note the following important statements.

Lemma 1 (Handshake Lemma)**.** The number of odd vertices in any graph is even.

Proof. Consider each edge of the graph as a «thread» connecting 2 vertices. The sum of all vertex degrees in the graph is the total number of «threads» emanating from all vertices. However, we counted each «thread» twice because it has 2 ends. Thus, the sum of the degrees of all vertices in the graph is even.

Assuming that our graph has an odd number of odd vertices. Then, the sum of the degrees of odd vertices is odd. This means that the sum of the degrees of all vertices in the graph is the sum of some even number equal to the sum of the degrees of even vertices and some odd number equal to the sum of the degrees of odd vertices, i.e., an odd number, which contradicts the evenness of the sum of vertices in the graph. This contradiction proves the lemma. □

A similar idea was used in the previous chapter to find the number of edges in a complete graph.

Let us illustrate the potency of the proved lemma through the use of examples.

Example 2.1. Prove that the number of people who have ever held this book and made an odd number of handshakes with other such people is even.

Solution: Let the individuals who have held this book be represented as vertices in a graph, and the vertices that correspond to the individuals who shook hands with each other be connected by edges.

Then, according to the Handshake Lemma, the number of odd vertices in such a graph is even. Therefore, the number of people corresponding to these vertices is also even, which is what we needed to prove! □

Example 2.2. During his pre-election campaign, a candidate for mayor of the city of Beavers promised to install new telephone lines between houses as follows: there are a total of 25 houses in the city, and each house must be connected to exactly 5 others. Prove that he will not be able to keep his promise.

Solution: Let the houses be vertices and the telephone lines be edges of the graph. Then, the degree of each vertex is 5. The sum of all degrees is $5 \times 25 = 125$, which should be equal to twice the number of edges. However, this is impossible, as 125 is an odd number. □

There are numerous errors that occur when graph problems are solved because only specific cases are considered, and no attempt is made to solve the problem in general.

Example 2.3. Mrs. Owless's entire family, who were renowned for their conviviality and unfriendly character, attended her birthday celebration. At the party, each of the attendees had enough patience to greet exactly n others (and for each attendee, this number n is the same and greater than one), while Mrs. Owless, closely watching her relatives, counted a total of 109 handshakes. What could n be, and what is the number of attendees if the latter was more than two?

Solution: Let's denote the number of attendees as s. Then the number of handshakes is:

$$\frac{s \cdot n}{2} = 109.$$

The number 109 is prime, and since $s > 2$, the unique solution is $n = 2$, $s = 109$. □

Example 2.4. (Leningrad): We ascertain the possibility of drawing 9 segments on a plane so that each intersects exactly with three others.

Solution: Consider a graph where the vertices represent the segments. Such «transformation» of a graph is sometimes called a «dual graph». Vertices are connected by an edge if the corresponding segments intersect. In this case, we have an odd number of vertices (9) of odd degree (3), which contradicts the Handshake Lemma. Therefore, drawing 9 segments on a plane such that each intersects exactly with three others is impossible. □

Problem Set

Problem 2.1. (MF — 1992.7.3): A resident of a foreign intelligence agency reported to the center about the impending signing of a series of bilateral agreements between fifteen former Soviet republics. According to his report, each of them will conclude an agreement with exactly three others. Does the resident deserve trust?

Problem 2.2. (COM — 2014.6.4): A group of friends exchanged letters in such a way that each letter was received by everyone except the sender. Each person wrote the same number of letters, resulting in a total of 440 letters received by everyone together. How many people could be in this group?

Problem 2.3. (COM — 2013.6.8): Can 1006 different 2012-gons be drawn, with all vertices in common, such that no two have a common side?

Problem 2.4. (MF — 2017.7.6): Among 49 schoolchildren, each is acquainted with at least 25 others. Prove that they can be divided into groups of 2 or 3 people, each person being acquainted with everyone in their group.

Problem 2.5. (Leningrad): A certain king has 19 vassal barons. Is it possible that each vassal barony has one, five, or nine neighboring baronies?

Problem 2.6. (Ivanov): In a graph, each vertex is either blue or green. Moreover, each blue vertex is connected to five blue vertices and ten green vertices, while each green vertex is connected to nine blue vertices and six green vertices. Which type of vertices is more prevalent— blue or green?

Problem 2.7. (Ivanov): In a graph, each vertex has three edges that are incident to it. Can this graph have 1990 edges?

Problem 2.8. (Leningrad): At a conference, there are 50 scientists, each of whom is acquainted with at least 25 other participants of the conference. Prove that there are four of them who can be seated at a round table so that each sits next to acquaintances.

Problem 2.9. (Ivanov): In a club, each member has one friend and one enemy. Prove that

a) The number of members is even;
b) The club can be divided into two neutral groups.

Problem 2.10. (Mos2ARSO — 2006.6.5): In a burrow, there is a family of 24 mice. Every night, exactly four of them go to the warehouse for cheese. Is it possible that at some point, each mouse visited the warehouse exactly once, along with every other mouse?

Problem 2.11. (AMC — 2004.10A.13): At a party, each man danced with exactly three women, and each woman danced with exactly two men. Twelve men attended the party. How many women attended the party?
(A) 8 (B) 12 (C) 16 (D) 18 (E) 24

Problem 2.12. (University of South Carolina High School Math Contests — 1993.21): Suppose that each pair of eight tennis players either played exactly one game last week or did not play at all. Each player participated in all but 12 games. How many games were played among the eight players?
(A) 10 (B) 12 (C) 14 (D) 16 (E) 18

Problem 2.13. (AMC — 2017.10A.8): At a gathering of 30 people, there are 20 people who all know each other and 10 people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur within the group?
(A) 240 (B) 245 (C) 290 (D) 480 (E) 490

Problem 2.14. (AHSME — 1990.16): At one of George Washington's parties, each man shook hands with everyone except his spouse, and no handshakes took place between women. If 13 married couples attended the party, what was the total number of handshakes between these 26 individuals?
(A) 78 (B) 185 (C) 234 (D) 312 (E) 325

Problem 2.15. (Indonesia MO — 2005.8): There are 90 contestants in a mathematics competition. Each contestant gets acquainted with at least 60 other contestants. One of the contestants, Amin, stated that at least four contestants have the same number of new friends. Prove or disprove his statement.

Problem 2.16. (AHSME — 1951.31): A total of 28 handshakes were exchanged at the conclusion of a party. Assuming that each participant was equally polite toward all the others, the number of people present was:
(A) 14 (B) 28 (C) 56 (D) 8 (E) 7

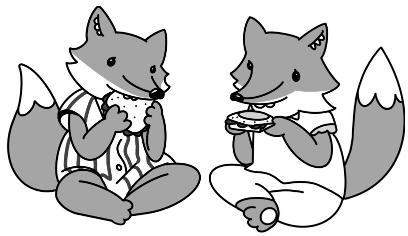
Problem 2.17. (Nederlandse Wiskunde Olympiade): There are n guests at a party. Any two guests are either friends or not friends. Every guest is friends with exactly four of the other guests. Whenever a guest is not friends with two other guests, those two other guests cannot be friends with each other either. What are the possible values of n?

Problem 2.18. (Pink Kangaroo): At a conference, the 2016 participants were registered from P1 to P2016. Each participant from P1 to P2015 shook hands with exactly the same number of participants as the number on their registration form. How many hands did the 2016th participant shake?

(A) 1 (B) 504 (C) 672 (D) 1008 (E) 2015

Problem 2.19. (HMO): A number of couples met, and each person shook hands with everyone else present but not with themselves or their partners. There were 31,000 handshakes altogether. How many couples were there?

Skill Assessment Problems



Skill Assessment Problem 2.1. In a dormitory, there are 214 students. Every hour, exactly 4 of them go to the kitchen to have a snack. Is it feasible that each student has encountered each other in the kitchen exactly once at some point in time?

Skill Assessment Problem 2.2. Can the degrees of vertices in a graph be:
a) $6, 5, 4, 3, 2, 2$;   b) $5, 5, 4, 3, 2, 1$;   c) $5, 5, 4, 3, 2, 2$?

Solutions to Skill Assessment Problems

Solution to Problem 2.1: Notice that there are a total of

$$\frac{214 \cdot 213}{2} = 107 \cdot 213$$

pairs of students. Every hour, 4 of them meet in the kitchen, which makes $\frac{4\times 3}{2} = 6$ pairs. However, since $107 \cdot 203$ is odd, it cannot be divisible by 6. Therefore, it is impossible for each student to have encountered each other exactly once at some point.

We can restate this problem using graphs: there is a complete graph on 214 vertices (where all edges are present), and we want to know if it can be divided into complete subgraphs of 4 vertices each. $\square$

Solution to Problem 2.2: a) Assume that such a graph exists. The sum of degrees of vertices in this graph is even, so there is no contradiction according to this criterion. How many vertices are there in this graph? Six. However, the first vertex has degree 6, which means it has more edges than possible. Thus, such a graph does not exist.

b) Assume such a graph exists. If a graph on 6 vertices has a vertex with degree 5, it means it is connected to all other vertices. Thus, vertices 1 and 2 are connected to all other vertices, including vertex 6. Then, the degree of vertex 6 must be at least 2. This contradiction therefore proves the non-existence of such a graph.

c) The sum of degrees of vertices in this graph is odd, leading to a contradiction. $\square$

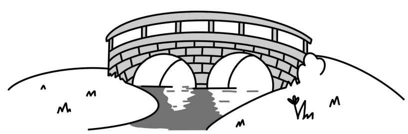
Euler Tracks and Königsberg Bridges.

> Many mathematicians believe that the intersection of two flat jokes yields a single thin one.
>
> —Popular joke

Theory and Practice

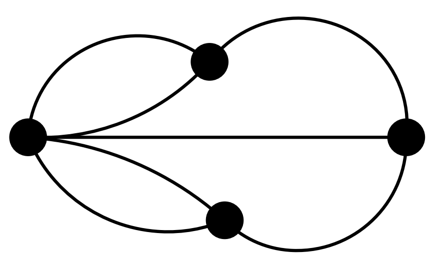
This chapter addresses a historically important problem.

Example 3.1 (The Seven Bridges of Königsberg)**.** There were 7 bridges in Königsberg. Find a way to cross all seven bridges of Königsberg exactly once, without crossing any of them twice.



Solution: This problem was first solved by Leonhard Euler, a great mathematician. His idea is therefore reproduced in this section.

The vertices of a graph are represented here as the areas of the city that are bounded by water, while the edges of the graph are the bridges that connect various parts of the city. At that period, Königsberg could be depicted as follows:



As proved in the previous chapter, the number of odd vertices in a graph is even.

However, Leonhard Euler further concluded that:

- If all vertices in the graph are even, then it is possible to draw the graph without lifting the pencil from the paper, starting and ending at any vertex.
- If there are exactly two odd vertices in the graph, then it is possible to draw the graph without lifting the pencil from the paper, starting at one odd vertex and ending at the other odd vertex.
- A graph with more than two odd vertices cannot be drawn without lifting the pencil from the paper.

