Tatiana Babicheva is an inspiring teacher who believes that teaching is not just about imparting knowledge but also about being a lifelong learner. This passion for growth and discovery led her to earn two PhDs: one in applied mathematics and its applications in Russia, and another in computer science from Université Paris-Saclay in France. She has published 10 books in Russian and French in the field of popular science, including mathematics at the competition level.

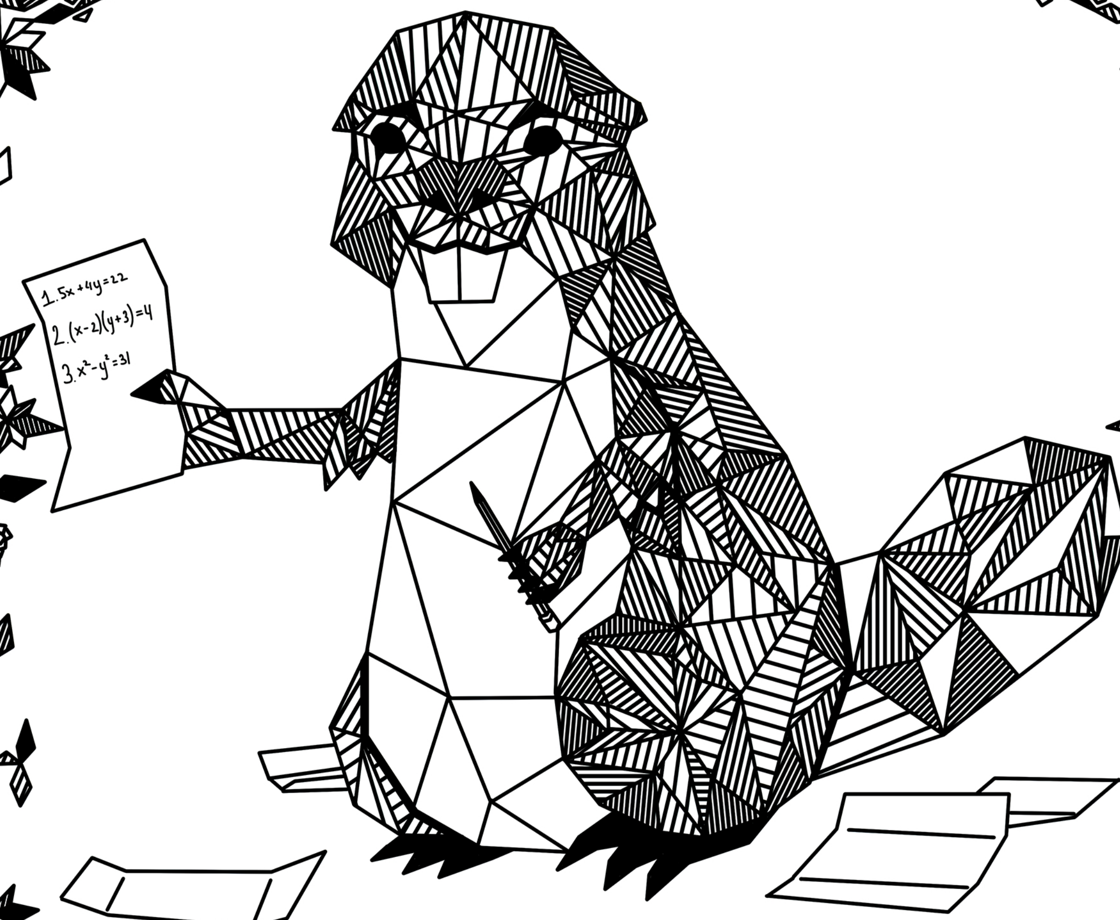
Egor Lazarevich is a nationally recognized musician and academic with a diverse range of music, mathematics olympiad, and entrepreneurship accomplishments. As a pianist, he has performed at Carnegie Hall, served as a guest artist at Harvard, and earned top awards in international competitions. Egor has inspired students through private math tutoring and leadership as founder of the Boston University Academy Stock Market Club. Additionally, Egor is a dedicated community volunteer.

Edvard Olsen, a senior at Sevenoaks School, combines his enthusiasm for mathematics with a keen interest in its real-world applications. His journey into mathematical problem-solving began at an early age, and he particularly enjoys tackling complex and challenging puzzles. Currently, Edvard is exploring the fields of machine learning and artificial intelligence, driven by a passion for mathematical research and its impact on practical problems. He aspires to continue delving into mathematics and its applications in areas such as science and financial systems.

