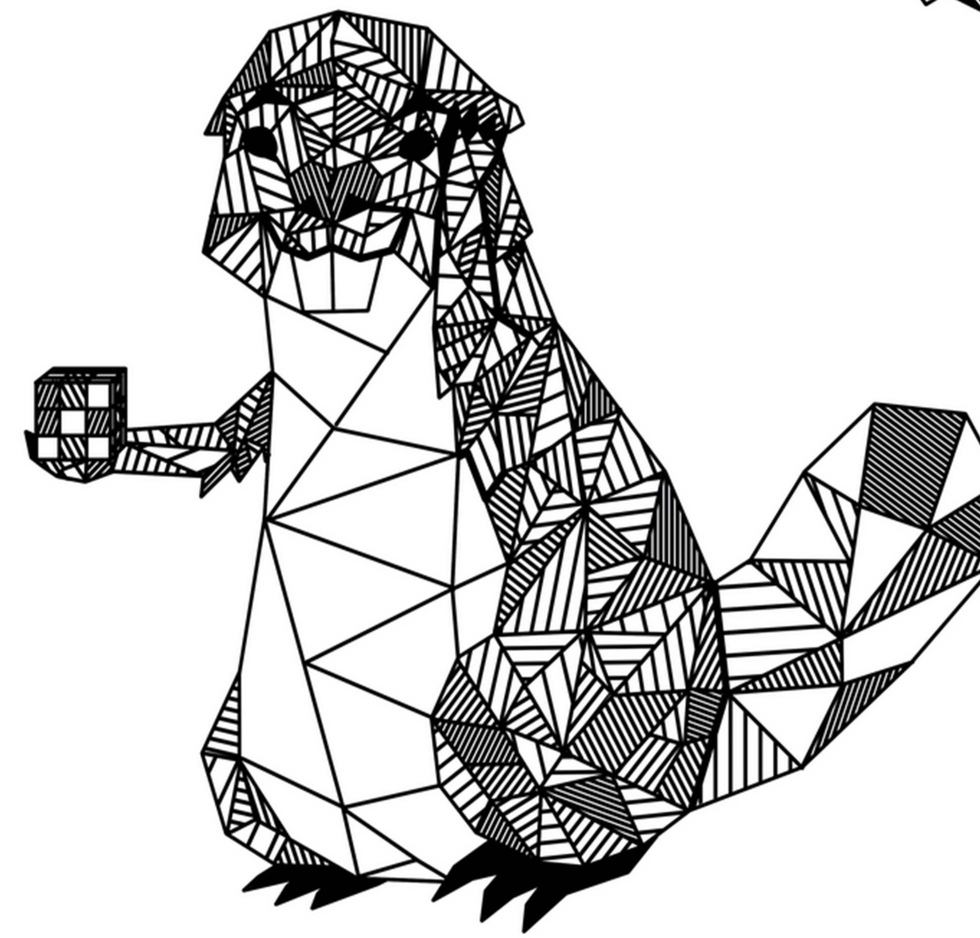
Tatiana Babicheva is an inspiring teacher who believes that teaching is not just about imparting knowledge but also about being a lifelong learner. This passion for growth and discovery led her to earn two PhDs: one in applied mathematics and its applications in Russia, and another in computer science from Université Paris-Saclay in France. She has published 10 books in Russian and French in the field of popular science, including mathematics at the competition level.

Dmitry Babichev is a gold medalist of the 2008 International Mathematical Olympiad. After completing his Master's degree, he pursued a PhD in Machine Learning and Parametric Statistics under the supervision of Francis Bach at DI/ENS. His thesis, titled On Efficient Methods for High-Dimensional Statistical Estimation, focuses on innovative solutions for complex statistical challenges. Currently, Dmitry works in the field of machine learning. He believes in the importance of explaining complex ideas in simple terms and is passionate about popularizing mathematics.

INTRODUCTION TO COMPETITIVE GEOMETRY

Mathematical Competitions. Levels A1-A2

4.Introduction to Competitive Geometry



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Dedication

> "Dedicated to our students, who know how to fold a cube and what is the net of the brain.

Introduction

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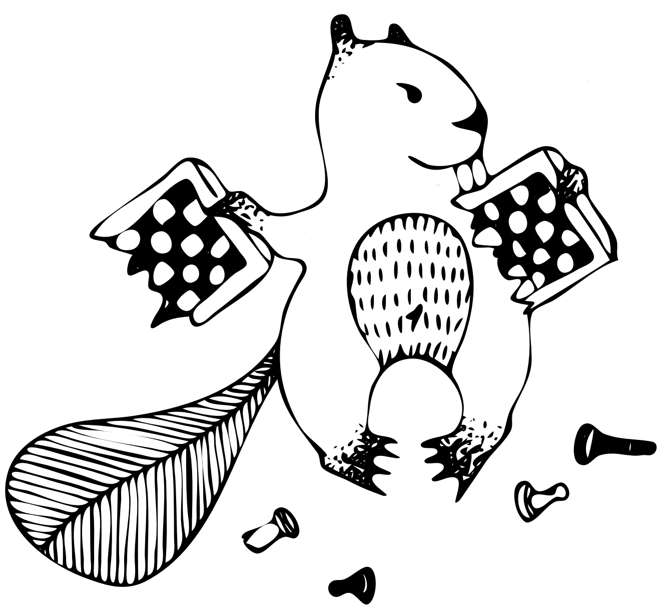


Introduction to the series

Begin your preparation for Competition Mathematics with our carefully crafted series. These books are designed to inspire a love for problem-solving and foster critical thinking. They are ideal for both budding mathematicians and passionate enthusiasts.

Inside, you will find a wide range of challenges, puzzles, and problems. Each one is selected to enhance your mathematical abilities. Experience the challenge of solving complex equations and gain confidence by deciphering complex geometric puzzles. Every book has engaging content to stimulate your mind and expand your skills.

If you're preparing for regional competitions, national tournaments, or simply want to deepen your mathematical knowledge, this series is an invaluable resource. The books provide clear explanations, strategic insights, and numerous practice problems. They aim to build your confidence and equip you with the skills needed to tackle any mathematical challenge.

While school mathematics forms a foundation, this series goes beyond it without requiring advanced knowledge to understand the material. Our course covers a wide range of topics, reflecting the diverse nature of Olympiad problems. Solving a geometry problem may require knowledge of combinatorics, while a number theory problem might involve understanding invariants and the pigeonhole principle.

Olympiad problems are generally not restricted to specific grade levels, making these books suitable for high school students. Some of the problems included have been featured in the final stages of national math Olympiads for higher grades. The goal is to demonstrate how to solve problems using straightforward and elegant methods, avoiding unnecessary complexity.

We have categorized competition mathematics into levels similar to the international standards used for foreign language proficiency. This approach is based on the concept of the «language» of competition mathematics. Traditional grade-based divisions are often outdated, as understanding a topic might only require elementary-level math. Moreover, the topics in these books are interconnected. Without a grasp of a topic at level A1, understanding its expanded form at level A2 can be challenging.

Here's what to expect at each level:

Let's use an analogy with foreign languages:

Level A1. You understand (generally) foreign speech and can talk about family, activities, hobbies, travels, weather, and buying things. In short, the standard tourist set. Can you conjugate basic verbs and be familiar with different tenses? The question «How are you?» doesn't stump you. Congratulations! You have a good A1 level! This is enough for survival.

Similarly, in olympiad math — you can «survive» at beginner-level olympiads, understand what is required in problems, and formulate solutions. You likely won't need math knowledge beyond seventh grade to understand topics at this level. (The problem might be from an 11th-grade olympiad, but the solving method remains the same.)

At level A2, you can discuss preferences in art, cultural differences, and main social trends, etc. You form complex sentences («This is Peter, whose dad works at the bank. I've already told you about him»), can write to a friend on Facebook, describe a vacation, and understand the essence of any conversation in the language.

You can recognize and solve middle-level Olympiad problems. You will be able to avoid common mistakes and present your solutions effectively. Topics at this level typically require knowledge up to the eighth grade.

This series of books generally covers levels A1 and A2 of competition math: you will understand any problem from most competitions, formulate your solution, and even change the solution of ChatGPT to match the real competition problem. However, you are still far from being a native speaker.

What is in these books?

This series uses a proof-based approach to problem-solving, which is usually reserved for advanced levels in countries like the USA and the UK. However, this method helps build a solid foundation in mathematics.

Each chapter is divided into four parts:

1. The first part covers the theoretical background and provides detailed solutions to typical problems.

2. The second part presents a problem set labeled by source. Olympiad problems are marked with notations like «Year.Grade/Round.Number.» For example, «ACM 2016.10A.5» is the fifth problem from the 10th-grade 10A variant of the ACM Olympiad 2016. Grade numbering may vary between countries, so adjust accordingly. Non-grade-specific Olympiads, like AIME, are marked by version (I or II) instead of grade.

You will encounter many problems from the Russian Olympiads (a country with a strong tradition in Olympiad mathematics) and various US mathematical competitions (such as AMC and AIME). We sincerely recommend not only finding the correct answer from the given AMC options but also approaching these problems from a proof-based perspective.

The problem number usually provides a sense of difficulty; generally, a higher number indicates a more challenging problem. However, this labeling doesn't always apply to some «independent» Olympiads, which can sometimes confuse genuine Olympiad participants.

3. The third part includes problems for independent solving, with some original problems introduced here.

4. Solutions are found in the fourth part.

The series consists of the following books:

1. Competitive Arithmetics
2. Ideas and Methods
3. Introduction to Discrete Mathematics
4. Introduction to Competitive Geometry
5. Competitive Number Theory
6. Competitive Geometry

This series is designed for both experienced Olympiad participants and newcomers to mathematical problem-solving. It offers a journey where theory and application meet, providing a rewarding experience. Welcome to a unique math adventure!

Introduction to this book

Welcome to an exciting journey into the captivating world of geometry, where abstract concepts come to life through hands-on exploration and creative problem-solving. This book brings together a collection of geometric and pre-geometric challenges, many of which are accessible without requiring prior knowledge of formal school-level geometry.

We will begin by cultivating geometric thinking through engaging puzzles like Sudoku, which will sharpen your logical reasoning and spatial awareness. As we progress, we'll venture into three-dimensional space, exploring the fascinating world of polyhedron nets and the art of unfolding shapes.

Prepare to engage in practical activities, such as cutting paper as if it were a cake, to deepen your understanding of spatial relations and geometry's inherent logic. This book emphasizes discovering the underlying logic within geometry, even in areas that may initially seem abstract or complex.

The chapters are carefully structured to guide you from foundational warm-up puzzles through to more intricate topics, such as tessellations and geometry on grid paper. Each section builds upon the previous one, offering a steady progression of challenges that will enhance your comprehension and appreciation of geometric principles.

Whether you're new to geometry or looking to refine your skills, this book is designed to be an accessible and enriching experience. By the end of our journey, you'll have developed a robust understanding of geometry and an appreciation for its elegance and utility.