The graph shown above contains 4 odd vertices. Therefore, it cannot be drawn without lifting the pencil from the paper, which implies that it is not possible to cross all the bridges of Königsberg exactly once without crossing any of them twice. □

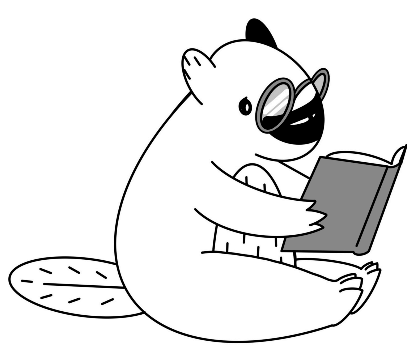
To continue our study, we need to introduce some new concepts.

Definition 5. A *path* in a graph is an ordered set of vertices in which each successive vertex is connected to the previous one by an edge.

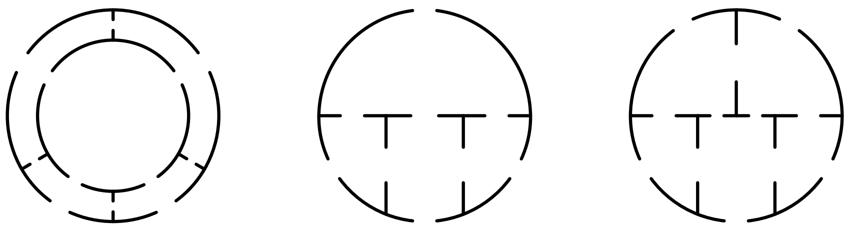
Definition 6. The length of a path is the number of edges forming that path.

Definition 7. Two vertices in a graph are *adjacent* if they are connected by an edge. Otherwise, they are called non-adjacent.

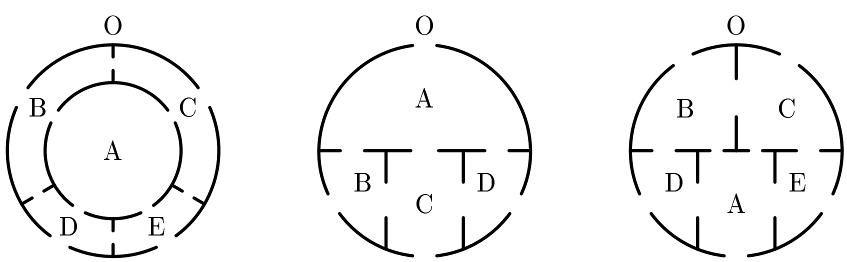
Definition 8. A *cycle* in a graph is a path in which the initial and terminal vertices coincide.



Example 3.2. Umberto decided to build a library and ordered several layout options for the rooms (shown in the figure below), with the mandatory condition of having exactly one door in each wall separating two rooms or a room and the exterior space. In what circumstances is it possible to pass through all the doors without passing through any one of them more than once?



Solution: Consider a graph with rooms as vertices (one of the vertices representing the exterior space) and edges as doors.



a) Here, the vertex A is connected with vertices B, C, D and E, thus having the degree equal to 4. Vertex B has a degree equal to 4. The degrees of vertices C, D, E, and O are respectively 4, 4, 4 and 4.

All vertex degrees are even, so an Eulerian path is possible.

b) Two vertices (B and D) have odd degrees, so there exists a path with endpoints at these vertices.

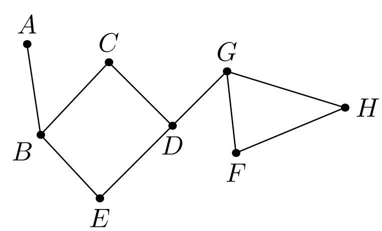
c) Four vertices (O, D, A, and E) have odd degrees, so an Eulerian path is not possible.

□

After solving the problem of the Königsberg bridges, a new branch of mathematics emerged — «Graph Theory», which we are explored in this book.

Definition 9. A *simple cycle* in a graph is a path that starts and ends at the same vertex and does not pass through any other vertex more than once.

For example, in the figure below, $B - C - D - E - B$ and $F - G - H - F$ are cycles.



Each of the ideas outlined in the Königsberg bridges problem solution has a name.

Definition 10. An *Eulerian path* in a graph is a path that traverses each edge exactly once.

Definition 11. An Eulerian cycle is a closed Eulerian path.

Definition 12. An Eulerian graph is a graph in which all edges can be traversed to form an Eulerian cycle.

Problem Set

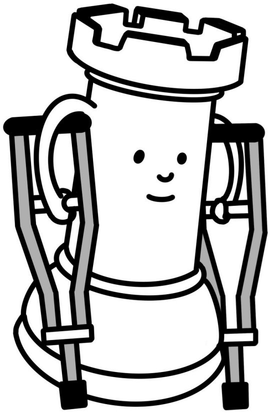
Problem 3.1. (Leningrad): A group of islands is interconnected by bridges, allowing for the access of any island from any other. A tourist visited all the islands, crossing each bridge exactly once. He visited the Triple Island three times. How many bridges lead from the Triple Island if the tourist:

a) did not start from or end on it;
b) started from it but did not end on it;
c) started and ended on it?

Problem 3.2. (Leningrad): a) A piece of wire is 120 cm long. Is it possible to frame a cube with edges of 10 cm without breaking the wire?

b) What is the minimum number of times the wire needs to be broken to make the required frame?

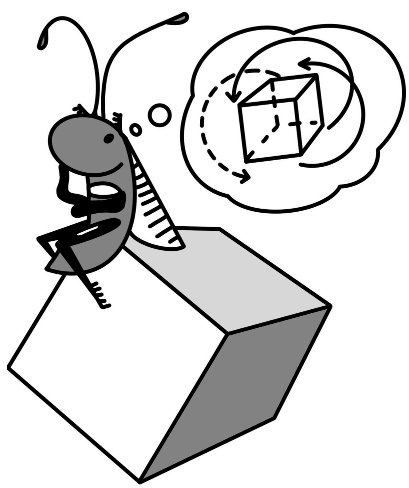
Problem 3.3. (TOT — 1992/1993.8-9.4): An ant crawls along the wireframe of a cube, never turning back. Is it possible that it visited one vertex 25 times and each of the others 20 times?



Problem 3.4. (MMO — 1961.8.1.4): Given that a rook is only allowed to move one square at a time. Prove that it can go through all the squares of a rectangular chessboard, visiting each square exactly once, and return to the initial square if and only if the number of squares on the board is even.

Skill Assessment Problems

Skill Assessment Problem 3.1. If a cockroach is seated on one of the vertices of a cube, can it crawl along all its edges exactly once and return to the initial vertex?

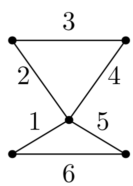


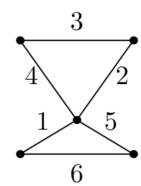
Skill Assessment Problem 3.2. Is there an Eulerian graph from which an Eulerian cycle may be extracted in several ways?

Solutions to Skill Assessment Problems

Solution to Problem 3.1: Let's represent the cube as a graph, where the vertices of the graph are the vertices of the cube, and its edges are the edges of the cube. Then the question can be formulated as «Is the resulting graph Eulerian?» In this graph, there are 8 vertices, each with degree 3. Therefore, there is no Eulerian cycle in this graph, and the cockroach cannot crawl along all its edges exactly once and return to the initial vertex. □

Solution to Problem 3.2: Yes, such graphs exist. An Eulerian cycle can be extracted from the given graph in multiple ways, as shown in the figures below.





□

Connected Graphs

> "The problems will be interesting. The whole department is solving one of them right now. If they solve it, we will include it in the exam paper.
>
> —Popular joke

Theory and Practice

The discussion on graphs is continued in this section. Another important characteristic is introduced.

Definition 13. A graph is said to be *connected* if any two of its vertices can be connected by a path, i.e., a sequence of edges, each starting where the previous one ends.

Definition 14. A *tree* is a connected graph without cycles.

There is an important fact about the trees. A tree always has one less edge than the number of its vertices. For now, we will use this fact without proof.

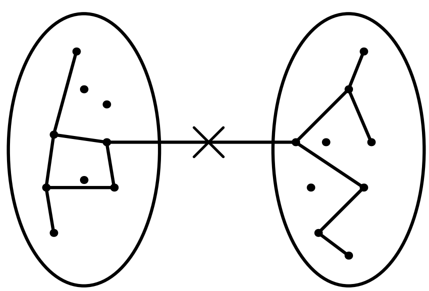
Example 4.1. Mrs. Owless, who resides in Beaverland, has 8 neighbors. Roads exist between certain pairs of nine neighboring homes, each of which connects two houses. It is known that each house is connected to at least four others. As the Harvest Festival approaches, Mrs. Owless has prepared pork pies in accordance with the Foxville recipe to share with her neighbors, to whom she can walk along the road (the pie is big; it's not easy to carry; besides, Mrs. Owless, being somewhat stubborn, does not want to soil her festive coat). Prove that Mrs. Owless will have to serve her neighbors with pies and congratulate them on Harvest Day.

Solution: Suppose the assertion is false, and there is some house A that Mrs. Owless, living in house B, cannot reach. In that case, among the remaining 7 houses, there are at least 4 houses connected to Mrs. Owless's house by roads, and there are also 4 houses connected by roads to house A. Then one of these houses is connected by a road to both A and B, which contradicts the assumption. Therefore, Mrs. Owless can walk along the roads to reach any house and congratulate any of the hosts on Harvest Day. □

Any graph that is not connected can be divided into *connected components* of the graph.

Example 4.2. In a certain country, each city is connected to exactly 10 others, and from any city, you can reach any other. One road was closed for repairs. Prove that you can move from one city to another.

Solution: Again, imagine that the cities are vertices and the roads are edges. Then, the problem considers a connected graph, where the degree of each vertex is 10. Suppose that the graph ceased to be connected after removing the road (edge). This means that the graph has split into two disconnected parts (as shown in the figure below).



The left part of the graph, which is still a graph, can be used to calculate the sum of degrees, albeit lesser. Assume that there are k remaining cities in the dataset. The sum of the degrees of vertices in this graph is $10k - 1$, as 10 roads continue to emerge from all cities except one. However, this is an odd number, which means we have obtained a contradiction. Thus, after eliminating the edge, the graph remained connected, which is required to be proved. □

Notice that an exact depiction of the graph is impossible — if we were to draw 10 edges from each vertex, the drawing would be cumbersome and unattractive. As a result, when drawing graphs, all edges and vertices are often omitted, only retaining the important structure of the graph.

The previous chapter explained the concept of Eulerian graph. Therefore, the theorem is formulated below.

Theorem 1. A graph is Eulerian if and only if it is connected and all its vertices have even degrees.

The proof of this theorem will be provided in the book dedicated specifically to graph theory; however, it can be used in solving the problems.

Example 4.3. Given a volleyball net with dimensions 12 by 25 squares. What is the maximum number of ropes that can be cut without the net falling apart?