COMPETITIVE NUMBER THEORY

Mathematical Competitions. Levels A1-A2

5.Competitive Number Theory



TATIANA BABICHEVA

Edvard Olsen;Egor Lazarevich

llustrators: Kilyanova Natalia; Yuliya Chayka

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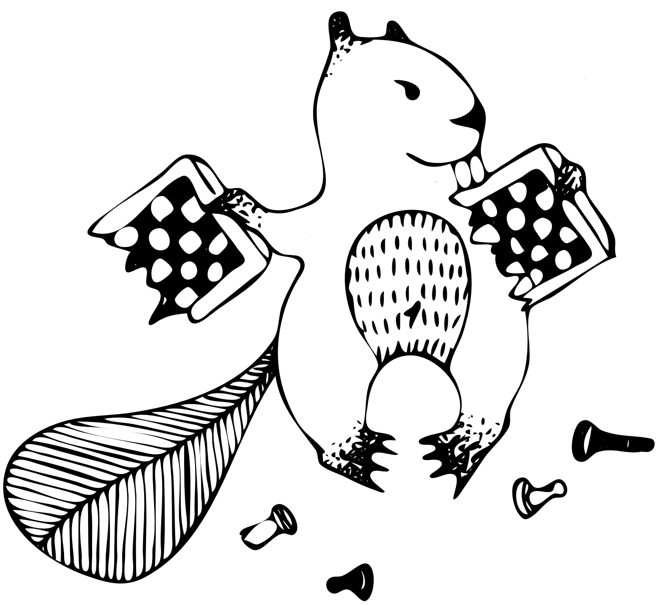
Dedication

“ To my parents.

Introduction



Introduction to the series

Begin your preparation for Competition Mathematics with our carefully crafted series. These books are designed to inspire a love for problem-solving and foster critical thinking. They are ideal for both budding mathematicians and passionate enthusiasts.

Inside, you will find a wide range of challenges, puzzles, and problems. Each one is selected to enhance your mathematical abilities. Experience the challenge of solving complex equations and gain confidence by deciphering complex geometric puzzles. Every book has engaging content to stimulate your mind and expand your skills.

If you're preparing for regional competitions, national tournaments, or simply want to deepen your mathematical knowledge, this series is an invaluable resource. The books provide clear explanations, strategic insights, and numerous practice problems. They aim to build your confidence and equip you with the skills needed to tackle any mathematical challenge.

While school mathematics forms a foundation, this series goes beyond it without requiring advanced knowledge to understand the material. Our course covers a wide range of topics, reflecting the diverse nature of Olympiad problems. Solving a geometry problem may require knowledge of combinatorics, while a number theory problem might involve understanding invariants and the pigeonhole principle.

Olympiad problems are generally not restricted to specific grade levels, making these books suitable for high school students. Some of the problems included have been featured in the final stages of national math Olympiads for higher grades. The goal is to demonstrate how to solve problems using straightforward and elegant methods, avoiding unnecessary complexity.

We have categorized competition mathematics into levels similar to the international standards used for foreign language proficiency. This approach is based on the concept of the «language» of competition mathematics. Traditional grade-based divisions are often outdated, as understanding a topic might only require elementary-level math. Moreover, the topics in these books are interconnected. Without a grasp of a topic at level A1, understanding its expanded form at level A2 can be challenging.

Here's what to expect at each level:

Let's use an analogy with foreign languages:

Level A1. You understand (generally) foreign speech and can talk about family, activities, hobbies, travels, weather, and buying things. In short, the standard tourist set. Can you conjugate basic verbs and be familiar with different tenses? The question «How are you?» doesn't stump you? Congratulations, you have a good A1 level! This is enough for survival.

Similarly, in olympiad math — you can «survive» at beginner-level olympiads, understand what is required in problems, and formulate solutions. You likely won't need math knowledge beyond seventh grade to understand topics at this level. (The problem might be from an 11th-grade olympiad, but the solving method remains the same)

At level A2, you can discuss preferences in art, cultural differences, main social trends, etc. You form complex sentences («This is Peter, whose dad works at the bank. I've already told you about him»), can write to a friend on Facebook, describe a vacation, and understand the essence of any conversation in the language.