List of competitions used in this book

- «Математический праздник», in English mean «Mathematical festival». We note it in the book as «MF». The official site (in Russian) is https://olympiads.mccme.ru/matprazdnik/
- Городская устная математическая олимпиада для 6—7 классов, mean «City Oral Mathematical Olympiad for 6—7 grades». We note it in the book as «COM». The official site (in Russian) is https://olympiads.mccme.ru/ustn/
- Всероссийская олимпиада по геометрии им. И. Ф. Шарыгина (Москва), mean «Sharygin Geometry Olympiad». We note it in the book as «Sharygin». The official site (in Russian) is https://geometry.ru/
- Турнир городов, mean «Tournament of Towns». We note it in the book as «TOT». The official site is https://www.turgor.ru/en/
- Школьный этап Всероссийской олимпиады школьников, mean «first stage of All-Russian School Olympiad». We note it in the book as «1ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- Муниципальный этап Всероссийской олимпиады школьников, mean «second stage of All-Russian School Olympiad». We note it in the book as «2ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- Муниципальный этап Всероссийской олимпиады школьников (Москва), mean «second stage of All-Russian School Olympiad in Moscow». We note it in the book as «Mos2ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- American Mathematics Competitions. We note it in the book as «AMC». The official site is https://maa.org/math-competitions
- Московская математическая олимпиада, mean «Moscow Mathematical Olympiad». We note it in the book as «MMO». The official site (in Russian) is https://mmo.mccme.ru/
- Турнир Архимеда, mean «Archimedes Tournament». We note it in the book as «AT». The official site (in Russian) is http://www.arhimedes.org/
- Московская математическая регата, mean «Moscow mathematical regatta». We note it in the book as «MMG». The official site (in Russian) is https://olympiads.mccme.ru/regata/
- «Покори Воробьёвы горы», mean «Conquer Vorobyovy Gory», competition of MSU. We note it in the book as «PVG». The official site (in Russian) is

https://pvg.mk.ru/

- UK Maths Trust Pink Kangaroo. We note it in the book as «Pink Kangaroo». The official site is https://ukmt.org.uk/
- UCT Invitational Mathematics Challenge and Olympiad. We note it in the book as «UCT». The official site is https://www.uctmathscompetition.org.za/invitationalchallenge

Contents

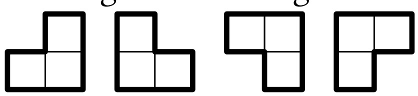
Warm-Up Puzzles: Cuttings

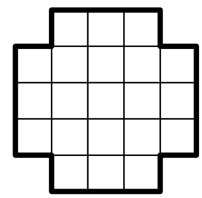
> It is true that a mathematician who is not somewhat of a poet will never be a perfect mathematician.
>
> —Karl Weierstrass

Theory and Practice

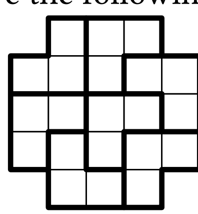
Cutting problems are among the simplest to present in competitions. Depending on the accuracy of the resulting drawing, candidates typically receive either 0 points or full marks for the given problem based on specific criteria. Let's consider an example from a middle-level mathematics Olympiad in Russia.

Example 1.1. Cut the figure on the right into corners.





Solution: In this problem, we already know the shapes into which we need to cut the original figure, so we can derive the following solution.



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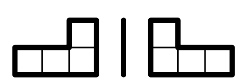
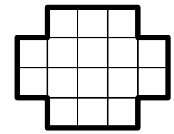
In problems involving cutting, geometric figures are typically considered equal if they can be superimposed on each other in a way that they completely coincide.



This may occur after the «flipping» of the paper from which they are cut. For example, the figures shown on the right are considered equal in terms of classic cutting problems.

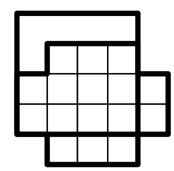
Sometimes, problems involving cutting take on a slightly different form.

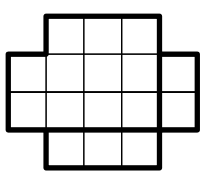
Example 1.2. Cut the figure obtained from a 4×5 rectangle by removing four corner cells of size 1×1 into three non-square pieces in such a way that these pieces can be rearranged to form a square.

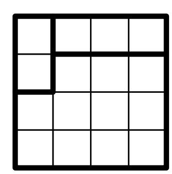


Solution: To solve a problem where we need to obtain a certain figure from parts of another, it's crucial to first determine the exact figure we aim to obtain. Let's calculate the number of cells in the original figure; there are 16 of them. We aim to obtain a square, which means the resulting square should be of size 4×4.

Let's try to understand which parts will remain in their places by drawing a square on this figure, initially attempting to «fill» as much of its area as possible. Since the figure is symmetrical, we'll get the following diagram. Then, by keeping the part shared with the square intact, we achieve the required cutting and can assemble a square from these pieces.







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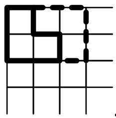
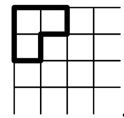
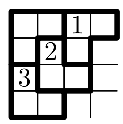
There are some problems where additional constraints are imposed on the cuttings.

Example 1.3. (MF — 2011.6.2): Cut the 6×6 square into corner pieces consisting of three cells each, ensuring that no two corners form a 2×3 rectangle.

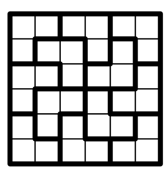
Solution: In these problems, there are no specific solving methods; it's enough to use common sense and logic, essentially conducting a sort of «thoughtful trial and error.» For example, it can be quickly understood that a figure that «touches» a corner cell cannot be positioned in such a way

, but should be positioned only like this: .

By repeating the reasoning for all four corners of the 6 by 6 board, we have essentially «cut out» four figures from the square. Trying to cover cell 1, we can find that the figure can be placed, for example, as shown in the diagram. Then, attempting to cover cell 2, we obtain the only option that does not contradict the condition. Next, we cover cell 3.



Next, we will obtain a similar pattern in each corner.



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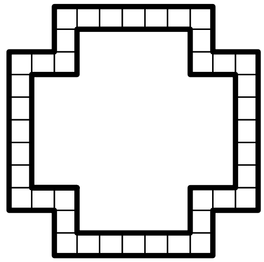
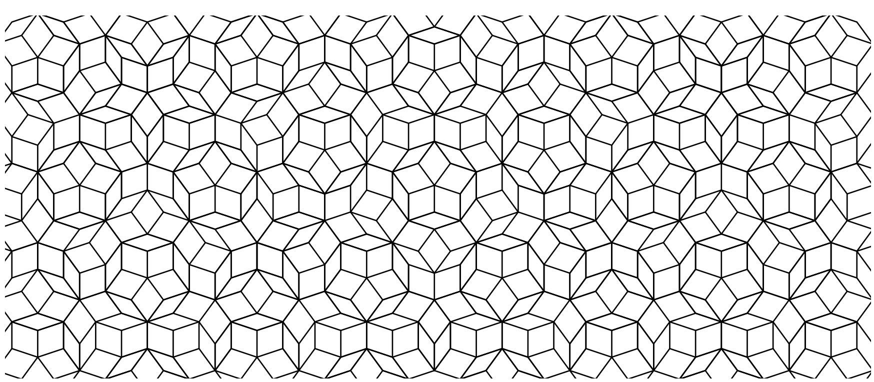
Problem Set

Problem 1.1. (MF — 2012.6.1): Cut the following frame into 16 equal parts.



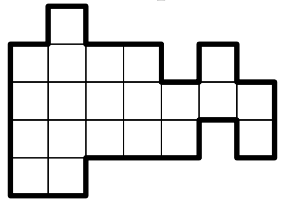
Problem 1.2. (COM — 2012.6.1): Show how to cut a square of size 5×5 cells into «corners» each one cell wide so that all «corners» consist of a different number of cells. (The «sides» of the «corners» can be of equal or different lengths).

Problem 1.3. (MF — 2011.7.1): The diagram below shows a fragment of a mosaic consisting of two types of rhombuses: «wide» and «narrow.» Draw how to cut out a shape, precisely composed of 3 «wide» and 8 «narrow» rhombuses, along the mosaic lines. (The shape should be connected along the sides.)



Problem 1.4. (MF — 2015.1): Max wanted to cut 12 rectangles of size 1×2 from an 8×8 grid so that it would be impossible to cut a 1×3 rectangle from the remaining part of the grid. (Cutting is only allowed along the grid lines.) And he succeeded! Show on the diagram how he could have done this.

Problem 1.5. (MF — 2016.1): Color one cell on the diagram, then cut the uncolored part along the grid lines into two identical pieces.

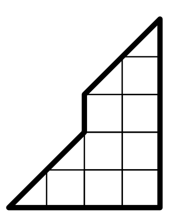


Problem 1.6. (MF — 2016.7.1): Esther instructed Jean to cut a square into 7 rectangles (not necessarily distinct), each with one side twice as long as the other. Is it possible to accomplish this task?

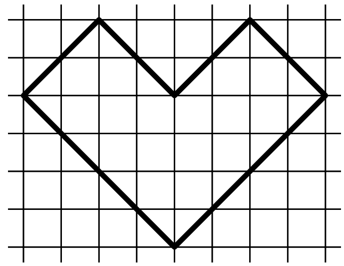
Problem 1.7. (COM — 2004.7.1): Draw a hexagon that the jury cannot cut into two quadrilaterals.

Problem 1.8. (MF — 2003.7.1): On a grid paper, a square is drawn. It is known that it can be cut into rectangles of size 1×6 cells. Prove that this square can also be cut into corner pieces with three cells each.

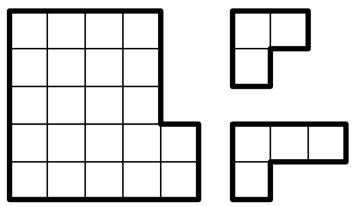
Problem 1.9. (MF — 2006.6.2): Cut the figure below into two identical parts.



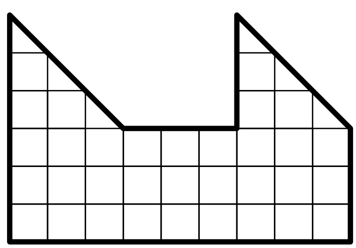
Problem 1.10. (MF — 2009.6.2): Cut the figure depicted in the drawing into eight identical parts.



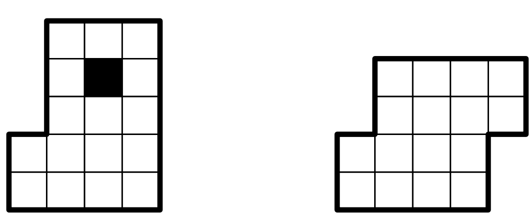
Problem 1.11. (MF — 2002.6.2;7.2): Max cut the figure into three-cell and four-cell corners, drawn to the right of the figure. How many three-cell corners could have been obtained?



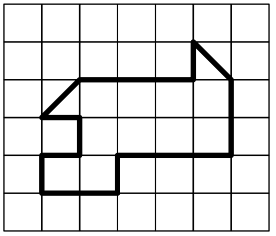
Problem 1.12. (MF — 1995.6.2): Cut the figure depicted in the drawing into two identical parts.



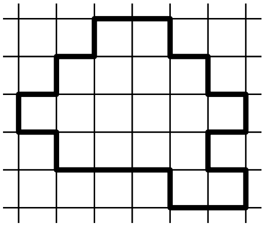
Problem 1.13. (COM — 2008.6.2): Cut the figure with the cut-out square into two identical parts that can be rearranged to form the second figure. Parts can be rotated and flipped.



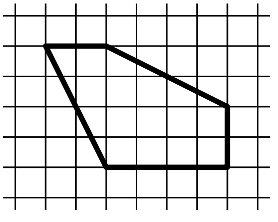
Problem 1.14. (COM — 2006.6.2): Add two cells (along the grid lines) to the figure depicted in the drawing such that it can be cut along the grid lines into two equal parts.



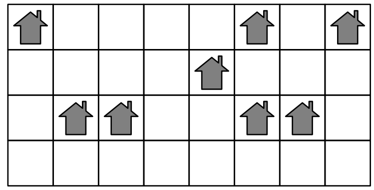
Problem 1.15. (MF — 1999.7.2): Cut the figure below (along the grid lines) into three equal (identical in shape and size) parts.



Problem 1.16. (MF — 2006.7.2): Cut the pentagon depicted in the drawing into two identical parts.



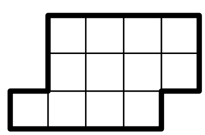
Problem 1.17. (AT — 2012.2): Divide the land plot into 8 identical country plots (that is, matching both in area and shape) such that the boundaries of the plots follow the grid lines, and each plot has a small house.



Problem 1.18. (MF — 2012.7.2): A square was cut into several pieces. After rearranging these pieces, they were assembled into a triangle. Subsequently, an additional piece was introduced to these pieces — leading to the surprising revelation that the new set of pieces could be arranged to form both a square and a triangle. Illustrate how this intriguing transformation could have occurred by depicting how the two squares and two triangles might have been composed from the pieces.

Problem 1.19. (COM — 2008.7.2): A square was cut into twelve right-angled triangles. Can ten of them be identical to each other, while the remaining two differ both from the equal triangles and from each other?

Problem 1.20. (COM — 2006.7.2): Cut the figure depicted in the picture into two identical parts using three different methods (cutting is only allowed along grid lines).