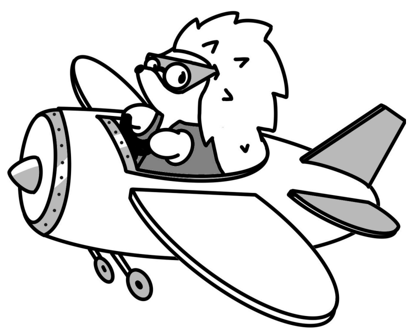
Solution: Let's represent the net as a graph, where the nodes are vertices and the ropes are edges. We need to remove as many edges as possible while keeping the graph connected. We will proceed as follows: as long as there is a cycle in the graph, we can remove one edge from this cycle. The graph will remain connected. We will «cut» cycles until there are none left. A connected graph without cycles is a tree. The remaining graph will have $13 \times 26 = 338$ vertices, so it will have $338 - 1 = 337$ edges. Initially, the graph had $13 \times 25 + 12 \times 26 = 637$ edges. Therefore, we can cut a maximum of $637 - 337 = 300$ ropes. It is impossible to cut more than one rope — a tree becomes disconnected when any of its edges are removed. □

Note that the problem statement includes the word «maximum», which means it requires both an upper bound and an example. In this case, the upper bound and the example are interdependent, and constructing the example facilitates the proof of the upper bound.

Problem Set

Problem 4.1. (PVG 2017.5-6.6;7-8.5;9.4): In the Empire of Beaverland, there were 1000 cities and 2017 roads (each road connects two cities). From each city, it was possible to travel to every other city. Once, an evil wizard enchanted N roads, making them impassable. As a result, 7 kingdoms were formed, each of which provided the ability to travel from one city to another using the roads. However, it was impossible to travel between kingdoms using the roads. What is the maximum value of N for which this is possible?

Skill Assessment Problems



Skill Assessment Problem 4.1. In the country of Hedgehogland, a presidential candidate promises to establish air travel after the elections. It is planned that there will be 7 flights departing from the capital, 6 from each of the other cities, and only one flight from the city of Beavers. The residents of Beavers are of the opinion that they will be unable to travel to the capital, even with transfers. Prove that they are mistaken.

Skill Assessment Problem 4.2. (MMO — 1990.9.1) In a group of seven boys, each has at least three brothers among the others. Prove that all seven boys are brothers.

Skill Assessment Problem 4.3. In the country of Beaverland, there are 99 cities, each connected by airlines to at least 50 other cities. Prove that it is possible to fly to any other city from each city (possibly with transfers).

Solutions to Skill Assessment Problems

Solution to Problem 4.1: Suppose that it is impossible to fly from the city of Beavers to the capital. This implies that the graph, where the cities are vertices and the airlines are edges, is disconnected. Moreover, the capital and the city of Beavers belong to different connected components. Consider the connected component containing the city of Beavers: the degrees of all vertices, except one, are 6, and the degree of one vertex is 1. However, in this case, the sum of all degrees is an odd number, which is impossible. Therefore, the residents of the city of Beavers are mistaken, and it is indeed possible to fly from there to the capital. □

Solution to Problem 4.2: Suppose this is not true, and there are at least two boys who are not brothers. Then, according to the problem setting, each of the remaining 5 boys has at least three brothers among them. However, in this case, at least one of them will be a common brother to the two non-brother boys, meaning they are also brothers. This contradiction proves the claim of the problem. □

Solution to Problem 4.3: If there are a pair of cities that are not connected by a chain of edges (airlines), then the graph is disconnected. Therefore, it has at least 2 connected components. Since each city is connected to at least 50 other cities, one connected component cannot have fewer than $1 + 50 = 51$ vertices (the city itself and 50 cities connected to it by edges). Thus, since there are at least 2 connected components in the graph, there are at least $51 \cdot 2 = 102$ vertices in total, but there are only 99 cities. This contradiction completes the proof.

In fact, a more general fact holds: if a graph has n vertices, and each vertex has a degree of at least $\frac{n-1}{2}$, then such a graph is connected. □

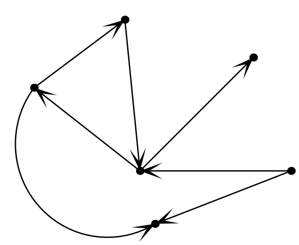
Directed Graphs

> A professor is testing a very weak applicant. Eventually, it turns out that the applicant cannot even solve the simplest inequalities.
> «Tell me», he asks, losing all hope, «what is greater: -1 or 0?»
> «Of course, minus one!» the applicant confidently replies.
> «But why?!» the professor exclaims, grasping his head.
> «Well, -1 is something at least, while 0 is absolutely nothing.»
>
> —Popular joke

Theory and Practice

Definition 15. A graph is said to be *directed* if its edges have a direction.

This is usually represented by an arrow pointing in a certain direction (as shown in the diagram below). Unless otherwise specified, in a directed graph, two edges between the same pair of vertices, where one moves in one direction and the other in the opposite direction, are usually not allowed.



In various rivers, flows, or one-way roads are often mentioned in problems involving directed graphs. Sometimes, one-way relationships between people are also described (e.g., Max owes money to Leo, who owes money to Solomon; Alice likes Jean, who likes Beatrice, etc.).

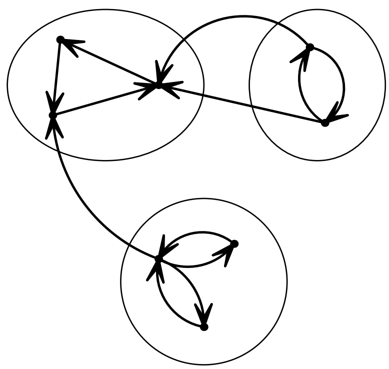
Similarly to the concept of vertex degree in an undirected graph, the concepts of *in-degree* and *out-degree* are introduced in a directed graph — the number of edges entering a vertex and leaving a vertex, respectively. In this case, the previously discussed handshake lemma is slightly modified.

Lemma 2 (Directed Handshake Lemma)**.** In any directed graph, the sum of the in-degrees of all vertices is equal to the sum of the out-degrees of all vertices.

Definition 16. Vertices u and v of a graph are said to be *strongly connected* if there is a path along the edges from u to v and from v to u (movement is only allowed in the direction of the arrows!).

Definition 17. A graph is regarded as *strongly connected* if any two of its vertices are strongly connected.

Any graph can be divided into strongly connected components, between which directed edges are drawn (as shown in the figure below). Regrettably, the proof for this claim exceeds the scope of this book; therefore, we request that you accept it on faith.



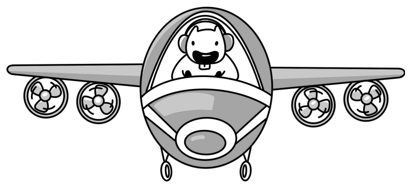
Example 5.1. In a certain country, cities are connected by one-way airline routes. It is known that the condition is fulfilled: after leaving any city, it is not possible to return to it using these airline routes. Prove that it is possible to supplement the airline system in order to establish a connection between any two cities, noting that it is impossible to return to a city after leaving it.

Informal Solution: To solve this problem, we will construct a new graph where each vertex represents a city and each directed edge represents a one-way airline route.

To add new airlines to establish a link between any two cities while maintaining the one-way restriction, we can simply add a new directed edge between each pair of cities that are not already directly connected.

This addition of new directed edges ensures that any two cities are now connected by an airline. Moreover, since we only added new directed edges without removing any existing ones, the condition that after leaving any city, it is not possible to return to it through the original airline routes still holds.

Therefore, by adding the necessary directed edges between cities, we have satisfied the requirements of the problem. □



Formal Solution: The problem can be rephrased in terms of graphs: we are given a directed graph in which there are no cycles, as the condition holds: after leaving a city, it is not possible to return. Consider any two arbitrary cities A and B between which there is no edge; it can be proven that they can be connected by either a directed edge AB or a directed edge BA. Suppose this is not the case. If we cannot draw a directed edge AB, it means that a cycle is created when it is added: there exists a path S_1 from B to A. Similarly, if we cannot draw a directed edge BA, then there exists a path S_2 from A to B. If we leave vertex A and first follow path S_2 and then path S_1, we will return to vertex A— this is a contradiction. Therefore, it is possible to draw some directed edge between vertices A and B. This procedure is continued until a directed edge is drawn between any two vertices, which completes the proof. □

Definition 18. A directed complete graph (in which every vertex is connected to every other) is called a *tournament*.

The theory of directed graphs can be useful, for example, in solving problems related to divisibility.

Example 5.2. Seven natural numbers are written around a circle. It is known that in each pair of neighboring numbers, one divides the other. Prove that there exists a pair of non-neighboring numbers with the same property.

Solution: Let's represent the numbers as vertices of a graph and the divisibility as directed edges: if the first number divides the second, we draw an arrow from the first to the second. Then, between any two neighboring numbers, there must be an arrow in some direction.

We will prove that there are two neighboring arrows directed in the same direction (either clockwise or counterclockwise). Assume that some arrows are pointing clockwise, counterclockwise, and so forth. The seventh arrow will therefore point in a clockwise direction, and we discovered two neighboring arrows pointing in the same direction. Therefore, there is a number a that divides its neighboring number b, and b divides its neighboring number c. Thus, $a \vdots c$ — there exist two non-neighboring numbers, one of which divides the other. This completes the proof. □

We introduced the concept of strong connectivity, which means that there is also a concept of weak connectivity.

Definition 19. A graph is said to be *weakly connected* if every vertex can be reached from every other vertex, possibly by violating the direction of the edges (by moving against the arrows).

There is yet another type of connectivity.

Definition 20. A graph is said to be *one-way connected* if, for any pair of vertices, one of them is reachable from the other.

Problem Set

Problem 5.1. (Leningrad): Max, who came from Falseland, stated that the region is home to numerous lakes that are interconnected by rivers. Each lake has three rivers flowing out of it, and four rivers flow into each lake. Prove that he has lied.

Problem 5.2. (Mccme — 35501): Arrows are arranged on the sides of a certain polygon. Prove that the number of vertices where two arrows enter is equal to the number of vertices where two arrows exit.

Problem 5.3. (Leningrad): Prove that you can arrange arrows on the edges of a connected graph in such a way that you can reach any other vertex from a given vertex using arrows.

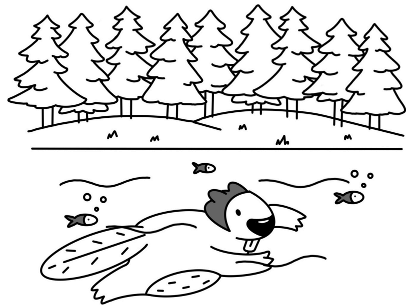
Problem 5.4. (Leningrad): In a certain country, there is a capital city and 100 other cities. Some cities (including the capital) are connected by roads with one-way traffic. Each non-capital city has 20 roads leading out of it, and 21 roads lead into each such city. Prove that it is not possible to drive to the capital from any city.