You can recognize and solve middle-level Olympiad problems. You will be able to avoid common mistakes and present your solutions effectively. Topics at this level typically require knowledge up to the eighth grade.

This series of books generally covers levels A1 and A2 of competition maths: you will understand any problem from most competitions, formulate your solution, and even change the solution of ChatGPT to match the real competition problem. However, you are still far from being a native speaker.

What is in these books?

This series uses a proof-based approach to problem-solving, which is usually reserved for advanced levels in countries like the USA and the UK. However, this method helps build a solid foundation in mathematics.

Each chapter is divided into four parts:

1. The first part covers the theoretical background and provides detailed solutions to typical problems.

2. The second part presents a problem set labeled by source. Olympiad problems are marked with notations like «Year.Grade/Round.Number.» For example, «ACM 2016.10A.5» is the fifth problem from the 10th-grade 10A variant of the ACM Olympiad 2016. Grade numbering may vary between countries, so adjust accordingly. Non-grade-specific Olympiads, like AIME, are marked by version (I or II) instead of grade.

You will encounter many problems from the Russian Olympiads (a country with a strong tradition in Olympiad mathematics) and various US mathematical competitions (such as AMC and AIME). We sincerely recommend not only finding the correct answer from the given AMC options but also approaching these problems from a proof-based perspective.

The problem number usually provides a sense of difficulty; generally, a higher number indicates a more challenging problem. However, this labeling doesn't always apply to some «independent» Olympiads, which can sometimes confuse genuine Olympiad participants.

3. The third part includes problems for independent solving, with some original problems introduced here.

4. Solutions are found in the fourth part.

The series consists of the following books.

1. Competitive Arithmetics
2. Ideas and Methods
3. Introduction to Discrete Mathematics
4. Introduction to Competitive Geometry
5. Competitive Number Theory
6. Competitive Geometry

This series is designed for both experienced Olympiad participants and newcomers to mathematical problem-solving. It offers a journey where theory and application meet, providing a rewarding experience. Welcome to a unique math adventure!

Introduction to this book

This book is dedicated to the topic of number theory, which encompasses subjects such as divisibility, the fundamental properties of integers, and the fundamental theorem of arithmetic. You will also gain a better understanding of what an irreducible fraction is, how many prime numbers exist, and why we use Arabic numerals.

Number theory is perhaps one of the most loved and most dreaded topics in mathematical competitions. In this book, we will prove many of the properties of numbers that you have likely studied before, as well as expand our arsenal with new topics, methods, and types of problems.

Do not be intimidated by the intertwined structure of this book; it is designed this way on purpose. By examining different approaches to various theorems from completely different perspectives, you will learn how to avoid circular reasoning in proofs and how to bypass these logical loops.

In our view, the first 8 chapters are accessible to typical (but math-enthusiastic) 8th-grade students, while the remaining chapters are slightly more challenging.

List of competitions used in this book

- «Математический праздник», in English mean «Mathematical festival». We note it in the book as «MF». The official site (in Russian) is https://olympiads.mccme.ru/matprazdnik/
- Городская устная математическая олимпиада для 6—7 классов, mean «City Oral Mathematical Olympiad for 6—7 grades». We note it in the book as «COM». The official site (in Russian) is https://olympiads.mccme.ru/ustn/
- Турнир городов, mean «Tournament of Towns». We note it in the book as «TOT». The official site is https://www.turgor.ru/en/
- Школьный этап Всероссийской олимпиады школьников, mean «first stage of All-Russian School Olympiad». We note it in the book as «1ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- Муниципальный этап Всероссийской олимпиады школьников, mean «second stage of All-Russian School Olympiad». We note it in the book as «2ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- Муниципальный этап Всероссийской олимпиады школьников (Москва), mean «second stage of All-Russian School Olympiad in Moscow». We note it in the book as «Mos2ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- Региональный этап Всероссийской олимпиады школьников, mean «third stage of All-Russian School Olympiad». We note it in the book as «3ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- Всероссийская олимпиада школьников, mean «All-Russian School Olympiad». We note it in the book as «ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- American Mathematics Competitions. We note it in the book as «AMC». The official site is https://maa.org/math-competitions
- American Invitational Mathematics Examination. We note it in the book as «AIME». The official site is https://www.maa.org/math-competitions
- Mock American Invitational Mathematics Examination. We note it in the book as «Mock AIME».
- American High School Mathematics Examination. We note it in the book as «AHSME». The official site is https://www.maa.org/math-competitions/amc