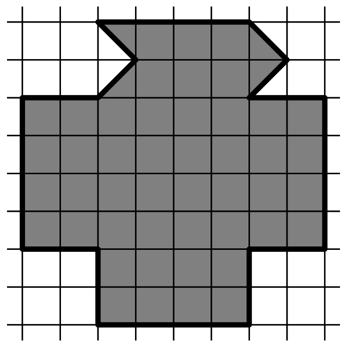


Problem 1.21. (MF — 1991.6.3): How can you cut two square pancakes lying in the frying pan into two equal parts each with a single straight cut?

Problem 1.22. (COM — 2003.6.3): Does there exist a 10-sided polygon that can be cut into 5 triangles?

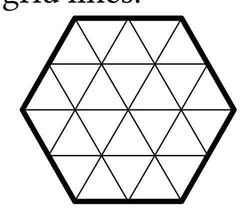
Problem 1.23. (MF — 2008.6.4): Cut any square into squares of two different sizes so that there are as many small squares as there are large ones.

Problem 1.24. (MF — 2017.6.4): Cut the figure into twelve identical parts.



Problem 1.25. (COM — 2013.6.5): Jean ordered Esther to sew a square blanket from five rectangular pieces in such a way that the side lengths of all the pieces were pairwise distinct and formed whole numbers of inches. Can Esther complete the task without the help of her fairy godmother?

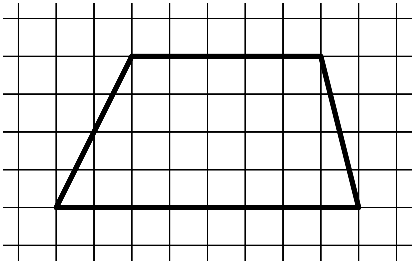
Problem 1.26. (MF — 2015.6.4;7.3): Cut the drawn hexagon into four identical figures. You can only cut along the grid lines.



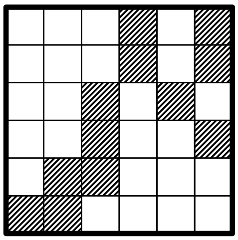
Problem 1.27. (COM — 2005.7.3): Cut a 5×5 square along the grid lines into three parts with equal perimeters.

Problem 1.28. (MF — 2003.6.4): A rectangle is divided into several rectangles, each with a perimeter that is a whole number of meters. Is it true that the perimeter of the original rectangle is also a whole number of meters?

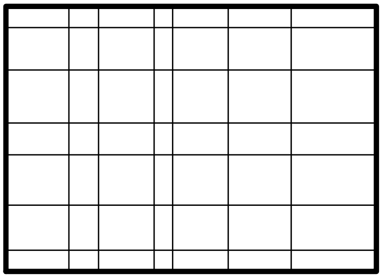
Problem 1.29. (MF — 1998.6.4): Cut the figure shown in the picture into two parts so that a triangle can be assembled from them.



Problem 1.30. (MF — 1997.6.4): Cut the given figure into four equal parts, each containing three shaded cells.



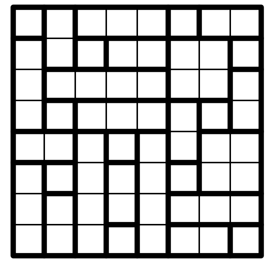
Problem 1.31. (MF — 2003.7.4): The rectangle was cut with six vertical and six horizontal cuts into 49 rectangles, as shown in the figure below. It turned out that the perimeter of each of the resulting rectangles is an integer number of meters. Is the perimeter of the original rectangle necessarily an integer number of meters?



Problem 1.32. (MF — 2003.7.4): Does there exist a decagon that can be divided into 6 parts with a single straight line?

Problem 1.33. (MF — 1994.6.5): Cut a square into three parts that can be assembled into a triangle with three acute angles and three distinct sides.

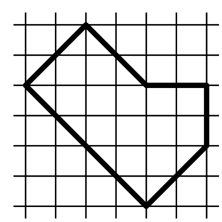
Problem 1.34. (MF — 2010.6.5): Leo cut an 8×8 chessboard along the cell boundaries into 30 rectangles, ensuring that equal rectangles did not touch even at the corners (see the figure at the right). Improve his achievement by cutting the board into a greater number of rectangles while still satisfying the same condition.



Problem 1.35. (MF — 1996.6.5): a) Can we cut a convex pentagon into four acute-angled triangles?

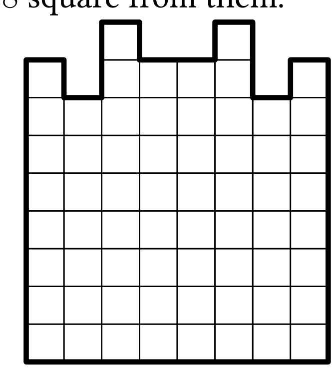
b) Is it possible to cut a regular pentagon into four acute-angled triangles?

Problem 1.36. (COM — 2009.6.5): Cut the figure in the picture into three equal parts (not necessarily along the grid lines).



Problem 1.37. (COM — 2013.7.5): Cut a 7×7 square into nine rectangles (not necessarily distinct) such that you can assemble any rectangle with sides not exceeding 7 from these nine rectangles.

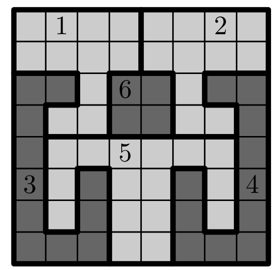
Problem 1.38. (MF — 1995.7.6): Cut the given figure into two parts such that you can assemble a complete 8×8 square from them.



Problem 1.39. (COM — 2010.6.7): Leo calls a rectangle whose sides differ by 1 an «almost-square». (For example, a rectangle with sides 5 and 6 is an almost-square.) Is there an almost-square that can be cut into 2010 almost-squares?

Problem 1.40. (COM — 2016.6.8): How many equal octagons can a square of size 8×8 be cut into? (All cuts must follow the grid lines.)

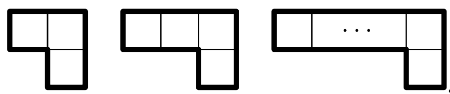
Problem 1.41. (COM — 2011.6.9): Leo cut a cardboard square 8×8 along the cell boundaries into six pieces (see the figure). It turned out that the square remained *strong*: if you place it on the table and pull (along the table) any part in any direction, the entire square will stretch along with that part. Show how to cut such a square along cell boundaries into at least 27 pieces, so that the square remains strong and each part has no more than 16 cells.



Problem 1.42. (COM — 2009.6.9): The square $2n \times 2n$ is given. Leo colors any two cells in it. Can Peter always cut this square into two equal parts so that the colored cells are in different halves?

Skill Assessment Problems

Skill Assessment Problem 1.1. The letter «L» is a grid figure consisting of $2, 3, 4, \ldots$ cells arranged in a row and one additional cell that shares a common side with one of the outer cells:



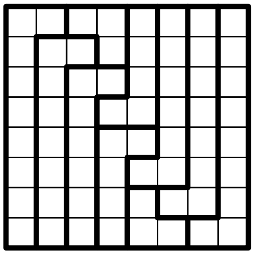
There is a set consisting of «L» letters, in which each size of the figure occurs exactly twice. Create a square 8×8 using figures from the set in such a way that there is no figure of any size used exactly once.

Skill Assessment Problem 1.2. One student, sitting in prison for a brawl in a pub, really wanted to make a flag for his organization. He planned for the flag to be square, measuring 5 by 5 meters, with a hole in the center measuring 1 by 1 meter. However, he only had a rectangular piece of fabric measuring 6 by 4 meters with gridlines in a square pattern with a step of 1 meter. The lines were parallel to the sides of the rectangle and started from them. Additionally, he wanted to cut the original piece into no more than 4 parts, which also had to be identical for easier concealment. He wanted to make the cuts only along the gridlines. Moreover, if the guards noticed rectangular pieces of fabric, they might become suspicious. Help the young man solve this problem.

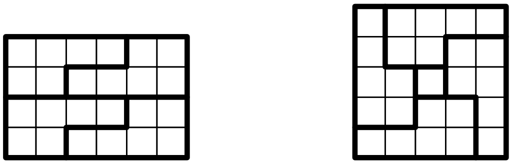
Skill Assessment Problem 1.3. There are rectangles $1 \times 1, 1 \times 2, 1 \times 3, \ldots, 1 \times 13$. Using these pieces, create a rectangle such that each side is greater than one.

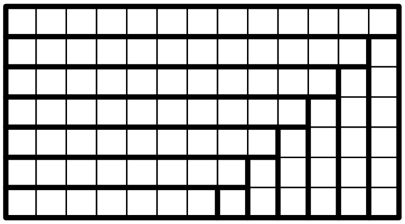
Solutions to Skill Assessment Problems

Solution to Problem 1.1: One of the possible solutions is provided in the Figure. □



Solution to Problem 1.2: One of the possible solutions is provided in the Figure. □





Solution to Problem 1.3: Let's find the size of the desired triangle. Its area should be equal to the sum of the areas of all rectangles, i.e., $S = 1 + 2 + \ldots + 13 = 91$. To obtain a rectangle, it must be the product of two numbers, each greater than one: $91 = 7 \cdot 13$. It is possible to arrange a 1×13 figure only along the long side. Then notice that $1+12 = 2+11 = \ldots = 5+7$, from which we obtain an elegant partition (see the figure). □

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Coloring

> Many who have had an opportunity to know any more about mathematics confuse it with arithmetic and consider it an arid science. In reality, however, it is a science that requires a great deal of imagination.
>
> —Sofia Kovalevskaya

Theory and Practice

Sometimes, in Olympiads, there are problems where you don't need to prove anything but rather present something, for example, a figure colored in different colors. Beginners usually like that nothing needs to be proven, but finding a solution is not always easy. In 1852, the Scottish physicist Francis Guthrie, while working on a map of the counties of England, noticed that it was sufficient to use four colors to color it. His brother informed the famous mathematician Auguste De Morgan about this observation, and later Arthur Cayley formulated the hypothesis (1878) — the «four-color problem» — whether it is possible to color any map of the world using four colors in such a way that adjacent countries are colored differently. Despite the apparent simplicity of this hypothesis, its proof turned out to be challenging. Only in 1976, the problem was solved by Kenneth Appel and Wolfgang Haken, with a significant portion of the calculations performed by a computer, and the proof text turned out to be so extensive (hundreds of pages and thousands of diagrams) that few mathematicians were able to read it in its entirety.

One of the classic non-Olympiad puzzle themes for coloring problems is Sudoku. Let's explain what Sudoku is for those who haven't encountered it.

When playing Sudoku, the game field is a square of size 9×9, divided into smaller squares with sides of 3 cells. Thus, the entire game field consists of 81 cells. Some numbers (from 1 to 9), called hints, are already placed in some cells at the beginning of the game. The player is required to fill in the empty cells with numbers from 1 to 9 so that in each row, each column, and each small 3×3 square, each digit appears only once. Classical Sudoku is usually recognized as a puzzle with 9 digits. There are also variations with larger and smaller game fields; for example, a 4×4 field divided into 4 squares of 2×2. Let's go through the simplest methods of solving such problems.

Example 2.1. Solve the Sudoku.

1			
	2		
		3	1
			4

Solution: Number the cells of the board, such as in chess or in the game of battleship, for ease of describing the steps for the solution. Note that in the bottom-right square, 3 out of 4 cells are already marked. Therefore, the cell $C1$ can only contain the number 2.

Consider the bottom-left square. It must contain the digit 1 in one of its cells. This digit cannot be in column A since 1 is already in cell $A4$. Additionally, the digit cannot be in row 2 since 1 is already in cell $D2$. Thus, 1 can only be in cell $B1$. Similarly, in the top-right square, 1 can only be in cell $C3$.

Repeat the reasoning for the digit 2 in the bottom-left square, and we conclude that it can only be in cell $A2$. Therefore, in row 2, with 3 numbers already marked, cell $B2$ can only contain the digit 4. This leaves only the cell $A1$ for the digit 3 in the bottom-left square. Consequently, in column A, the digit 4 must be placed in cell $A3$. By repeating the reasoning, we obtain the solution. The completed Sudoku concludes the solution to the problem.

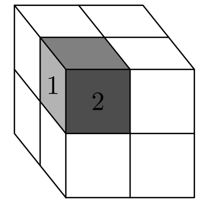
4	1	3	4	2
3	4	2	1	3
2	2	4	3	1
1	3	1	2	4
	A	B	C	D

□

We can also meet in competitions some problems involving proofs.