Problem 5.5. (Ivanov): In a country, every two cities are connected by a one-way road. Prove that there is a city from which one can drive to any other city using at most two roads.

Problem 5.6. (AMC — 2016.10B.22): A set of teams held a round-robin tournament in which every team played against every other team exactly once. Every team won 10 games and lost 10 games; there were no ties. How many sets of three teams $\{A, B, C\}$ were there in which A beat B, B beat C, and C beat A?
(A) 385 **(B)** 665 **(C)** 945 **(D)** 1140 **(E)** 1330

Skill Assessment Problems

Skill Assessment Problem 5.1. In the land of Beaverland, there are 11 dams, and all of them are connected by rivers with one-way and strong currents (the weak paws of beavers cannot withstand such speed). Each dam is connected to each river, and each dam receives water from 5 rivers and releases water into 5 rivers. Prove that it is possible to swim from any dam to any other dam, crossing a maximum of two rivers in the process.



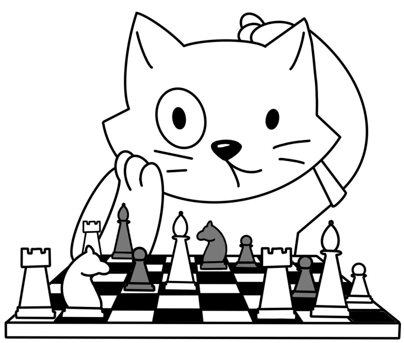
Skill Assessment Problem 5.2. In the kingdom, every city is connected to every other city by a road. Can a mad king introduce one-way traffic on the roads in such a way that after leaving any city, one cannot return to it?

Skill Assessment Problem 5.3. At some point in a single-round tournament with 100 people, it turned out that all players except Max won 26 games and lost 25 games each. Prove that Max does not know how to play at all.

Solutions to Skill Assessment Problems

Solution to Problem 5.1: Consider two dams numbered X and Y. Assuming that there is a river flowing from dam X to dam Y. Then, one can reach from X to Y by traversing only one river. Can one reach from Y to X? From Y, rivers flow into 5 dams, and into X, rivers flow from 5 dams. If these dams are not the same, then there must be at least $5+5+1+1=12$ dams in Beaverland (five dams for each of the X and Y dams and the dams themselves). However, there are only 11 dams in the country, leading to a contradiction. Hence, some of the dams coincide, and we can swim from Y to X through one of them. □

Solution to Problem 5.2: In this section, we address the introduction of orientation (with specified properties) to an undirected graph. Let us enumerate all vertices and arrange all arrows from a lesser number to a larger one. Therefore, it is impossible to arrive at a city with a smaller number from a city with a larger number. □



Solution to Problem 5.3: The total number of wins in this tournament is $26 \cdot 99 + x$, where x is the number of Max's wins. The total number of losses in this tournament is $25 \cdot 99 + y$, where y is the number of Max's losses. Since the total number of wins in the tournament equals the total number of losses, then $26 \cdot 99 + x = 25 \cdot 99 + y$, thus $y - x = 99$. Max played a total of 99 games, so the sum of his wins and losses does not exceed 99. Thus, $x = 0$. Therefore, Max did not win a single game but lost 99 times. Hence, we can conclude that he does not know how to play at all. □

Combinatorics. Enumeration of Options

> “A phone rings in the mathematics faculty dean’s office. The deputy dean, an associate professor from the calculus department, picks up the receiver.
> «Tell me, how can I construct an angle of 50 degrees?» the question comes.
> «Just a moment», says the deputy dean, covering the receiver with his hand, and begins to ponder aloud, «So, 50 degrees is approximately one radian...» Then he starts recalling about pi, the circumference of a circle, and so on. Seeing his struggles, another deputy dean, an associate professor from the geometry department, joins the discussion. He authoritatively states that such an angle cannot be constructed with a compass and a ruler.
> At that moment, the dean enters the room. They decide to ask him. In response, he decisively takes the receiver:
> «And who exactly is asking?»
> «It’s from the philology faculty», the voice says.
> «Take a protractor», the dean cuts in and hangs up.
>
> —Popular joke

Theory and Practice

The emergence of many traditional branches of mathematics, such as algebra, geometry, and arithmetic, can be attributed to practical necessity (bookkeeping, land division, etc.). However, there is a branch of mathematics whose emergence is more closely related to the entertainment aspect of society — combinatorics and probability theory. Various games have been known to people since ancient times — Egyptian senet, chess, checkers, Go, and later, cards. In each of these games, it was necessary to consider various combinations of moving pieces. Moreover, at a more modern stage, players — amateur mathematicians — contributed to combinatorics as a science. For example, the correspondence between the avid player Chevalier de Méré and Pierre de Fermat and Blaise Pascal is known, where very subtle combinatorial questions were addressed.

It is likely that you have heard about exhaustive enumeration before. Let us review the fundamental principles behind employing exhaustive enumeration to solve combinatorial problems:

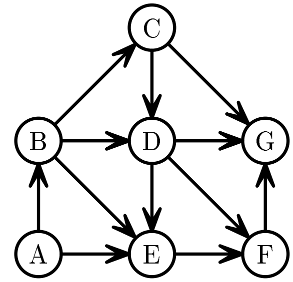
- It is necessary to denote combinations with letters or numbers so that each combination is represented by its unique sequence of letters or numbers.
- Combinations should be listed in alphabetical order (when denoted by letters) or in ascending order (when denoted by numbers).

These rules will ensure that you do not overlook any options, but they also eliminate the possibility of options being repeated.

After reviewing the topic of graphs, we can expand the class of problems that can be solved using exhaustive enumeration methods.

One of the typical problems on simple graph theory tests is finding the number of paths from one vertex of a directed graph to another.

Example 6.1. (State Exam in Informatics, Russia) The diagram shows the road network connecting cities **A**, **B**, **C**, **D**, **E**, **F**, **G**. Each road allows travel in only one direction, as indicated by the arrows. How many different paths exist from city **A** to city **G**?



Solution: We enumerate the possible paths by listing the sequences of visited vertices and arrange them in alphabetical order to ensure that nothing is missed: **ABCDEFG; ABCDFG; ABCDG; ABDFG; ABDG; ABEFG; AEFG**.

Notice that the exhaustive enumeration solution to this problem became feasible only because the road network was narrow. There is a method for solving such problems for larger networks, but it goes beyond the scope of this topic. □

The graph in the previous problem did not contain any cycles. In another case, there will be an infinite amount of paths. We can provide the following definition:

Definition 21. A **directed acyclic graph** (DAG) is a directed graph with no directed cycles.

Example 6.2. In a nursery, there are four little beavers. How many ways are there to choose two of them to send to a forestry Olympiad?

Solution: Let's number the beavers with the digits $1, 2, 3, 4$. The possible pairs are: $12, 13, 14, 23, 24, 34$. There are six options in total. □

Example 6.3. In a nursery, there are four little beavers. In how many ways can they be paired up for a game of ball?

Solution: This problem is similar to the previous one but is fundamentally different in its concept. Let us assign the beavers with the digits $1, 2, 3, 4$. The possible pairs are: $12 - 34, 13 - 24, 14 - 23$. There are three options in total. □

Example 6.4. Alice, Beatrice, and Clarice received 3 different cloaks from the foxes of Beaverland, each distinguished by the color of the brooch. In how many ways can they divide the cloaks among themselves (so that each receives exactly one cloak)?

Solution: Let's number the cloaks and enumerate all possible arrangements: 123, 132, 213, 231, 312, 321. The answer is 6. □

Example 6.5. Assuming that the foxes offered Alice, Beatrice, and Clarice a choice of one of two cloaks as a gift. In how many ways can they choose the cloaks?

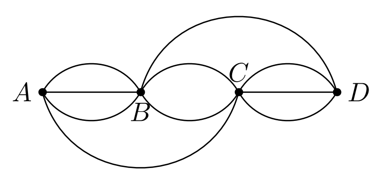
Solution: Let's number the cloaks and enumerate all possible arrangements. Here, there will be repetitions of numbers: 111, 112, 121, 211, 221, 212, 122, 222. The answer is 8. □

Problem Set

Problem 6.1 (Euler's Hat Problem)**.** : Four gentlemen went into a restaurant and handed their hats to the waiter, and upon leaving, they received their hats back. How many possibilities exist such that each of them receives someone else's hat?

Problem 6.2. (MIPT — 2014.7;8): In how many ways can a committee of seven people be formed from its members from eight married couples, while ensuring that members of the same family do not serve on the committee simultaneously?

Problem 6.3. (LT — 2012.7): Cities A, B, C, and D are connected by roads, as shown in the diagram.



In how many ways can one travel from city A to city D, visiting each city exactly once?

Problem 6.4. (MF — 1997.7.1): Which type of rectangle with integer side lengths is more numerous: those with a perimeter of 1996 or those with a perimeter of 1998? (Rectangles with sides $a \times b$ and $b \times a$ are considered identical.)

Problem 6.5. How many numbers exist greater than 3528, each of which can be obtained by permuting the digits of the given number?

Problem 6.6. How many three-digit numbers are there whose sum of digits does not exceed 4?

Problem 6.7. How many four-digit numbers are there whose sum of digits exceeds 33?

Problem 6.8. Four distinct points are marked on a circle. How many arcs are formed as a result?

Problem 6.9. How many two-digit numbers are there where the first digit is less than the second digit?

Problem 6.10. Six acquaintances shake hands with each other. How many handshakes were made in total?

Problem 6.11. Seven chess players participated in a round-robin tournament, where each player played two games against each other (one game with white pieces and one with black pieces). How many games were played in the tournament?

Problem 6.12. Nine sixth graders received grades of A and B in math, literature, and geography. Prove that at least two of them have identical grades in these subjects.

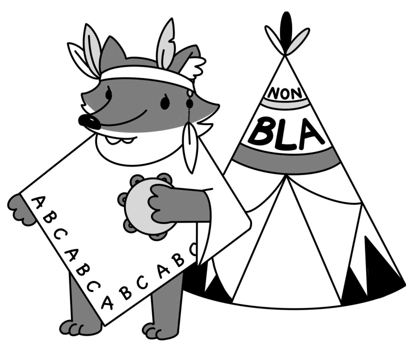
Problem 6.13. Fifteen sixth graders received grades of A and B in math, literature, geography, and physical education. Can we now claim that at least two of them have identical grades in these subjects?

Problem 6.14. Two doctors need to visit four patients, with each doctor needing to visit two patients. In how many ways can the visits be distributed between the two doctors?

Problem 6.15. There are two white, two red, and two pink carnations. In how many ways can they be arranged in three vases so that each vase contains two carnations of different colors?

Problem 6.16. Jean and Esther are playing table tennis, and the match continues until one player wins three games. How many possible outcomes are there for the match?

Problem 6.17. Two roads connect the Yellow Country to the Blue Country, and four roads connect the Blue Country to the Pink Country. Three roads connect the Yellow Country to the Purple Country, and three roads connect the Purple Country to the Pink Country. There are no direct roads from the Yellow Country to the Pink Country or from the Blue Country to the Purple Country. How many paths are there to get from the Yellow Country to the Pink Country? And from the Blue Country to the Purple Country?



Problem 6.18. The alphabet of the Non-Bla-Bla tribe contains only three letters: A, B, and C. A word is any sequence consisting of no more than three letters. How many words are there in the language of this tribe?

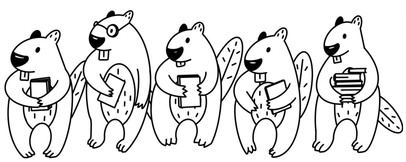
Problem 6.19. (AMC — 2015.10A.10): How many rearrangements of $abcd$ are there in which no two adjacent letters are also adjacent letters in the alphabet? For example, no such rearrangements could include either ab or ba.
(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Problem 6.20. (AMC — 2005.10A.18): Team A and team B play a series. The first team to win three games wins the series. Each team is equally likely to win each game; there are no ties, and the outcomes of the individual games are independent. If team B wins the second game and team A wins the series, what is the probability that team B wins the first game?
(A) $\frac{1}{5}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$

Problem 6.21. (AMC — 2002.12B.14): Four distinct circles are drawn in a plane. What is the maximum number of points where at least two of the circles intersect?
(A) 8 (B) 9 (C) 10 (D) 12 (E) 16

Skill Assessment Problems

Skill Assessment Problem 6.1. There are six beavers in the kindergarten. In how many ways can two of them be chosen to be sent to the Forestry Olympiad?



Skill Assessment Problem 6.2. There are six beavers in the kindergarten. In how many ways can they be paired up for a game of ball if Alice wants to play only with Beatrice or Clarice?

Skill Assessment Problem 6.3. Alice, Beatrice, and Clarice received 4 different cloaks from the foxes of Beaverland, each distinguished by the color of the brooch. In how many can they divide the cloaks among themselves (so that each receives exactly one cloak)?

Solutions to Skill Assessment Problems

Solution to Problem 6.1: Let's number the beavers from 1 to 5. Then they can be chosen for the Forestry Olympiad in the following ways:

1. 1 and 2;
2. 1 and 3;
3. 1 and 4;
4. 1 and 5;
5. 2 and 3;
6. 2 and 4;
7. 2 and 5;
8. 3 and 4;
9. 3 and 5;
10. 4 and 5.

This means there are a total of 10 options. □

Solution to Problem 6.2: Let's number the beavers from 1 to 6. Suppose Alice is numbered 1, Beatrice is numbered 2, and Clarice is numbered 3. Then, we can pair them up in the following ways:

1. $1+2$, $3+4$, $5+6$;
2. $1+2$, $3+5$, $4+6$;
3. $1+2$, $3+6$, $4+5$;
4. $1+3$, $2+4$, $5+6$;
5. $1+3$, $2+5$, $4+6$;
6. $1+3$, $2+6$, $4+5$.

Thus, there are a total of 6 ways. □

Solution to Problem 6.3: In each case, one of the cloaks will remain unchosen. If we discard the fourth cloak, the remaining three can be divided in six ways. Likewise, there will be six ways if we discard the third, second, or first cloak. Therefore, we have $6 \cdot 4 = 24$ possibilities. □

Sum and Product Rules

> One philosopher was shocked when Bertrand Russell told him that a false proposition implies any proposition. He said, «You mean that from the statement that two plus two equals five, it follows that you are the Pope?» Russell replied «Yes». The philosopher asked, «Can you prove this?» Russell replied, «Certainly», and contrived the following proof on the spot:
>
> (1) Suppose $2 + 2 = 5$.
> (2) Subtracting two from both sides of the equation, we get $2 = 3$.
> (3) Transposing, we get $3 = 2$.
> (4) Subtracting one from both sides, we get $2 = 1$.
>
> Now, the Pope and I are two. Since two equals one, then the Pope and I are one. Hence, I am the Pope.
>
> —Raymond Smullyan, «What Is the Name of This Book?»

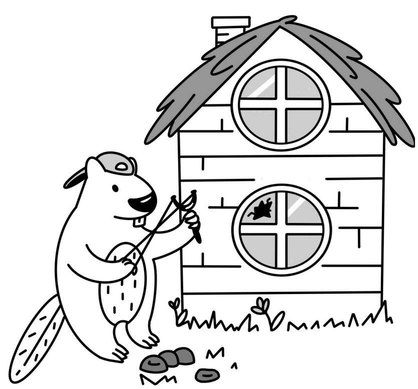
Theory and Practice

This chapter continues the discussion of combinatorics that was initiated in the preceding chapter. The fundamental approach to resolving combinatorial problems was examined in the preceding chapter: the enumeration of alternatives. In order to proceed, we will create two statements which we intend to employ in the future.

Claim 1 (Sum Rule)**.** If object A can be selected in n ways, and object B can be selected in m ways, then selecting either object A *or* B can be done in $m + n$ ways.

Claim 2 (Product Rule)**.** If object A can be selected in n ways, and object B can be selected in m ways, then selecting both object A *and* B can be done in $m \cdot n$ ways.

If you're new to this topic, when solving a problem, you should verbalize to yourself each time whether you need to use «and» or «or», and multiply or add accordingly.



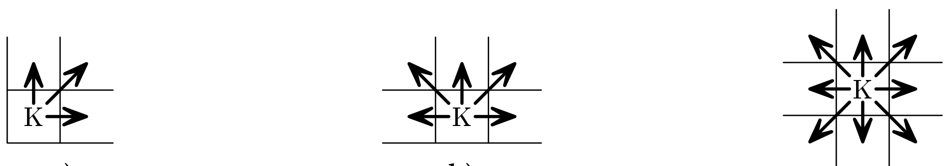
Example 7.1. Mischievous Jean has a collection of 5 different stones, one of which he chooses and throws into one of his friend Leo's 4 windows. In how many ways can he do this?

Solution: We will use the multiplication rule: we need to choose the stone he throws, **and** the window into which he throws it. So, there are $5 \cdot 4 = 20$ ways. □

Example 7.2. How many different ways can two kings, one black and one white, be placed on a chessboard without attacking one other?

Solution: Let us attempt to the multiplication rule: the first king can be placed in 64 ways — on any square. However, the number of ways to position the second king will be contingent upon the placement of the first king: in a corner, on a border, or on any other square.

First, we will employ the addition rule: the first king is placed in a corner **or** on a border, **or** on any other square.



a) b) c)

In the first case, the king attacks three squares and occupies one more, so the second king can be placed in $64 - 4 = 60$ ways. Since there are only 4 corner squares, we can place the first king in 4 ways **and** the second king in 60 ways.

There are 24 border squares, so the first king can be placed in 24 ways **and** the second king in $64 - 6 = 58$ ways (the king on the border attacks 5 squares and occupies one more). Thus, in this case, there are $24 \cdot 58$ ways.

In the third case, there are $36 \cdot 55$ ways, following similar reasoning. Therefore, the total is $4 \cdot 60 + 24 \cdot 58 + 36 \cdot 55 = 3612$ ways. □

Using the above rules, you can once again turn to the foxes from the previous chapter.

Example 7.3. Alice, Beatrice, Clarice, Dorice, and Fabrice received 5 different cloaks as gifts from the foxes of Beaverland, each distinguished by the color of its clasp. How many ways can they divide the cloaks among themselves (so that each receives exactly one cloak)?

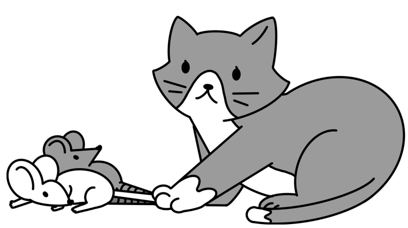
Solution: It would obviously take a long time to solve this problem by enumeration. However, if the foxes choose their cloaks in turn, Alice can choose hers in five different ways, which will limit Beatrice's choice to four, Clarice's to three, Dorice's to two, and Fabrice's to none. Using the multiplication rule would yield the following result: $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways. □

It is possible to ask the following questions: what if the foxes choose cloaks in a different order? Will new cases arise? We recommend that you confirm that the cases under consideration will remain identical as a result of the symmetry of the problem.

Additionally, another important rule can be formulated.

Claim 3 (Division Rule)**.** If each option has been counted n times during the calculation, then the result should be divided by n.

Such situations have not yet occurred in the problems we have explored thus far — each option has been counted exactly once, which is not the case for the subsequent problem.



Example 7.4. Sophie the cat caught 20 mice. In how many ways can she choose her dinner if she eats 2 mice for dinner?

Solution: Using the multiplication rule, she can choose the first mouse in 20 ways and the second one in 19 ways, since one mouse has already been chosen. So the answer is $19 \cdot 20$? Not quite. The problem is that we counted each method of choosing mice twice. Indeed, consider two mice, Mickey and Minnie. There are two ways: first, Mickey is chosen, then Minnie, or vice versa — first Minnie, then Mickey. Therefore, the desired number of ways is

$$\frac{19 \cdot 20}{2} = 190.$$

□

Example 7.5. Max and Leo received 3 cloaks as gifts from the foxes of Beaverland, distinguished by the color of their clasps: one emerald and two silver. In how many different ways can they divide the cloaks between themselves (so that each gets exactly one cloak)? (It is worth noting that the silver cloaks are indistinguishable from each other).

Solution: Let's try to enumerate the options: SE, ES, SS (E — Emerald, S — Silver), for a total of 3 possibilities.

We then apply the multiplication rule. Max chooses one of the three cloaks, and Leo chooses one of the remaining two: $3 \cdot 2 = 6$. The answer that was obtained is twice as large as the previous one. The reason is that the last six options contain repeating cases due to the two identical silver cloaks. If we denote them S_1 and S_2, it becomes clear: ES_1, ES_2, $\mathrm{S}_1\mathrm{E}$, $\mathrm{S}_2\mathrm{E}$, $\mathrm{S}_1\mathrm{S}_2$, $\mathrm{S}_2\mathrm{S}_1$. □

Problem Set

Problem 7.1. (PVG 2017.5-6.2;7-8.1): How many natural numbers from 1 to 2017 have exactly three distinct natural divisors?

Problem 7.2. (PVG 2017.5-6.2): How many three-digit natural numbers have an even number of distinct natural divisors?

Problem 7.3. (PVG 2017.5-8.6;9.5): In how many ways can the number 10000 be factored into three natural factors, none of which is divisible by 10? Consider that factorizations that differ only in the order of factors are not considered distinct.

Problem 7.4. (PVG 2017.5-6.6): In how many ways can the number 1024 be factored into three natural factors, such that the first factor is a multiple of the second and the second is a multiple of the third?

Problem 7.5. (PVG 2017.5-6.5;7-8.4;9.3): Max is creating a four-digit password for a lock. He dislikes the digit 2, so he does not use it. He also dislikes the placement of two identical numbers together. Furthermore, he wants the first digit to be the same as the last one. How many variants does Max need to try to guarantee guessing his password?