Example 2.2. (TOT — 1988.7-8.4): Each face of a cube is divided into four equal squares, and these squares are colored with three colors in such a way that squares sharing a common side have different colors. Prove that each color is applied to exactly 8 squares.

Solution: Let's draw the resulting cube and examine it closely. Consider the «neighbors» of a black square.
One can notice that neighbors 1 and 2 are each other's neighbors. Therefore, all 3 squares emanating from one vertex of the cube must be of different colors. The cube can be «cut» into 8 small cubes, each with 3 colored faces, all in different colors. Thus, each color is applied to exactly 8 squares. □

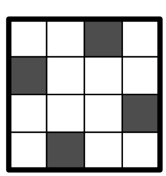


In principle, the methods for solving coloring problems often closely relate to different topics of mathematics and even to some arithmetics.

Example 2.3. (MF — 1999.6.3): A 4 by 4 square is divided into 16 cells. Color these cells in black and white so that each black cell has three white neighbors, and each white cell has exactly one black neighbor. (Cells sharing a common side are considered neighbors.)

Solution: Let's assume there are a total of x boundaries between black and white cells. Then, each white cell has exactly one boundary, which is the boundary between cells of different colors. Therefore, there are a total of x white cells. At the same time, each black cell has 3 sides, which are boundaries between it and its surrounding white cells. Thus, the total number of black cells is $\frac{x}{3}$. Since each cell in the square is either black or white, we get the equation $x + \frac{x}{3} = 16$, which yields $x = 12$.

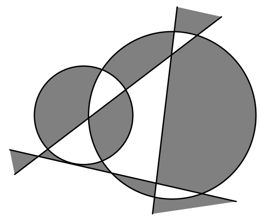
A black cell cannot be in the corner, as corner cells have only 2 neighbors. At the same time, since corner cells are white and each white cell has one black neighbor, considering the fact that there are only 4 black cells, we can arrive at the solution.



For this problem, providing only the diagram is sufficient as a solution. □

Finally, let's mention an interesting fact (without proof).

Suppose there is a set of intersecting, non-intersecting, and tangent lines and circles on the plane. It turns out that the plane can be colored in 2 colors so that the boundaries of the colors form these lines and circles!

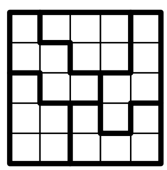


Problem Set

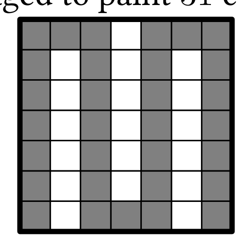
Problem 2.1. (MF — 2000.6.2;7.1): In the 7×7 square, color some cells so that each row and each column contains exactly three colored cells.

Problem 2.2. (MF — 1990.6.1;7.1): Color the plane in three colors so that on each line there are points of no more than two colors, and each color is used.

Problem 2.3. (MF — 1996.6.6): Color the cells of a 5×5 board in five colors so that each horizontal row, each vertical row, and each highlighted block contain all the colors.

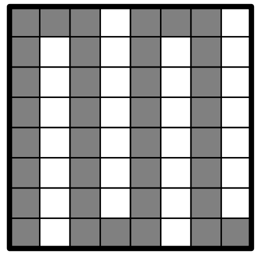


Problem 2.4. (MF — 2002.6.4): Avant-garde artist Leo Cellin painted several cells on a 7×7 board, following the rule: each subsequently painted cell must be adjacent to the previously painted cell by a side but should not be adjacent to any other previously painted cell. He managed to paint 31 cells.

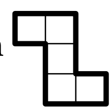


Beat his record: a) paint 32 cells; b) paint 33 cells.

Problem 2.5. (MF — 2002.7.5): Avant-garde artist Leo Cellin painted several cells on an 8×8 board, following the rule: each subsequently painted cell must be adjacent to the previously painted cell by a side but should not be adjacent to any other previously painted cell. He managed to paint 36 cells. Beat his record! (The authors can paint 42 cells!)



Problem 2.6. (MF — 2001.6.6): We will alternately color the cells of an 8×8 grid in red so that after coloring each subsequent cell, the figure consisting of colored cells has an axis of symmetry. Show how it is possible to color a) 26 cells; b) 28 cells. (As an answer, place numbers from 1 to 26 or 28 in the order in which coloring was done on the cells that should be colored.)

Problem 2.7. (MF — 2004.6.7): The cells of a notebook sheet are colored in eight colors. Prove that there is a figure of the form , inside of which there are cells of the same color.

Problem 2.8. (Pink Kangaroo): A 5×5 square is divided into 25 cells. Initially, all its cells are white. Neighboring cells are those that share a common edge. On each move, two neighboring cells have their colors changed to the opposite color (white cells become black, and black ones become white). What is the minimum number of moves required in order to obtain the chess-like coloring (with the corners colored in white)?

Problem 2.9. A four-deck ship is hidden on a 7×7 field (4 cells in a row or column). Is it possible to make 12 shots to hit this ship exactly? If possible, show where to shoot. If not, explain why.

Skill Assessment Problems

Skill Assessment Problem 2.1. A wooden cube was painted white on the outside. Each of its edges was divided into 10 equal parts, after which the cube was cut in such a way that small cubes were obtained, with each edge 10 times smaller than the original cube. How many small cubes are there with at least one painted face?

Skill Assessment Problem 2.2. Solve the Sudoku.

	5	6		3	
			6	4	5
1		4			
			1		4
2	3	5			
	4		5	2	

Solutions to Skill Assessment Problems

Solution to Problem 2.1: To solve this problem (as well as most problems where you need to calculate something introduced by the condition «at least»), it is easier to solve it «in reverse.» To do this, first find the number of cubes with no painted faces. The total number of cubes is $10 \cdot 10 \cdot 10$, and the unpainted ones are all the cubes that are behind the «top layer,» forming a cube $8 \cdot 8 \cdot 8$. Thus, there are $10^3 - 8^3 = 488$ cubes with at least one painted face. □

Solution to Problem 2.2: By following the actions suggested in the theoretical part, we get:

4	5	6	2	3	1
3	1	2	6	4	5
1	6	4	3	5	2
5	2	3	1	6	4
2	3	5	4	1	6
6	4	1	5	2	3

The completed Sudoku completes the solution to the problem. □

Method of Coloring

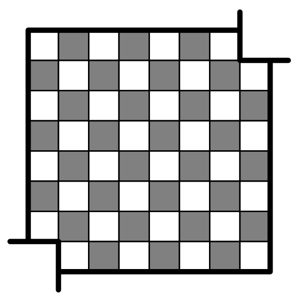
> Mathematics is the most beautiful and most powerful creation of the human spirit.
>
> —Stefan Banach, Polish mathematician

Theory and Practice

Well-thought-out coloring of figures can work wonders. Most of us have been familiar with the chessboard since childhood and the arrangement of black and white squares. It turns out that this coloring helps to solve some problems in a much more visual and understandable way.

As a warm-up, let's consider one of the most famous cutting problems in this context.

Example 3.1. Given a board of size 8×8 cells, with the bottom-left and top-right corners removed (see the figure below). Can such a board be cut into dominoes of size 1×2?

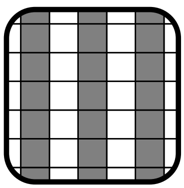


Solution: Trying to find an example of such a partition, we constantly fail, making us wonder — maybe it's impossible at all? But how to prove that it's impossible? After all, there are a huge number of attempts to arrange dominoes, and neither the participants of the competition nor the most powerful computer will be able to iterate through them in the time of the competition. Here, the chessboard coloring comes to our rescue.

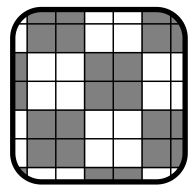
First, consider the board without the removed corners: in it, the number of white and black squares is the same — 32 squares each. Now, notice that the bottom-left and top-right corners are of the same color. Let's assume this color is black (the case with the white color is absolutely analogous). Then, the board with two removed corners has 32 white squares and 30 black squares. But each domino consists of one white and one black cell! Therefore, the board cut into dominoes must contain an equal number of black and white cells, which is not the case. Hence, it is impossible to cut this board into dominoes. □

The discussed coloring is far from the only one used in solving problems of this type. For example, the following colorings are also quite common:

- «Striped coloring» — in this coloring, the first column is entirely black, the next one is white, then black again, and so on (see Figure a));
- «Large chessboard coloring» — almost the same as the regular chessboard coloring, but everything is divided into 2×2 squares (see Figure b)).



a) «Striped coloring»



b) «Large chessboard coloring»

Colorings are usually used to prove the non-existence of the required partition. Usually, it is not immediately clear which coloring to use for the solution, so the first step is to determine which coloring will help solve the problem. It may be necessary to try several different coloring methods for this. Let's consider the following problem.

Example 3.2. Given a board of size 10 by 10. Can it be cut into rectangles of size 1 by 4?

Solution: Let's try using chessboard coloring — then there will be 50 black and white squares each. However, each rectangle occupies 2 black and 2 white squares, so theoretically, 25 rectangles of size 1 by 4 would give 50 black and 50 white squares, and there is no contradiction in this case. But this does not imply that such a partition exists.

Let's use the «large» chessboard coloring. Then the board is divided into 25 squares of 2 by 2, and of them, 13 will be black and 12 will be white. Therefore, there will be a total of 52 black and 48 white squares. However, each rectangle of size 1 by

4 occupies exactly 2 black and exactly 2 white squares. The resulting contradiction shows that it is impossible to cut the board into the specified rectangles. □

Problem Set

Problem 3.1. (COM — 2002.6.6): How many checkers can be placed on a chessboard if any square consisting of nine cells contains exactly one checker?

Problem 3.2. (MMG — 2013/14.7.4.3): A central square of size 2×2 was cut out from a chessboard (8×8). Can the remaining part of the board be cut into equal figures in the form of the letter «L», consisting of four cells each?

Problem 3.3. (Mos2ARSO — 2014.9.6): A square of size 2×2 was cut out from a chessboard (8×8) in such a way that the remaining board could be cut into rectangles of size 1×3. Determine which square could be cut out.

Problem 3.4. (MMG — 2017/18.11.2.3): Eight rectangles of size 2×1 were cut out from an 8×8 grid. After that, from the remaining part, it is required to cut out a square of size 2×2. Is this always possible?

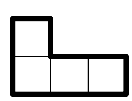
Problem 3.5. (UCT): Prove that it is not possible to cover a 9×9 square with 13 2×3 rectangles and one L-shaped tromino placed in any orientation.

Skill Assessment Problems

Skill Assessment Problem 3.1. There is a figure called a «beaver» on a checkerboard. It can move three squares horizontally and one square vertically or five squares vertically and one square horizontally. Can the «beaver,» after making several moves, reach a square adjacent to the initial one?

@@@@

Skill Assessment Problem 3.2. Can a 10×10 square be cut into the shapes shown below?



Skill Assessment Problem 3.3. Can a 8×8 square with a corner cell removed be cut into rectangles of size 1×3?

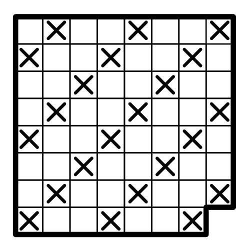
Solutions to Skill Assessment Problems

Solution to Problem 3.1: This is a straightforward problem if we notice that on a chessboard, the «beaver» never changes the color of the square it lands on. All squares neighboring the initial square have a different color. So, the answer is «impossible.» □

Solution to Problem 3.2: Consider the striped coloring with horizontal stripes. Note that each figure of this type occupies either 1 black cell and 3 white cells or 1 white cell and 3 black cells. There are a total of 100 cells in the 10×10 square, and each figure has 4 cells. Therefore, if the square can be cut into these figures, there must be 25 of them. The entire board has 50 white squares. Each figure occupies either 1 or 3 white squares, but the sum of 25 odd numbers cannot be even. Therefore, it is impossible to cut the 10×10 square into these figures. □

Solution to Problem 3.3: Our goal is to color the figure in such a way that, no matter how we place a 1×3 rectangle, it will cover one and exactly one colored cell (let's call this the «condition»). On the right side of the figure, you can see one possible coloring that satisfies this condition.