Problem 7.6. (PVG 2017.5-6.4;7-8.3;9.2): Max is creating a password for his smartphone. The password consists of four decimal digits. Max needs at least two (or more) identical digits in the password and wants the number 7 to be absent. In how many ways can Max do this?

Problem 7.7. (MF — 1998.6-7.1): If there are 17 parallels and 24 meridians on a globe. In how many parts does the surface of the globe divide into? A meridian is an arc that connects the North Pole with the South Pole, and a parallel is a circle parallel to the equator (the equator is also considered a parallel).

Problem 7.8. (MF — 1996.6.3): Which category of five-digit numbers is more prevalent: those that are not divisible by 5 or those in which neither the first nor the second digit from the left is a 5?

Problem 7.9. (LT — 2016.5-8.4): How many numbers, divisible by 4 and less than 1000, do not contain any of the digits 6, 7, 8, 9, or 0?

Problem 7.10. (PVG 2017.5-6.5;7-8.6;9.4): On a grid paper, a right triangle with legs of length 7 grid cells each (along the grid lines) is drawn. Then, all the grid lines inside the triangle are traced. What is the maximum number of triangles that can be found in this drawing?

Problem 7.11. (COM — 2009.6.5): Maria buys 16 balloons for the birthday celebration. The store has three colors of balloons: blue, red, and green. How many different purchases of 16 balloons would be possible if Maria wanted each color of the balloon to make up at least one-quarter of the total number of balloons?

Problem 7.12. (COM — 2009.6.8): A father says to his son:

- Today, we both have birthdays, and you have become exactly 2 times younger than me.
- Yes, and this is the eighth time in my life when I am younger than you by a whole number of times.

How old is the son if the father is not older than 75 years?

Problem 7.13. (AMC — 2020.8.23): If five different awards are to be given to three students, and each student will receive at least one award. In how many different ways can the awards be distributed?
(A) 120 **(B)** 150 **(C)** 180 **(D)** 210 **(E)** 240

Problem 7.14. (AMC — 2017.10A.19): Alice refuses to sit next to either Bob or Carla. Derek refuses to sit next to Eric. In how many ways can the five of them sit in a row of 5 chairs under these conditions?
(A) 12 **(B)** 16 **(C)** 28 **(D)** 32 **(E)** 40

Problem 7.15. (AMC — 2012.12B.8): A dessert chef prepares the dessert for every day of the week, starting with Sunday. The dessert each day is either cake, pie, ice cream, or pudding. The same dessert may not be served two days in a row. There must be cake on Friday because of a birthday. How many different dessert menus are possible for the week?
(A) 729 **(B)** 972 **(C)** 1024 **(D)** 2187 **(E)** 2304

Problem 7.16. (iTest — 2008.81): Compute the number of 7-digit positive integers that start *or* end (or both) with a digit that is a (nonzero) composite number.

Problem 7.17. (iTest — 2008.67): At lunch, the seven members of the Kubik family sit down to eat lunch together at a round table. In how many distinct ways can the family sit at the table if Alexis refuses to sit next to Joshua? (Two arrangements are not considered distinct if one is a rotation of the other).

Problem 7.18. (AMC — 2007.10B.20): A set of 25 square blocks is arranged into a 5×5 square. How many different combinations of 3 blocks can be selected from that set so that no two are in the same row or column?
(A) 100 **(B)** 125 **(C)** 600 **(D)** 2300 **(E)** 3600

Problem 7.19. (AIME — 2003.II.3): A good word as a sequence of letters that consists only of the letters A, B, and C, with the exception of some of these letters that may not appear in the sequence. In this sequence, A is never immediately followed by B, B is never immediately followed by C, and C is never immediately followed by A. How many seven-letter good words are there?

Skill Assessment Problems

Skill Assessment Problem 7.1. Sophie the cat caught 20 mice. In how many ways can she choose her breakfast, lunch, and dinner if:

- These mice were large, and she needed only one to satisfy her hunger.
- These mice were medium-sized, and she needed 2 for each meal.
- The cat is on a diet and skips one of the meals, but for one of the remaining meals, she wants to eat two mice, and for the other, she wants to eat one.

Skill Assessment Problem 7.2. In how many ways can a 2×3 table be colored with 2 colors (white and black) so that at least one cell is white and at least one cell is black?

Skill Assessment Problem 7.3. If 24 students participated a «Year-round Olympiad Schools» project, in how many ways can we choose:

a) Two lucky winners who will receive special prizes from the project coordinator (the prizes are the same);
b) Three lucky winners who will receive special prizes from the project coordinator (the prizes are different)?

Skill Assessment Problem 7.4. How many five-digit numbers can be formed using the digits 0, 2, 4, 6, 8, with no repetition of digits?

Solutions to Skill Assessment Problems

Solution to Problem 7.1: 1. For breakfast, the cat can choose one of the 20 mice, for lunch, one of the remaining 19, and for dinner, one of the 18 mice, a total of $20 \cdot 19 \cdot 18$ options.

2. For breakfast, the cat must choose 2 out of 20 mice, for lunch, 2 out of the remaining 18, and for dinner, 2 out of 16, i.e., a total of

$$\frac{20 \cdot 19}{2} \cdot \frac{18 \cdot 17}{2} \cdot \frac{16 \cdot 15}{2} \text{ options.}$$

3. There are 3 options for the meal she will forgo. After that, she can choose the meal where she will eat 2 mice — 2 options. During the last remaining meal, she will eat one mouse. During the meal where she eats 2 mice, there are $\frac{20 \cdot 19}{2}$ options to choose, and during the meal where she eats one mouse — 18 options. Based on the rule of multiplication, we obtain

$$3 \cdot 2 \cdot \frac{20 \cdot 19}{2} \cdot 18 \text{ options.}$$

□

Solution to Problem 7.2: There are 2 options for coloring each cell, and each cell is colored independently. Therefore, based on the rule of multiplication, the total number of coloring options is 2^6, of which 2 cases are not suitable — when all cells are white and when all cells are black, so the number of suitable options is $2^6 - 2$. □

Solution to Problem 7.3: In the first case, we need to choose 2 out of 24 people, irrespective of the order, i.e., there will be $\frac{24 \cdot 23}{2}$ options.

In the second case, the order matters, and there will be $24 \cdot 23 \cdot 22$ options. □

Solution to Problem 7.4: The first digit can be chosen in 1 out of 4 ways (zero cannot be in the first place), the second — 1 out of 4 (the remaining ones), the third — 1 out of 3, the fourth — 1 out of 2, and the fifth will be the remaining one, so in total $4 \cdot 4 \cdot 3 \cdot 2$ options. □

Method of Mathematical Induction

“ Once Euclid was asked:
«What would you prefer — two whole apples or four halves?»
«Four halves,» replied Euclid.
«But isn’t that the same thing?»
«Of course not. By choosing halves, I will immediately see whether these apples are wormy or not.»

—Popular joke

Theory and Practice

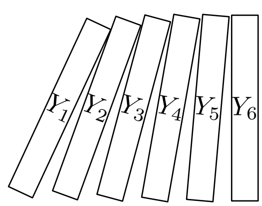
The method of mathematical induction is used to prove a series of statements, where each subsequent statement is derived from the previous one:

$$Y_1 \to Y_2 \to Y_3 \to \dots \to Y_n \to Y_{n+1} \to \dots$$

The first statement is called the *base of induction*.

Each arrow represents an *inductive step*: if the n-th statement is true and the transition is correct, then the $(n+1)$-th statement is also true. Thus, induction works as follows: the second statement follows from the first, the third from the second, and so on, up to any n. Typically, the correctness of all transitions is proven uniformly.

Induction is similar to the domino principle: if we push the first domino (the base of induction), then each domino will push the next (the inductive step), and the chain of fallen dominoes will reach any domino.



Example 8.1. (Shen): Prove by mathematical induction that (for any $n \geqslant 1$):

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

Solution: Base of induction: $n = 1$. It is obvious that

$$\frac{1 \cdot (1+1)}{2} = 1.$$

Inductive step: Using the statement

$$1 + 2 + \cdots + n = \frac{n \cdot (n+1)}{2},$$

let's prove that

$$1 + 2 + \cdots + n + (n+1) = \frac{(n+1) \cdot (n+2)}{2}.$$

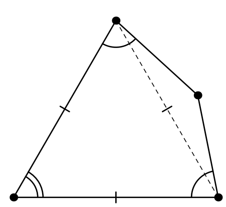
Indeed, since $1 + 2 + \cdots + n = \frac{n \cdot (n+1)}{2}$, we can add the term $(n+1)$ to both sides:

$$1 + 2 + \cdots + n + (n+1) = \frac{n \cdot (n+1)}{2} + (n+1) = \frac{(n+1) \cdot (n+2)}{2},$$

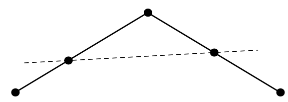
which is what we intended to prove. □

Example 8.2. Prove that for any $n \geqslant 4$, there exists a convex n-gon with exactly 3 acute angles.

Solution: Base of induction: $n = 4$. It is not difficult to construct a convex quadrilateral with 3 acute angles and 1 obtuse angles.



Inductive step: Using the statement about the existence of such an n-gon, let's prove the existence of an $(n+1)$-gon. We proceed as follows: take an n-gon with 3 acute angles (which exists; this is the n-th statement), and since n is at least 4, there is at least 1 obtuse angle. «Cut off» one of the obtuse angles, as shown in the figure below, resulting in 2 angles, each of which is greater than the one that was cut off, i.e., both are obtuse.



It is important to note that induction in this instance commences from the fourth statement, rather than the first. However, this does not affect the solution, as the analogy with dominoes is that they begin at 4. □

It should be noted that the base case of the induction, although often obvious, still requires careful verification. If there is no base case, there is no induction — there is no one to push the first domino to start the chain reaction.

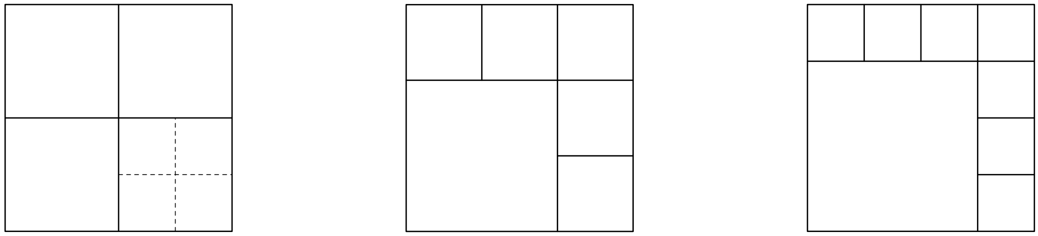
The peculiarity of the method of *complete* mathematical induction is that when proving the statement number $n+1$, one can use the truth of all statements from number 1 to number n. Formally, this can be written as:

$$\text{from } \{Y_1, Y_2, \ldots, Y_n\} \text{ follows } Y_{n+1}.$$

Typically, the method of comprehensive mathematical induction uses only a few previous statements, rather than all of them at once. An analogy can be made: if the method of mathematical induction could be represented as a set of dominoes, and for the current domino to fall, it had to be «pushed» by the previous one. And now the dominoes are somehow connected by hinges, and for the $(n+1)$-th domino to fall, all the dominoes to which it is connected must fall.