To cover all the cells marked with crosses, 22 rectangles are needed. However, the area occupied by them will be 66 cells, and we only have 63 cells available. Therefore, such a cut does not exist.



In solving this problem, we can find other colorings that satisfy the «condition.» If there were fewer such cells, we would need to present a solution (which would not be possible). That's why we chose, from the set of colorings satisfying the «condition,» the one in which the colored cells are more than 21.

□

Net of a Polyhedron

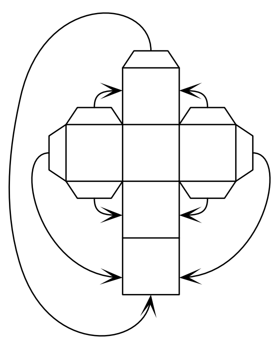
> “What is mathematics? It is only a systematic effort to solve puzzles posed by nature.
>
> —Shakuntala Devi

Theory and Practice

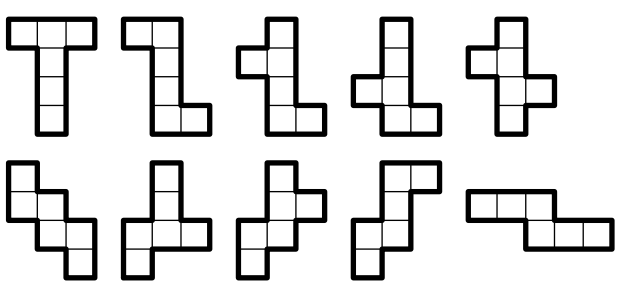
One of the important concepts in solid geometry problems (i.e., problems related to three-dimensional figures) is the concept of «unfolding» or «net.»

Let's consider a figure you've already encountered in previous math competitions — the cube. You can easily make a cube from paper.

Consider the following net of a cube, shown in the figure. If we fold this figure along the grid lines, connecting and gluing together the edges marked with arrows (using the convex parts of the net), we will obtain a three-dimensional figure.



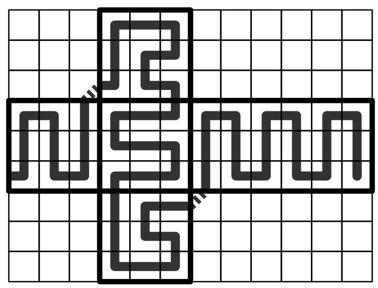
There are many possible nets of a cube made up of squares, and all the variants are shown in the figure below and the one above.



Sometimes, the net can help solve or illustrate a problem, for example:

Example 4.1. (TOT — 1988-1989.7-8.4): Can you draw a closed path on the surface of a Rubik's cube that passes through each square exactly once (the path does not pass through the vertices of the squares)?

Solution: Let's draw the net of the cube and try to draw a «continuous» path, keeping in mind how the sides of the cube are connected. We get the following picture.



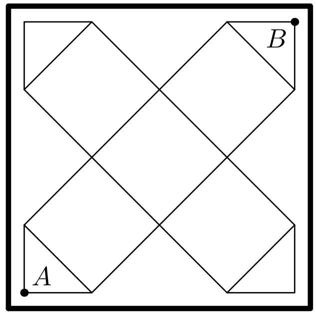
From this, we conclude that it is possible to draw such a path. □

Without the illustration of the net, the solution would be much longer and less clear.

The net will not always represent a figure that needs to be glued along the edges; sometimes, it can be something more complex.

Example 4.2. (MMO — 1954.8.1): Cut out from a 3×3 square one shape that represents the net of the complete surface of a cube with an edge length of 1.

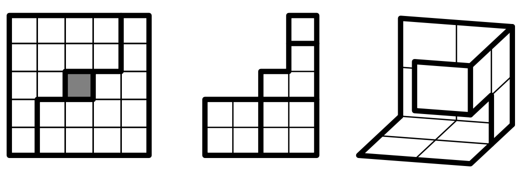
Solution: Notice that the net is cut from a symmetric figure. Let's try to make a symmetric net of a cube. Cut one of the faces of the cube along the diagonals, and then make a cut along the 4 edges coming out of this face, resulting in:



The distance AB is equal to four times the length of the cube's edge, i.e., it is equal to 4. Notice that the diagonal of the 3×3 square has a length of $3\sqrt{2} > 4$, so our net does not exceed its boundaries. □

Example 4.3. (MF — 1998.7.6): From a 5×5 square, cut out the central cell. Cut the resulting figure into two parts that can be folded into a $2 \times 2 \times 2$ cube.

Solution: Notice that the square can be cut as shown in the figure in the middle.



Each part folds as shown on the right and forms with the second one a $2 \times 2 \times 2$ cube. □

Problem Set

Problem 4.1. (Sharygin — 2005.23): Cover a cube with five convex pentagons of equal size in a single layer.

Problem 4.2. (Sharygin — 2011.10.8): There is a sheet of metal measuring 6×6. It is allowed to make cuts, but in such a way that it does not fall apart into pieces, and to bend it. How to make a cube with an edge of 2, divided by inner partitions into unit cubes?

Problem 4.3. (COM — 2007.8-9.1): Given a rectangular strip of size 12×1. Cover a cube with an edge of 1 using this strip in two layers (you can bend the strip but not cut it).

Problem 4.4. (COM — 2010.6.4): Given a cube with an edge of 2. Show how to glue 10 squares with a side of 1 onto it without overlap so that no squares share a segment (either a side or a part of it). Bending squares are not allowed.

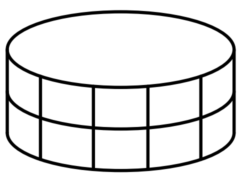
Problem 4.5. (MMG — 2011/12.10.2.2): A narrow ribbon is wound around a cylindrical column with a height of 20 meters and a diameter of 3 meters. The ribbon rises from the base to the top in seven complete turns. What is the length of the ribbon?

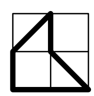
Skill Assessment Problems

Skill Assessment Problem 4.1. In the Guggenheim Museum in New York, there is a sculpture shaped like a cube. A beetle sitting on one of the vertices wants to explore the sculpture as quickly as possible to move on to other exhibits (it is sufficient to reach the opposite vertex of the cube). What path should it choose?

Skill Assessment Problem 4.2. Create the net of a regular quadrilateral pyramid.

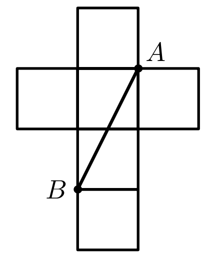
Skill Assessment Problem 4.3. (Mos2ARSO — 2014.5.4): Polina decided to paint her checkerboard bracelet, which is 10×2 in size (left picture), with a magical pattern of identical figures (right picture), alternating two colors in them. Help her do it.



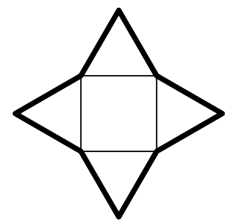


Solutions to Skill Assessment Problems

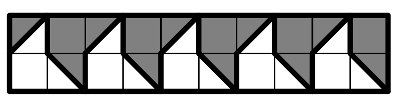
Solution to Problem 4.1: When choosing the net of the cube with the given arrangement of the initial and final vertices, it becomes clear that the shortest path on the cube will correspond to the shortest path, i.e., a straight line, on the net. □



Solution to Problem 4.2: The simplest form for the net will be to choose the base of the pyramid as its «center.» □



Solution to Problem 4.3: Let's create a net of the bracelet and carry out the partition and coloring (shown in the figure below).



□

Tessellations

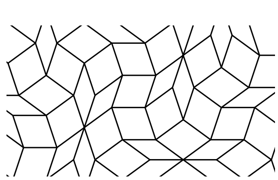
> Mathematics is not about numbers, equations, computations, or algorithms; it is about understanding.
>
> —William Paul Thurston, American mathematician

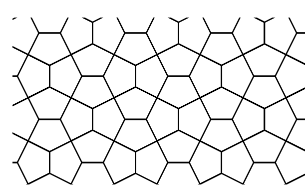
Theory and Practice

Modern mathematics, as a science, over time, becomes less accessible for understanding to a wide range of people. One reason for this is the abandonment of geometric, intuitive descriptions in favor of the formalization and algebraization of mathematics. However, despite the high level of abstraction in modern mathematics and the ongoing extreme increase in abstraction, discoveries are still made in this branch of science whose meaning is intuitively clear to non-professionals. One such example is related to the problem of tiling a plane with tiles (polygons of the same shape). Moreover, quite recently, discoveries in mathematics could be made by non-professionals.

For instance, the well-known story involves Marjorie Ruth Rice, a homemaker from San Diego. She was a mother of five and had no mathematical education. In the 1970s, she discovered a new type of pentagonal tiles, one of the solutions to the problem of finding convex pentagons that can tile the plane without gaps or overlaps. She came across an article in Scientific American by M. Gardner, who encouraged the search for such pentagons. She read it by chance and informed Gardner about her discovery.

Let's consider, for example, sidewalk tiles.

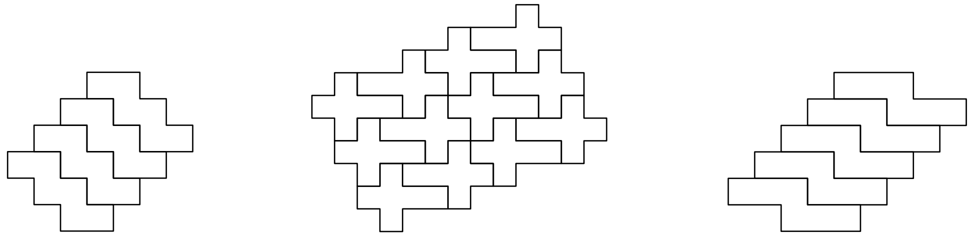




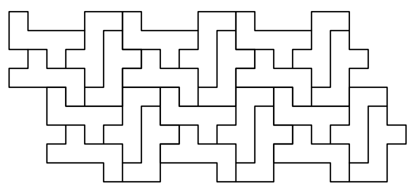
On the left picture, we tiled the plane with tiles of various types, and on the right, we tiled it with identical tiles.

In the previous topic, we discussed the net of a cube. Any net of a cube can tile the plane.

Here are a few examples.



One of the important principles in solving problems of tiling a plane is the principle of repeatability and symmetry. Essentially, in classical Olympiad mathematics, problems of tiling a plane are problems of dividing infinite space into identical parts.

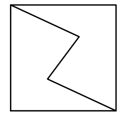


Example 5.1. (MF — 1990.5.4): Tile the plane with identical pentagons.

Solution: One might initially attempt to draw identical regular pentagons. For instance, a common «solution» might look like this:

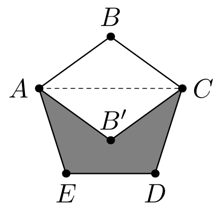
«Cut» the pentagon into 3 triangles. In each of them, the sum of angles is $180°$, so the sum of the pentagon's angles is $180° \cdot 3 = 540°$. Therefore, one angle is $540° : 5 = 108°$. After tiling the plane, several angles of the pentagon converge in each corner, but $360°$ cannot be divided by $108°$, so tiling the plane in this way is impossible.

If the problem explicitly stated regular pentagons, this would be a valid solution. However, the regularity of the pentagons was not specified in the problem set! In fact, tiling the plane in this way is possible. For example, we can first tile the plane with equal squares and then divide each of them into two identical pentagons.



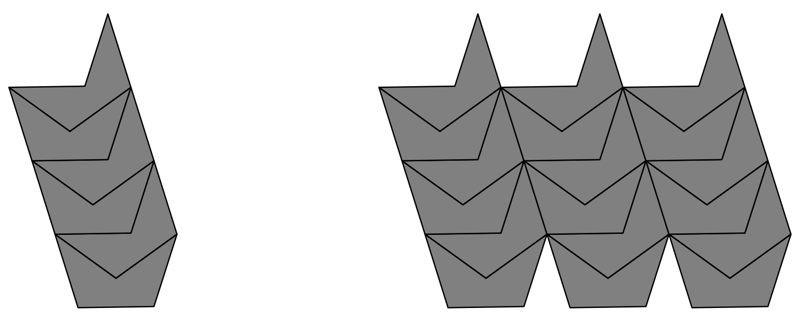
The obtained illustration completes the solution to the problem. □

Example 5.2. (Sharygin — 2005.10-11.3): $ABCDE$ is a regular pentagon. Point B' is symmetric to point B with respect to line AC (see the figure). Can the plane be tiled with pentagons congruent to $AB'CDE$?



Solution: Let's find the angles of the obtained pentagon. In the course of solving the previous problem, we already obtained that the angle of a regular pentagon is $108°$. Thus, $AB'CDE$ has angles equal to $36°$, $252°$, $36°$, $108°$, $108°$, respectively. Also, the sides are equal because $ABCDE$ is regular and $AB' = AB = BC = B'C$ since B' is symmetric to B with respect to AC.

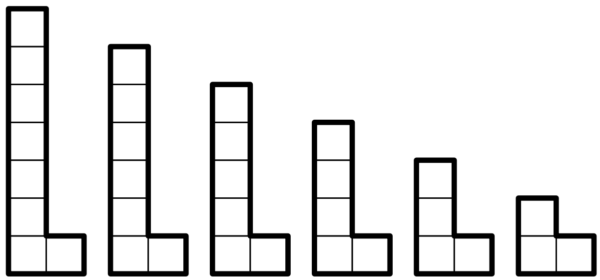
Therefore, the region $AB'CDE$ can tile a «strip» (left image). Then, using identical parallel strips, we can tile the entire plane (right image).



The obtained illustration completes the solution to the problem. □

Problem Set

Problem 5.1. (COM — 2005.6.3) Assemble a rectangle from the set of corner pieces shown in the diagram.

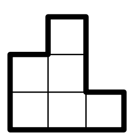


Problem 5.2. (MF — 1990.5.4): Tile the plane with identical heptagons (7-gons).

Problem 5.3. (MF — 2004.6.4;7.5): Assemble the following figures shown in the diagram:

а) a square with dimensions 9×9 with a 3×3 square cut out in its center;
б) a rectangle with dimensions 9×12.

The figures can be not only rotated but also flipped.



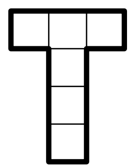
Problem 5.4. (TOT — 1996/97.8-9.2): For which integer values of n can a regular triangle with side length n be tiled with tiles in the shape of an isosceles trapezoid with sides 1, 1, 1, 2?

Skill Assessment Problems

Skill Assessment Problem 5.1. Tile the plane with identical octagons (8-gons).

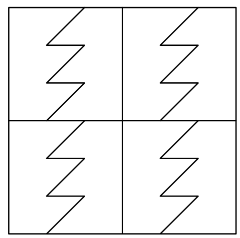
Skill Assessment Problem 5.2. Tile the plane with identical nonagons (9-gons).

Skill Assessment Problem 5.3. Tile the plane with the following net of a cube.



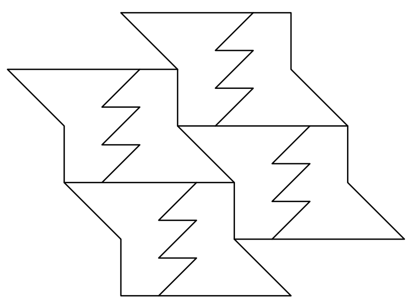
Solutions to Skill Assessment Problems

Solution to Problem 5.1: An example of tiling is shown in the figure below. Since the plane can be tiled with squares, the initial polygons can also be used for tiling.



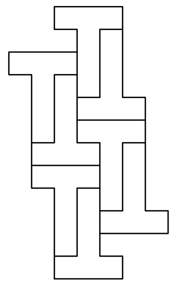
□

Solution to Problem 5.2: An example of tiling is shown in the figure. The constructed infinite strips with parallel edges allow for the completion of the plane tiling.



□

Solution to Problem 5.3: An example of tiling is shown in the figure below. The constructed infinite strips with parallel edges allow for the completion of the plane tiling.



□

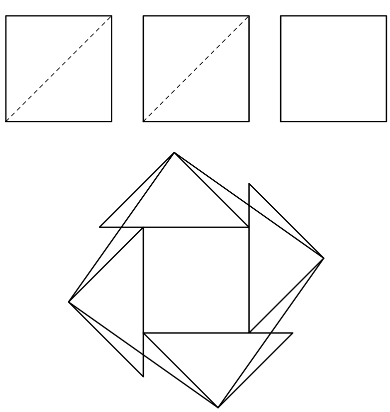
Tessellations and Cuttings with Restrictions

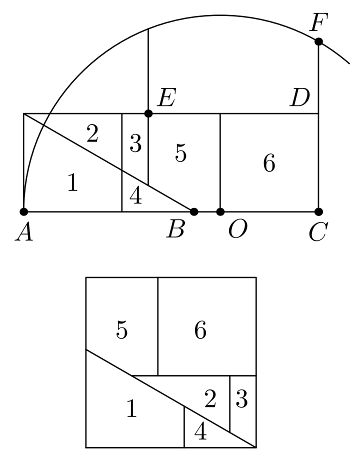
> The pure mathematician, like the musician, is a free creator of his world of ordered beauty.
>
> —Bertrand Russell, British philosopher

Theory and Practice

Tasks related to tiling were discussed in the previous chapter; however, there are also slightly modified assignments with additional conditions, such as constraints on the number or types of figures and restrictions on the way a figure can be manipulated — whether «flipped» (mirror-symmetric to the given figure), etc.

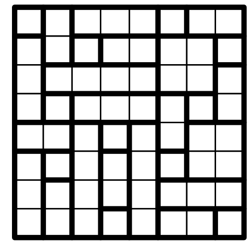
Tiling problems have been the subject of interest for many mathematicians since ancient times. While many simple problems of this type were solved in antiquity, Abu'l-Wafa, a renowned mathematician and astronomer of Persian origin from the 10th century living in Baghdad, is considered the author of the first systematic work on dissections. Only fragments of his works have survived. One of the most famous problems attributed to Abu'l-Wafa is his dissection problem: to cut three equal squares into pieces that can be assembled into one large square. The figure below illustrates the solution: cut two squares along their diagonals into four triangles each, then arrange them around the third square. Four more cuts complete the solution to the problem.



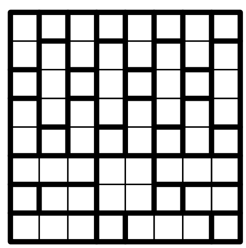


Geometers seriously started considering problems related to dissections, where the goal was to cut figures into pieces with a constraint on the minimum number of parts and then assemble them into a different figure, only in the early 20th century. One of the pioneers in this fascinating branch of geometry was the renowned puzzle creator Henry E. Dudeney. Dudeney solved the ancient Abu'l-Wafa problem with just 6 pieces, which is currently the partition with the smallest possible number of parts (in the figure on the right, $AB = ED = CF$, and O is the center of the circle).

Example 6.1. (MF — 2010.6.5): Sasha cut an 8×8 chessboard along the cell boundaries into 30 rectangles in such a way that equal rectangles did not touch even at the corners. Can this achievement be improved by cutting the board into a larger number of rectangles while preserving the same condition?



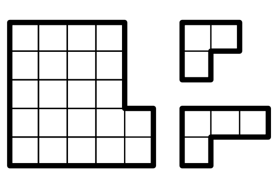
Solution: The board is cut into 35 rectangles, as shown in the figure.



It turns out that the board cannot be cut into a larger number of rectangles in the manner described above. However, the proof of this is quite cumbersome, and we will not present it here. It was not necessary to solve the problem.

□

Example 6.2. Nemo cut the figure into three-cell and four-cell corners, as shown to the right of it. How many three-cell corners could have been obtained?



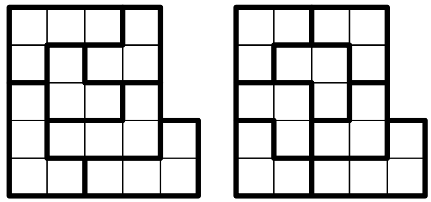


Figure 6.1: Cases for $x = 2$ on the left and for $x = 6$ on the right.

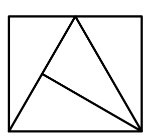
Solution: The figure consists of 22 cells. Let x be the number of obtained three-cell corners, and y be the number of four-cell corners. Then $3x + 4y = 22$. Note that x is even and $x < 8$ (since $3x$ cannot exceed 22), so $x = 0, 2, 4$, or 6. The values $x = 0$ and $x = 4$ do not fit, as y becomes non-integer. Both remaining options for $x = 2$ and $x = 6$ can be realized, as shown in the figure.

If you simply obtain the drawings by trial and error, you will not receive full marks for this problem. You need to explain why other options are not possible!

□

Example 6.3. (TOT — 2015.8-9.2): A rectangle was formed by arranging identical scalene right-angled triangles without any holes or overlaps. Is it necessary that there are two of these triangles that are positioned to form a rectangle?

Solution: Not necessarily. For example, take an equilateral triangle and attach right-angled triangles with a $60°$ angle to two of its sides with the hypotenuses in common at the vertex.

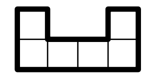


Now, we can cut the equilateral triangle along the altitude from the remaining vertex.

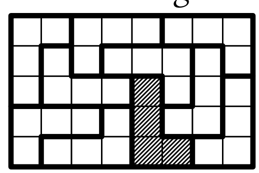
□

Problem Set

Problem 6.1. (MF — 2005.6.4): Neznayka placed only 13 bracket-shaped figures without overlaps in a 10×10 square, as shown in the figure. Try to find how to place more such figures in the same square.



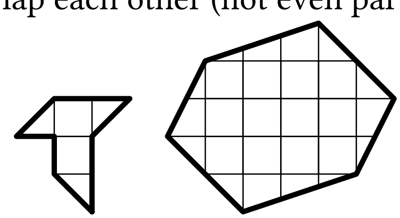
Problem 6.2. (MF — 2003.6.5): The young parquet layer has 10 identical tiles, each consisting of 4 squares, forming the shape of the letter «L» (all tiles are oriented the same way). Can he form a 5×8 rectangle with them? (Tiles can be rotated but not flipped. For example, the shaded tile in the figure is oriented incorrectly.)



Problem 6.3. (COM — 2016.6.5): Vasya drew with a pencil a partition of a rectangular grid into 3×1 rectangles (trimino), inked the central cell of each resulting rectangle, and then erased the pencil lines. Is it always possible to restore the original partition?

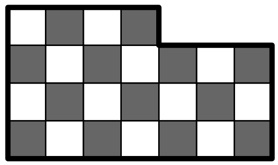
Problem 6.4. (MF — 1999.6.6): A gray square is drawn on the plane. There are seven square tiles of the same size. It is necessary to place them on the plane so that they do not overlap and each tile covers at least a part of the gray square (at least one point inside it). How can this be done?

Problem 6.5. (COM — 2008.7.8): You need to fit the four given identical figures (left picture) into the hexagon (right picture) so that they do not protrude beyond its boundaries and do not overlap each other (not even partially).



Skill Assessment Problems

Skill Assessment Problem 6.1. (COM — 2010.6.1;7.2) Grandma had a checkered cloth, as shown in the figure.

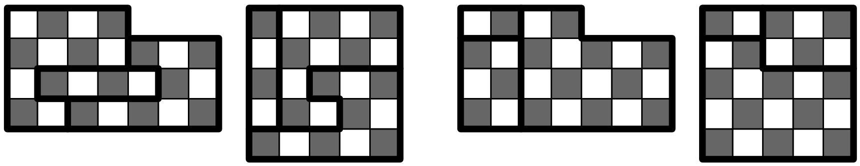


One day, she wanted to sew a square mat for the cat with dimensions 5×5 from it. Grandma cut the cloth into three pieces and sewed a square rug from them, which was also painted in a checkerboard order. Show how she could do it (the cloth has one side facing up and the other side facing down, so the pieces can be rotated but not flipped).

Skill Assessment Problem 6.2. Can a square be cut into four pieces in such a way that each piece has a common segment with any other?

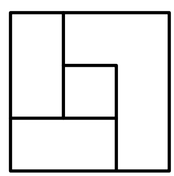
Solutions to Skill Assessment Problems

Solution to Problem 6.1: This can be done in several ways, some of which are shown in the figure.



□

Solution to Problem 6.2: The square can be cut as shown in the figure below.