Example 8.3. Prove that any square can be cut into any number of squares, starting from six.

Solution: First, we will attempt to cut the square into a minimum number of squares. It is obvious how to cut the square into 4 parts. After some thought, we can find a way to cut it into 6 and 8 parts (as shown in the figures below).

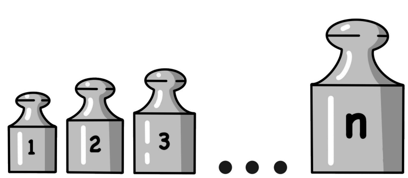


Suppose we want to cut the square into 7 parts. It is sufficient to first cut it into 4 parts, then cut one of the pieces into 4 parts again. Generally, if a square is cut into n parts, then when cutting any of its squares into 4 parts, we get $n+3$ parts, which means the implication $Y_n \to Y_{n+3}$ holds for any n. We have the following chains of implications:

- $Y_6 \to Y_9 \to Y_{12} \to Y_{15} \to \dots;$
- $Y_7 \to Y_{10} \to Y_{13} \to Y_{16} \to \dots;$
- $Y_8 \to Y_{11} \to Y_{14} \to Y_{17} \to \dots.$

Thus, to start the domino effect, we need Y_6, Y_7, and Y_8 to hold. But we have already established this. □

It is possible to assert that the problem under consideration contains three independent chains of dominoes. In order for all three to start, the base must be complex, comprised of three statements. When applying complete induction, one needs to carefully monitor the execution of this complex base; otherwise, the «chain reaction» will not start.



Example 8.4. For what values of $n > 3$ can a set of weights with masses $1, 2, 3, \dots, n$ grams be divided into three equal by-weight piles?

Solution: To divide the weights as required, the total mass of the weights must be divisible by 3. The sum of masses from 1 to n is equal to $n(n+1)/2$, and therefore n can only leave remainders 0 or 2 when divided by 3.

If we can divide a set of n weights into three equal piles by mass, then we can also divide a set of $n+6$ weights as required, because weights with masses $n+1, n+2, \ldots, n+6$ can be easily divided into three equal by mass piles, as $(n+1)+(n+6) = (n+2)+(n+5) = (n+3)+(n+4)$.

It is easy to verify that for the number of weights equal to 5, 6, 8, or 9, the required partitions exist:

- $1+4 = 2+3 = 5$;
- $1+6 = 2+5 = 3+4$;
- $1+2+3+6 = 4+8 = 5+7$;
- $1+2+3+4+5 = 6+9 = 7+8$.

Therefore, it is possible to divide all sets with a number of weights that result in remainders of 0 or 2 when divided by 3. □

Lastly, we provide a few examples of incorrect applications of the method of mathematical induction.

Example 8.5. We also prove that any n horses are of the same color.

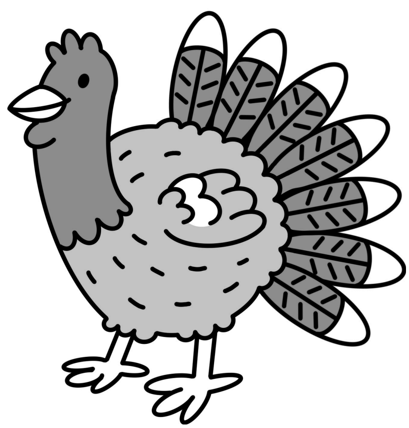
Solution: Base: $n = 1$. Indeed, the base is obvious — any single horse is of the same color as itself.

Induction: Assume that n horses are of the same color. Based on this, we will prove that any $n+1$ horses are of the same color. Indeed, consider any $n+1$ horses: $\mathrm{H}_1, \mathrm{H}_2, \ldots, \mathrm{H}_n, \mathrm{H}_{n+1}$. By the induction, the horses in the set $\mathrm{H}_1, \ldots, \mathrm{H}_n$ are of the same color, as are the horses in the set $\mathrm{H}_2, \ldots, \mathrm{H}_{n+1}$. But the horse H_2 is in both sets, thus all $n+1$ horses are of the same color. □

We encourage the reader to take a moment to reflect on the reasoning provided and determine the source of the error.

The error lies in the following: the transition $Y_1 \to Y_2$ is incorrect; in this case, H_2 is not present in both sets, and the sets under consideration do not intersect at all. Furthermore, the induction fails, despite the fact that the other transitions are accurate: the third statement follows from the second, and the fourth statement follows from the third, and so forth. The first domino fell, but it did not impact the second, resulting in the chain being broken.

Bertrand Russell, critiquing the idea of induction (from a philosophical standpoint), provides the following example in «Problems of Philosophy».



Example 8.6 (The Inductivist Turkey)**.** : Suppose a farmer feeds a turkey every day. The turkey gets used to this and expects to be fed when it sees the farmer. Suppose the turkey is a good inductivist and does not want to rush to conclusions. It starts observing the farmer and determines the time when food is brought. The turkey is confident in its assumption that the farmer will deliver food again tomorrow and, as a result, it swells with pride. The turkey fails to realize that the next day, instead of bringing food, the farmer will slaughter it, roast it, and eat it.

Solution: What is wrong with the turkey, and why is it still a bad inductivist? The turkey is wrong in assuming an inductive step, where from the fact that everything is fine today, it follows that everything will be the same tomorrow. □

Problem Set

Problem 8.1. (MMO — 1962.10.2.5): In a chess tournament, each participant played one game against each of the other participants. Prove that the participants can be numbered in such a way that no participant lost directly to the one following them.

Problem 8.2. (Shen): Consider all possible ordinary fractions with a numerator of 1 and any (positive integer) denominator:

$$\frac{1}{2}; \frac{1}{3}; \frac{1}{4}; \frac{1}{5}; \dots$$

Prove by mathematical induction that for any $n > 3$, one can represent 1 as the sum of n distinct fractions of this form. For example, for $n = 3$, one can write:

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}.$$

(It is clear that two fractions are not enough, since all fractions except the first one are less than half. It can also be verified that there is only one option for three fractions.)

Problem 8.3. (Shen): Prove that any square

$$2 \times 2, 4 \times 4, 8 \times 8, \dots, 2^n \times 2^n, \dots,$$

from which a corner square 1×1 is removed, can be cut into L-shaped pieces consisting of three cells.

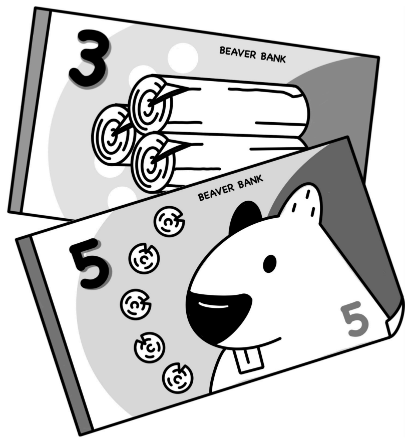
Problem 8.4. (Shen): In the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$$

each subsequent term is the sum of the two preceding ones. Prove that two numbers divisible by 7 in this sequence cannot be adjacent.

Problem 8.5. (MMCircle — 2016/2017.7): n lines are drawn on a plane, all passing through one point. Prove that they divide the plane into $2n$ regions.

Skill Assessment Problems



Skill Assessment Problem 8.1. Can any sum of whole numbers, greater than seven, be paid without change using banknotes of three and five beaver-euros?

Skill Assessment Problem 8.2. Prove that any natural number can be represented as a sum of distinct powers of two.

Solutions to Skill Assessment Problems

Solution to Problem 8.1: **First approach — transition $\mathbf{Y_n \to Y_{n+1}}$.** Base: $8 = 3 + 5$. Induction step: let the sum of n beaver-euros be payable using banknotes of three and five beaver-euros.

If there is at least one banknote of five beaver-euros among them, replace it with two banknotes of three beaver-euros: $(5) \to (3, 3)$, we get $n + 1$ beaver-euros. If all banknotes are of three beaver-euros, then, since there are at least three of them, we can replace $(3, 3, 3) \to (5, 5)$, obtaining $n + 1$ beaver-euros.

Second approach — transition $\mathbf{Y_n \to Y_{n+3}}$. We prove that any sum, starting from 8 beaver-euros, can be paid using banknotes of three and five beaver-euros. Base: $8 = 3 + 5$; $9 = 3 + 3 + 3$; $10 = 5 + 5$. Transition: from statement number n follows statement number $n + 3$.

Indeed, if n beaver-euros can be represented as the sum of $3-$ and $5-$ beaver-euros banknotes, then, by adding 3 beaver-euros, we can represent $n + 3$ beaver-euros. The triple base $n = 8, 9, 10$ and the transition $Y_n \to Y_{n+3}$ prove the statement of the problem. $\square$

Solution to Problem 8.2: We attempt to represent the number n as a sum of powers of two. We also choose the largest possible power that can be assigned to the number n. Let it be 2^k. Then $2^k \leqslant n < 2^{k+1}$. If $n = 2^k$, then the desired assumption is proven. If $n > 2^k$, then we only need to represent the number $n - 2^k$. This can be done by the induction hypothesis. The representation of the number n still needs to be checked for recurring powers of two. Since $n - 2^k < 2^k$, the powers of k cannot appear in the representation of the number $n - 2^k$, and the powers cannot be repeated.

Notice that the base in this problem consists of the numbers $2^0, 2^1, 2^2, \ldots$, and the transition has the form $n - 2^k \to n$.

Therefore, we have also proven the possibility of converting a number into the binary numeral system. $\square$

Invariant

> “ Today, I visited the profile of an old acquaintance, and he wrote there that he changed his life by 360 degrees. So, guys, this is why it’s important for you to study mathematics in school!
>
> —Popular joke

Theory and Practice

Assuming that there is some «process» in the problem; for example, there are numbers on the board that can be changed according to some rule or a position from which certain moves can be made. It can be said that this is a game in which one player participates. It is precisely through the use of such indicators that we can ascertain that the problem pertains to an invariant rather than another topic.

Definition 22. *Invariant* is a mathematical quantity or property that remains constant, i.e., does not change under a certain transformation.

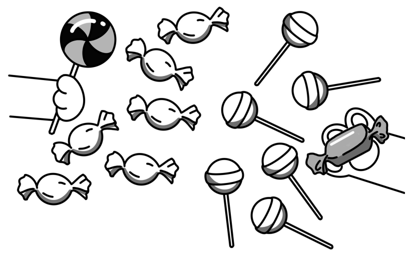
Often, the invariant is the parity (remainder when divided by 2) or the remainder of division by some number or a combination of some numbers related to the given problem. Let's consider one of the typical problems on invariants.

Example 9.1. There are 6 numbers written on the board — 3, 14, 15, 9, 2, 6. In a single operation, 1 can be added to any two numbers. Can all numbers be made the same?