□

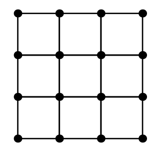
Geometry on Grid Paper

> “Just because we can’t find a solution, it doesn’t mean there isn’t one.
>
> —Andrew Wiles, English mathematician

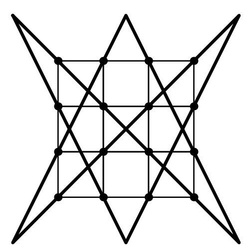
Theory and Practice

Geometry on checkered paper is already well-known to us from previous chapters. Let's move on to more complex or unusual problems related to the «cellularity» of space. To solve some problems in this chapter, you may need knowledge of the basics of the geometry course.

Example 7.1. (MF — 2005.7.3) Cross out all sixteen points shown on the figure using six segments, without lifting the pencil from the paper and without drawing segments along grid lines.



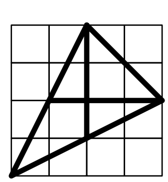
Solution: One possible solution is shown in the figure below:



□

Example 7.2. (MF — 1999.6.5): Draw on graph paper a triangle with vertices at the corners of cells, two medians of which are perpendicular. (A median connects a vertex of the triangle with the midpoint of the opposite side.)

Solution:



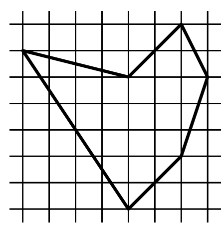
□

How to find the area of a figure on grid paper? Usually, such areas are calculated by dividing them into rectangles and triangles or by using complements to them. However, it can be calculated much easier! There is a remarkable formula called the **Pick's theorem**:

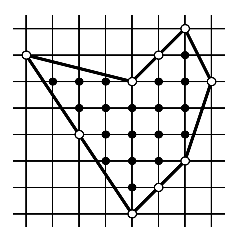
$$A = I + \frac{B}{2} - 1,$$

where I is the number of grid points inside the polygon, and B is the number of grid points on its boundary. From this formula, it follows that the area of a polygon with vertices at grid points is always an integer or a half-integer.

Example 7.3. What is the area of the following polygon?

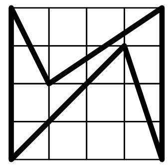


Solution: First, count the number of grid points inside (black) and on the boundary (white) of the polygon: $I = 19$, $B = 9$. Using the formula, we get $S = 19 + \frac{9}{2} - 1 = 22.5$. □



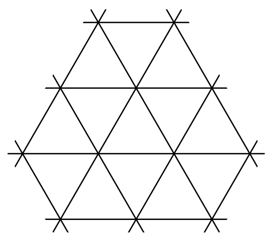
Example 7.4. (MF — 2007.7.4): On graph paper, four grid points forming a 4×4 square are marked. Mark two more grid points and connect them with a closed broken line to form a hexagon (not necessarily convex) with an area of 6 cells.

Solution: The solution should include both the drawing and an explanation of how the area of the resulting figure is calculated. There are 14 grid points on the boundary and 0 inside, so $S = 7 - 1 = 6$.



□

There are also problems on olympiads where the «cells» are not square but triangular, as shown in the figure:

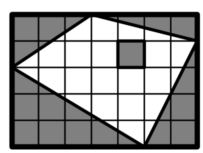


Problem Set

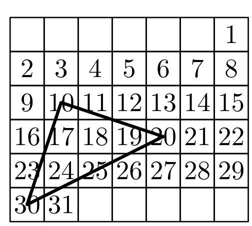
Problem 7.1. (MF — 1993.7.1): Can you drive nails into the centers of 16 cells of an 8×8 chessboard in such a way that no three nails lie on a straight line?

Problem 7.2. (MF — 2013.6.2): From each square with a side length of 3 cells, a figure is cut out consisting of five cells with the same perimeter as the square but with an area of 5 cells. Leo claims that he can cut out 7 such different figures (no two of them will coincide even if they are turned over). Is he right?

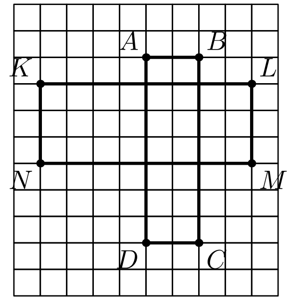
Problem 7.3. (AT — 2013.2): The estate of Marquis Carabas has the shape of a rectangle (shown in the figure). Part of the plot is occupied by a forest (highlighted in the dark), and the rest is pastures. What does the marquis have more of — forest or pasture? Explain the answer.



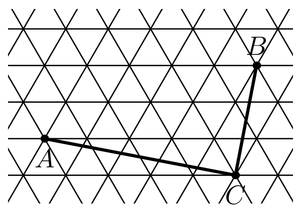
Problem 7.4. (MF — 2006.7.3): Alice made a calendar for January 2006 (shown in the figure below) and noticed that the centers of the cells on January 10, 20, and 30 form an isosceles right triangle. Alice assumed that this would be true in any other year, except for those years when the centers of the cells 10, 20, and 30 lie on a straight line. Is Alice correct?



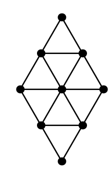
Problem 7.5. (COM — 2006.6.8): Rectangles $ABCD$ and $KLMN$ have parallel sides and are arranged as shown in the figure. Prove that the areas of quadrilaterals $ALCN$ and $KBMD$ are equal.



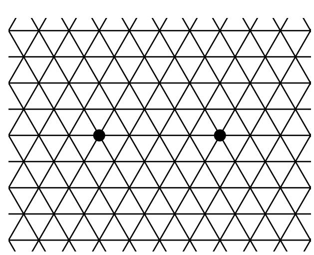
Problem 7.6. (COM — 2015.7.8): On a grid of equilateral triangles, an angle ACB is constructed. Find its magnitude.



Problem 7.7. (COM — 2012.6.2): A rhombus with a side length of two matches is folded from 16 matches and divided into triangles with a side length of one match. How many matches will be needed to fold a rhombus with a side length of 10 matches, divided into the same triangles with a side length of one match?

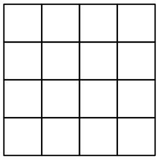


Problem 7.8. (COM — 2014.6.2): Leo and Max live in a city with a triangular road grid. In this city, they ride bicycles and are only allowed to turn left. Leo went to visit Max and, on the way, made exactly 4 left turns. The next day, Max went to Leo and arrived at him, making only one left turn. It turned out that the lengths of their routes were the same. Show how they could go (Leo's and Max's homes are marked on the figure).



Problem 7.9. (MF — 2006.7.6): Max painted one cell of a rectangle. Leo can paint other cells of this rectangle according to the following rule: you can paint any cell that has an odd number of painted neighbors (along the side). Can Leo paint all the cells of the rectangle (regardless of which cell Max chose) if the dimensions of the rectangle are: a) 8×9; b) 8×10 cells?

Problem 7.10. (COM — 2009.7.8): There are 40 identical shoelaces. If any shoelace is lit from one end, it burns, and if from the other, it does not burn. Max arranges the laces in the form of a square (in the figure, each lace is a side of a cell). Then Leo sets 12 matches. Can Max fold the laces so that Leo cannot burn all the laces?



Problem 7.11. (COM — 2010.7.8): A square with vertices at the nodes of the grid and sides of length 2009, going along the grid lines, was cut into several rectangles along the grid lines. Prove that among them, there is at least one rectangle whose perimeter is divisible by 4.

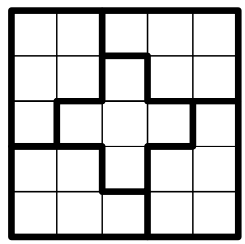
Skill Assessment Problems

Skill Assessment Problem 7.1. (Mos2ARSO — 2007.7.5): On graph paper, a square with side length 5 cells is drawn. It is required to divide it into 5 parts of equal area by drawing segments inside the square only along the grid lines. Is it possible that the total length of the drawn segments does not exceed 16 cells?

Skill Assessment Problem 7.2. Can you fill the cells of an infinite grid sheet with crosses and zeros in such a way that there are no three identical signs in a row on any horizontal, vertical, or diagonal line?

Solutions to Skill Assessment Problems

Solution to Problem 7.1: Yes, it is possible to divide the square in this way because the parts do not necessarily have to be the same; they just need to have the same area, i.e., occupy $25 : 5 = 5$ cells. One possible example is shown in the figure.



□

Solution to Problem 7.2: Yes, it is possible. On each horizontal line, the elements alternate in groups of 2; on the vertical lines, they alternate in groups of 1; and on the diagonals, they alternate in groups of 2.

X	X	O	O	X	X	O	O
O	O	X	X	O	O	X	X
X	X	O	O	X	X	O	O
O	O	X	X	O	O	X	X
X	X	O	O	X	X	O	O
O	O	X	X	O	O	X	X
X	X	O	O	X	X	O	O
O	O	X	X	O	O	X	X

□

Geometry of Squares and Rectangles

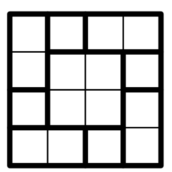
> The only way to learn mathematics is to do mathematics.
>
> —Paul R. Halmos, Hungarian-American mathematician

Theory and Practice

The geometry of squares and rectangles is a topic that can, in some sense, be considered a continuation of the topic «geometry on grid paper.» However, in this topic, the vertices of the shapes are not necessarily grid nodes.

Example 8.1. (MMG — 2012.7.2.2) Cut the 4×4 square along the grid lines into 9 rectangles so that equal rectangles do not touch either sides or vertices.

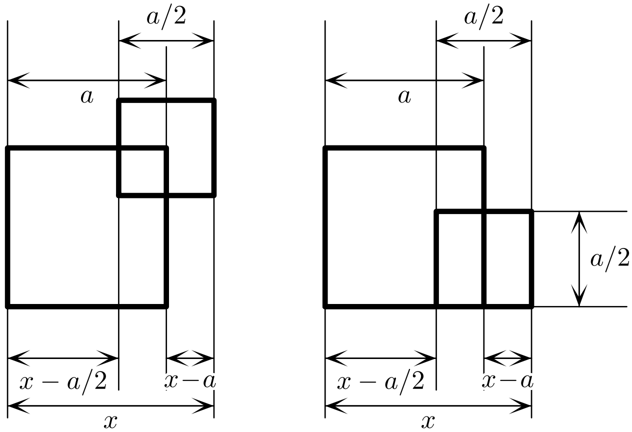
Solution: An example of such a cut is shown in the figure below.



□

Example 8.2. (COM — 2017.6.3): In a large square room, two square carpets were brought in, one with a side twice as long as the side of the other. When they were placed in opposite corners of the room, they covered an area of 4 m^2 in two layers, and when placed in adjacent corners, they covered an area of 14 m^2 in two layers. What are the dimensions of the room?

Solution: Let x be the length of the side of the room, and a be the length of the side of the larger carpet. Illustrate the problem setting with a diagram.



In the first case, the carpets cover a square with sides

$$x-(x-a)-\left(x-\frac{a}{2}\right)=\frac{3}{2}a-x.$$

In the second case, they cover a rectangle with sides

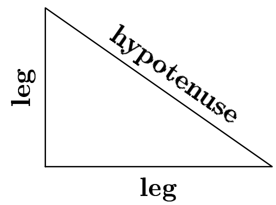
$$x-(x-a)-\left(x-\frac{a}{2}\right)=\frac{3}{2}a-x \text{ and } \frac{a}{2}.$$

Therefore, $\left(\frac{3}{2}a-x\right)\cdot\frac{a}{2}=14$ m^2. Since the side of a square with an area of 4 m^2 is 2 m, then $\frac{3}{2}a-x=2$ m. Substituting this into the equation above, we get: 2 m$\cdot\frac{a}{2}=14$ m^2, i.e., $a=14$ m, and $x=\frac{3}{2}a-2$ m $=19$ m. □

Example 8.3. (COM — 2005.6.4): The length of a rectangle was increased by 1 m, and the width was reduced by 1 mm. Could the area of the rectangle decrease?

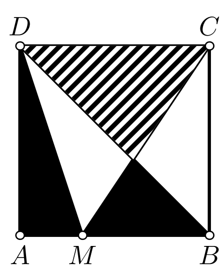
Solution: Let the initial length of the rectangle be 2 m, and the width be 2 mm, and their final values will be 3 m and 1 mm, respectively. Then the initial area was 4 m·mm, and the final one was 3 m·mm, i.e., the area decreased. □

Recall what a right-angled triangle is and what its sides are called. A triangle is called *right-angled* if one of its angles is a right angle, i.e., equals 90 degrees. The sides adjacent to the right angle are called *legs*, and the side opposite the right angle is the *hypotenuse*.



Let's assume that we know the lengths of the legs. How can we find the length of the hypotenuse? One of the most important formulas used in geometric problems involving right-angled triangles is the **Pythagorean theorem**: the sum of the squares of the legs is equal to the square of the hypotenuse. Triplets of integers satisfying this condition are usually called «Pythagorean triplets.» The most well-known Pythagorean triplet is $3, 4, 5$: $3^2 + 4^2 = 5^2$.

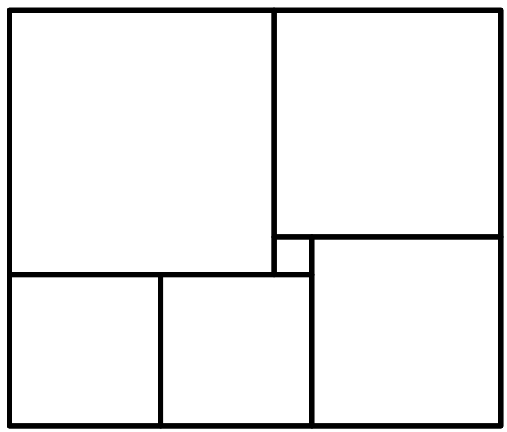
Example 8.4. (COM — 2005.6.9): On side AB of square $ABCD$, an arbitrary point M is marked. Prove that the area of the shaded triangle is equal to the sum of the areas of the black triangles.



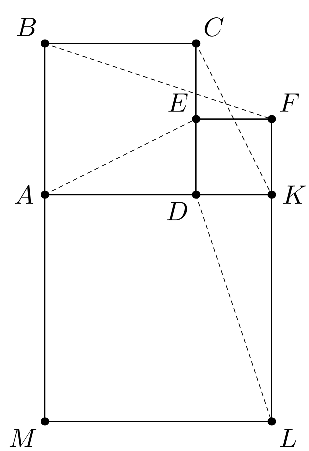
Solution: Let O be the point of intersection of DB and MC. Note that triangles ADB and DMC have equal bases and heights, so their areas are the same. Express their areas in terms of the areas of the triangles they contain: $A_{ADM} + A_{DMO} + A_{MOB} = A_{DMO} + A_{DOC}$, i.e., $A_{ADM} + A_{MOB} = A_{DOC}$, which is what needed to be proven. □

Problem Set

Problem 8.1. (MF — 1995.6.3): A rectangle is composed of six squares. Find the side length of the largest square if the side length of the smallest square is 1.



Problem 8.2. (MF — 1993.6.6): The square $ABCD$ with a side length of 2 and the square $DEFK$ with a side length of 1 are adjacent on the upper side AK of the square $AKLM$ with a side length of 3. Webs are stretched between pairs of points A and E, B and F, C and K, D and L. A spider climbs from bottom to top along the route $AEFB$ and descends along the route $CKDL$. Which route is shorter?

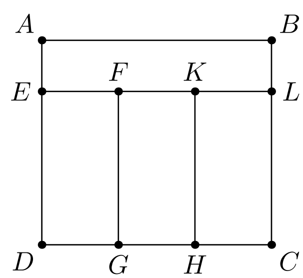


Problem 8.3. (MF — 2017.7.4): Given the square $ABCD$. On the extension of diag-

onal AC beyond point C, a point K is marked such that $BK = AC$. Find the angle $\angle BKC$.

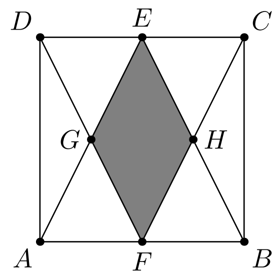
Problem 8.4. (COM — 2017.6.9): Vika has four figurines, Alina has a square, and Polina has a square of a different size. By joining forces, Alina and Vika can assemble a square using all five of their figurines. Is it possible for Polina and Vika to also assemble a square using all five of their figurines? (Squares are assembled without gaps or overlaps.)

Problem 8.5. (1ARSO — 2016.6.5): The rectangle $ABCD$ is divided into four smaller rectangles with the same perimeter. It is known that $AB = 18$ cm and $BC = 16$ cm. Find the lengths of the sides of the other rectangles.

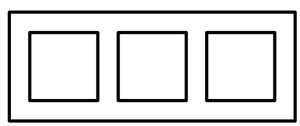


Problem 8.6. (PVG 2016.5–6.4; 7–8.3): A small garden measuring 6×7 meters was divided into 5 square beds. All the paths between the beds run parallel to the sides of the rectangle, and the side of each bed is an integer number of meters. Find the total length of the resulting paths. Consider the paths as lines without thickness.

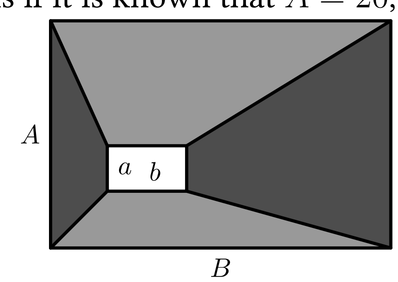
Problem 8.7. (PVG 2016.5–8.5): In the square $ABCD$, points F and E are the midpoints of sides AB and CD, respectively. Point E is connected to vertices A and B, and point F is connected to C and D, as shown in the figure. Determine the area of the rhombus $FGEH$ formed in the center if the side of the square $AB = 4$ is known.



Problem 8.8. (2ARSO — 2015.6.4): The frame for three square photos has a uniform width everywhere. The perimeter of one opening is 60 cm, and the perimeter of the entire frame is 180 cm. What is the width of the frame?



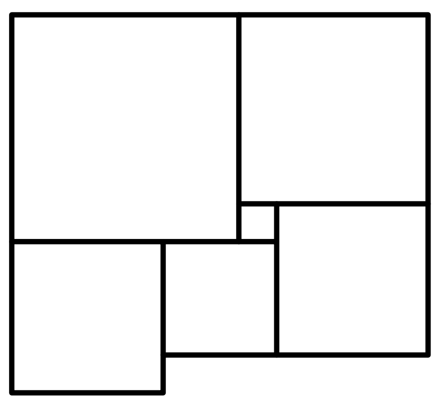
Problem 8.9. (PVG 2015.7.2): Inside a large rectangle of size $A \times B$, there is a small rectangle of size $a \times b$. Find the difference between the total area of the light-gray and dark-gray quadrilaterals if it is known that $A = 20$, $B = 30$, $a = 4$, $b = 7$.



Problem 8.10. (PVG 2014.7.3): Farmers Ivanov, Petrov, Sidorov, Vasiliev, and Ermolaev own rectangular plots of land, the area of which is indicated on the diagram. Find the area of the common pasture.

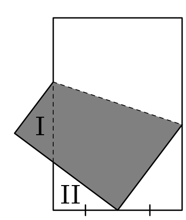
Ivanov 25 ha	forest	Ermolaev 30 ha
Petrov 28 ha	Common pasture	lake
wasteland	Sidorov 10 ha	Vasiliev 20 ha

Problem 8.11. (MF — 1995.7.3): The figure in the diagram is composed of squares. Find the side length of the bottom left square if the side of the smallest square is 1.



Problem 8.12. (COM — 2002.7.3): Is it possible to arrange four rectangles on the plane in such a way that no vertex is common to all rectangles, but any two rectangles share a common vertex? (Rectangles may intersect.)

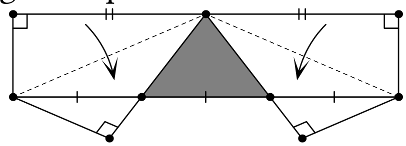
Problem 8.13. (MF — 2011.7.4): A rectangular sheet of paper is folded by matching one vertex with the midpoint of the opposite short side, as shown in the figure below. It turns out that triangles I and II are equal. Find the length of the long side of the rectangle if the short side is 8.



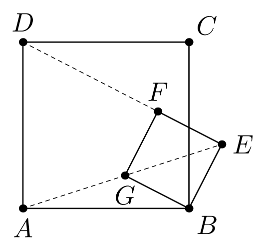
Problem 8.14. (MF — 1990.6–7.6): Inside the square $ABCD$, a square $KMXY$ is

located. Prove that the midpoints of segments AK, BM, CX, and DY are also vertices of a square.

Problem 8.15. (COM — 2014.7.7): Two corners of a rectangular sheet of paper are folded as shown in the figure. The opposite side is divided into three equal parts. Prove that the shaded triangle is equilateral.

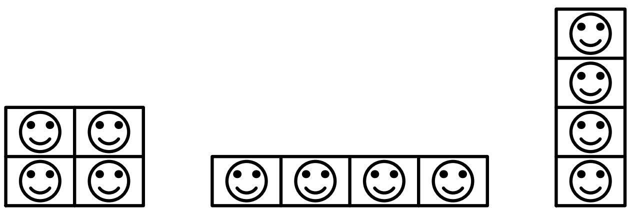


Problem 8.16. (COM — 2016.7.8): The squares $ABCD$ and $BEFG$ are arranged as shown in the figure. It turns out that points A, G, and E lie on the same line. Prove that the points D, F, and E also lie on the same line.



Skill Assessment Problems

Skill Assessment Problem 8.1. (2ARSO — 2014.6.3): From four photographs, three different rectangles can be formed, as shown in the diagram. The perimeter of one of them is 56 cm. Find the perimeters of the other two rectangles, given that the perimeter of a photograph is 20 cm.



Skill Assessment Problem 8.2. (Materials of mathematical circles, MCCMO) Given a rectangular piece of paper with dimensions 2×3. Having an infinite number of such rectangles at your disposal, show how to arrange them on the plane so that you can mark the vertices of a square with a side length equal to the diagonal of this rectangle without using drawing tools.

Solutions to Skill Assessment Problems

Solution to Problem 8.1: In the first figure, the obtained rectangle is 2 times higher and 2 times wider than an individual photograph, so its perimeter is exactly 2 times greater than the perimeter of one photograph, which is 40 cm.

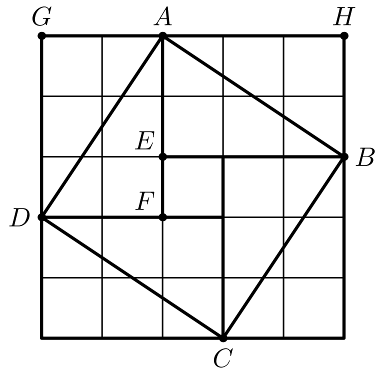
The rectangle shown in the second figure is 4 times wider than a photograph and has a height equal to the width of the photograph, so its perimeter is the sum of the perimeter of one photograph and its width taken 6 times.

The rectangle shown in the third figure is 4 times higher than a photograph and has a width equal to the height of the photograph, so its perimeter is the sum of the perimeter of one photograph and its height is taken 6 times.

Let the perimeter of the second rectangle be 56 cm. Since the perimeter of a photograph is 20 cm, then $56 - 20 = 36$ cm is its sixfold width. Therefore, in this case, the width of the photograph is 6 cm, and its height is $20 : 2 - 6 = 4$ cm. Consequently, the perimeter of the third rectangle is $20 + 6 \cdot 4 = 44$ cm.

If 56 cm is the perimeter of the third rectangle, then the height and width will swap places, so now the perimeter of the second rectangle will be 44 cm. □

Solution to Problem 8.2: Add 4 rectangles to form the shape shown in the figure. Sides AD and AB are equal as corresponding elements of equal rectangles $AFDG$ and $AHBE$, similarly $\angle DAF = \angle ABE$. Also, $\angle AEB = 90°$, $\Rightarrow$ $\angle EAB + \angle EBA = 90°$, $\Rightarrow$ $\angle EAB + \angle DAF = \angle DAB = 90°$. Similarly, it can be shown that all adjacent sides of quadrilateral $ABCD$ are equal and perpendicular, which means it is a square.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