Solution: Upon initial examination of the problem statement, the first idea would be to attempt to increase the smaller numbers in order to reach the larger ones. Nevertheless, it is possible to equalize five numbers after a bit of effort; however, the sixth number will be different. The problem cannot be solved through brute force, as there are an infinite number of possible outcomes (e.g., if they are equal at one million). The principle of the invariant will be used to prove that this is impossible.

We need to track a certain pattern in order to find the invariant. What didn't change when one was added to two numbers? It is easy to understand that the sum of all numbers will increase by 2, so the sum is **not** an invariant. However, if 2 is added to a number, its parity will not change. As a result, the parity of the sum of all numbers is constantly unchanged and equal to its initial value. But what was it? Since there is are odd numbers among the following numbers $3, 14, 15, 9, 2$, and 6, then, the sum of all numbers will be odd. However, we need all numbers to be equal to each other;

therefore, their sum must be divisible by 6 (their quantity), that is, it must be even. Hence, it follows that these numbers cannot be made equal. □



Example 9.2. A business is thriving in the 8^{th} grade: Max exchanges 1 chocolate candy for 6 lollipops, and Leo exchanges 1 lollipop for 6 chocolate candies. Solomon, a new student, comes to the class with 1 chocolate candy. He can have as many exchanges as he wants with his new classmates. Can he end up with an equal number of lollipops and chocolate candies, provided that he doesn't eat them?

Solution: We first attempt to find an invariant. At each moment in time, Solomon's financial position is characterized by two numbers: the number of chocolate candies X and the number of lollipops Y. For convenience, we denote it by the pair (X, Y). This pair could have changed in one move as follows: either it turned into $(X-1, Y+6)$, or into $(X+6, Y-1)$. Notice that the difference between the numbers changed by 7. This means that the remainder of the difference between the number of chocolate candies and lollipops divided by 7 did not change. In the end, the remainder and the difference, which are equivalent when divided by 7, must be zero. However, it was initially equal to 1. This is impossible, so Solomon cannot achieve his goal. □

Example 9.3. Jean took three exams. In Russian, he scored 5 points less than in physics, and in physics, he scored 9 points less than in math. A goldfish that appeared in Jean's dream promised to fulfill any number of wishes of the following types:

- add one point to each exam;
- reduce the points by 3 for one exam (of Jean's choice), and increase them by 1 for each of the other two.

The fish grants the wish if none of the results exceeds 100 points. Could John have scored 100 points more than one exam in his dream?

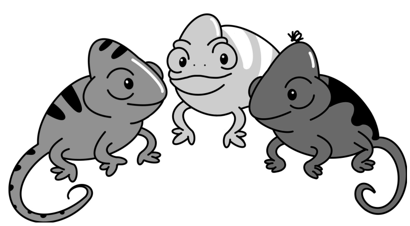
Solution: Suppose John managed to do this. Let's denote the number of points he got for these 2 exams (the points for both of which he turned into 100) by x and y, respectively. As a result of one act of magic, the following happens: either $(x, y) \to (x+1, y+1)$, or $(x, y) \to (x-3, y+1)$, or $(x, y) \to (x+1, y-3)$. Consider the remainder of $(x-y)$ under division by 4. It is easy to see that it is an invariant. But initially, the differences in points for all possible pairs of subjects are not multiples of four (they are 5, 9, and 14), and in the end, the difference in points for some two exams should be zero, i.e., divisible by four. Therefore, John's plan will not be successful under any circumstances. $\square$

Problem Set

Problem 9.1. (TOT — 1987,7-8): In the bottom left corner of an 8×8 chessboard, a 3×3 square is filled with nine chips. A chip can jump to an empty square through an adjacent chip, i.e., reflect symmetrically about its center (jumps can be made vertically, horizontally, and diagonally). Is it possible to rearrange all the chips into another 3×3 square, but in a different corner, after a certain number of such moves?

a) We need to rearrange the chips to the top left corner.
b) We need to rearrange the chips to the top right corner.

Problem 9.2. (TOT — 1987,7-10): There are two three-liter vessels. One contains 1 liter of water, and the other contains 1 liter of a two-percent solution of table salt. It is allowed to pour any part of the liquid from one vessel to the other, after which it is stirred. Can you, after several such pourings, obtain a one-and-a-half-percent solution in the vessel that initially contained water?



Problem 9.3. (TOT — 1985,7-10): The island of Graybrownrasps is inhabited by 13 gray, 15 brown, and 17 raspberry chameleons. If two chameleons of different colors meet, they simultaneously change their color to the third one (gray and brown both become raspberry, and so on). Is it possible that after some time, all chameleons will be of the same color?

Problem 9.4. (Mccme — 2004/2005.6): There are 20 scarves on a hanger. 17 girls approach the hanger one by one, and each either removes or hangs exactly one scarf. Can there be 10 scarves left on the hanger after the girls leave?

Problem 9.5. (Kozlova E. G., «Tales and Hints»): In the language of the Ancient Tribe, the alphabet consists of only two letters: «M» and «O». Two words are synonyms if one can be obtained from the other by excluding the sequences of letters «MO» and «OOMM» or by adding the sequence «OM» to any place. Are the words «OMM» and «MOO» synonyms in the language of the Ancient Tribe?

Problem 9.6. (Mccme — 2004/2005.7): Hooligans Max and Leo tore apart a poster, with Leo tearing each piece into 5 pieces and Max into 9 pieces. When attempting to assemble the poster, the teacher found 1988 scraps. Prove that he did not find all the pieces.

Problem 9.7. (Leningrad): There are seven glasses on the table, all upside down. In one moves, you can flip over any four glasses. Can you make many moves to ensure that all glasses are properly positioned?

Problem 9.8. The numbers 1, 2, $\ldots$, 100 are written on a blackboard. You may choose any two numbers a and b and erase them, replacing them with the single number $a + b1$. After 99 steps, only a single number will be left. What is it?

Problem 9.9. (Misha Lavrov, The Invariance Principle): A herd of 100 cows is divided into four pens: 10 cows in the north pen, 20 cows in the east pen, 30 cows in the south pen, and 40 cows in the west pen. The pens are connected through a gateway, which we may use to allow three cows out of one pen and distribute them among the others. For instance, if we let three cows out of the south pen, we end up with 11 cows in the north pen, 21 cows in the east pen, 27 cows in the south pen, and 41 cows in the west pen.

Prove that we can never use this gateway to split the herd into four equal groups, with 25 cows in each of the four pens.

Problem 9.10. (Misha Lavrov, The Invariance Principle): A bag contains 99 red marbles and 99 blue marbles. Taking two marbles out of the bag, you:

- put a red marble in the bag if the two marbles you drew are the same color (both red or both blue), and
- put a blue marble in the bag if the two marbles you drew are different colors.

Repeat this step (reducing the number of marbles in the bag by one each time) until only one marble is left in the bag. What is the color of that marble?

Problem 9.11. (University of Waterloo, Problem session of 2016): We begin with the numbers 1, 2, . . ., 50 written on the blackboard. At each step, we can erase any two of the numbers a and b and then write down the number $|a - b|$. We continue until one number remains. Determine whether this final number could be equal to 10.

Problem 9.12. (Nederlandse Wiskunde Olympiade): A frog starts at the point at coordinates $(0, 0)$ in the plane. He can make three kinds of jumps:

- from (x, y) to $(x,y - 5)$;
- from (x, y) to $(x - 2, y + 3)$;
- from (x, y) to $(x + 4, y + 9)$.

Ahead, there are three juicy snacks that the frog would like to eat: a worm at $(2013, 2018)$, a beetle at $(2018, 2019)$, and a snail at $(2018, 2023)$. Which of these snacks can the frog reach?

A) the worm and the snail
B) the beetle and the snail
C) the worm and the beetle
D) only the beetle
E) only the snail

Problem 9.13. (Nederlandse Wiskunde Olympiade): Julia constructs a sequence of numbers. She starts with two integers and chooses herself. Then, she calculates the next numbers in the sequence as follows: if the last number she wrote down is b and the number before that is a, then the next number will be $2b-a$. The second number in Julia's sequence is 55 and the hundredth number is 2015. What is the first number in Julia's sequence?

Problem 9.14. (Auckland Mathematical Olympiad): Starting with a list of three numbers, the «Make-My-Day» procedure creates a new list by replacing each number with the sum of the other two. For example, from $(1, 3, 8)$ «Make-My-Day» gives $(11, 9, 4)$ and a new «Make-My-Day» leads to $(13, 15, 20)$. If we begin with $(20, 1, 8)$, what is the maximum difference between two numbers on the list after 2018 consecutive «Make-My-Day»s?

Skill Assessment Problems

Skill Assessment Problem 9.1. The numbers $1, 2, 3, 2, 2, 0$ are written in a circle. In one move, you can add one to any two adjacent numbers. Is it possible to make all the numbers equal?

Skill Assessment Problem 9.2. There are six beavers sitting on six dams, with one beaver on each dam. The dams are arranged in a row with intervals of ten meters. If a beaver swims from one dam to another, then another beaver must swim the same number of meters in the opposite direction. Can all the beavers gather on one dam? What if there are seven beavers and seven dams?

Solutions to Skill Assessment Problems

Solution to Problem 9.1: We prove that the alternating sum of all the numbers written around the circle will be invariant, where the first number contributes with a plus sign, the second with a minus sign, the third with a plus sign, and so on. After adding one to any two adjacent numbers, such a function will remain constant: one of the two adjacent numbers will necessarily contribute with a plus sign, and one unit will be added to the total sum, while the other will contribute with a minus sign, subtracting one unit from the total sum. If all numbers become equal, then the alternating sum will be zero. However, initially, it is

$$1 - 2 + 3 - 2 + 2 - 0 = 2 \neq 0,$$

and it does not change after any operation, so it is impossible to make all the numbers equal. □

Solution to Problem 9.2: If we consider the position of each beaver from the same point on the river, then the sum of the distances of each beaver from this point will be an invariant. Let the leftmost dam be the reference point. Then the positions of the beavers will be as follows: $0, 10, 20, 30, 40, 50$. The sum of these numbers is 150. The beavers should be in the same place, which means the distances from each of them to the reference point must be equal. The sum of the distances of the six beavers is 150 meters, so they must gather at a point 25 meters away from the reference point. However, there is no dam there! Therefore, the answer to the first part of the problem is «no, they cannot».

We determine the meeting point by using the same reasoning to seven beavers. It will be at a distance of $(0 + 10 + 20 + 30 + 40 + 50 + 60)/7 = 30$ meters from the leftmost dam, i.e., near the fourth dam. Hence, we determine the correct movements for each beaver. The first beaver from the leftmost dam moves to the fourth dam, and the seventh beaver moves toward it to the same dam; the second and sixth beavers perform the same operation, and so do the third and fifth beavers. The fourth beaver stays in place. All movements are correct, and all beavers gather on one dam. □

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Thank you for being part of our journey to help create the next generation of math champions!