- American Junior High School Mathematics Examination. We note it in the book as «AJHSME». The official site is https://www.maa.org/math-competitions/amc
- Московская математическая олимпиада, mean «Moscow Mathematical Olympiad». We note it in the book as «MMO». The official site (in Russian) is https://mmo.mccme.ru/
- Олимпиада им. Леонарда Эйлера, mean «Leonhard Euler Math Olympiad». We note it in the book as «LEO». The official site (in Russian) is http://matol.ru/
- Объединённая межвузовская математическая олимпиада школьников, mean United Interuniversity Mathematical Olympiad for schoolchildren. We note it in the book as «OMMO». The official site (in Russian) is https://olympiads.mccme.ru/ommo/
- Cyprus Mathematical Olympiad. We note it in the book as «Cyprus MO». The official site is https://www.cms.org.cy/en/activities/cyprus-mathematical-olympiad
- University of Northern Colorado Math Contest. We note it in the book as «UNCO Math Contest». The official site is https://uncmathcontest.wordpress.com/
- iTest. We note it in the book as «iTest». The non official site is https://artofproblemsolving.com/wiki/index.php/ITest
- Junior Mathematical Olympiad. We note it in the book as «JMO». The official site is https://ukmt.org.uk/junior-challenges/junior-mathematical-olympiad
- Primary Mathematics World Contest. We note it in the book as «PMWC». The non official site is https://www.txst.edu/mathworks/PMWC/previous-pmwc-tests.html
- UNM-PNM Statewide High School Mathematics Contest. We note it in the book as «UNM-PNM». The official site is http://mathcontest.unm.edu/
- Indonesian Mathematics Olympiad. We note it in the book as «Indonesian MO». The official site is http://tomi.or.id/
- Турнир им. Ломоносова, mean «Lomonosov Tournament». We note it in the book as «LT». The official site (in Russian) https://turlom.olimpiada.ru/
- Турнир Архимеда, mean «Archimedes Tournament». We note it in the book as «AT». The official site (in Russian) is http://www.arhimedes.org/
- Олимпиада «Ломоносов», mean «Lomonosov Competition», competition of

Moscow State University. We note it in the book as «Lomonosov». The official site (in Russian) is https://olymp.msu.ru/

- Курчатов, mean «Kurchatov Competition». We note it in the book as «Kurchatov». The official site (in Russian) is https://olimpiadakurchatov.ru/
- Московская математическая регата, mean «Moscow mathematical regatta». We note it in the book as «MMG». The official site (in Russian) is https://olympiads.mccme.ru/regata/
- «Покори Воробьёвы горы», mean «Conquer Vorobyovy Gory», competition of MSU. We note it in the book as «PVG». The official site (in Russian) is https://pvg.mk.ru/
- The book of Altufova and Ustinov «Algebra and Number Theory. Collection of Problems for Mathematical Schools» in Russian. We note it in the book as «AU».
- Олимпиада «Физтех», means «MIPT mathematical competition». We note it in the book as «MIPT». The official site (in Russian) is https://olymp.mipt.ru/
- University of South Carolina High School Math Contests. We note it in the book as «University of South Carolina High School Math Contests». The official site is https://sc.edu/
- Журнал Квант, mean «Kvant Journal». We note it in the book as «Kvant». The official site (in Russian) is https://kvant.mccme.ru/
- Pan African Mathematics Olympiad. We note it in the book as «Pan African MO». The official site is https://www.africamathunion.org/AMU-pamo-official.php
- Colorado MATHCOUNTS. We note it in the book as «CMC». The official site is https://mathcounts.coloradomath.org/
- International Mathematical Olympiad. We note it in the book as «IMO». The official site is https://www.imo-official.org/
- Canadian Mathematical Olympiad. We note it in the book as «Canadian MO». The official site is https://cms.math.ca/Competitions/CMO/
- Alabama American Regions Mathematics League Team Selection Test. We note it in the book as «Alabama ARML TST». The official site is https://arml3.com/
- Mock American Regions Mathematics League. We note it in the book as «Mock ARML». The official site is https://arml3.com/

Contents

Parity: The Concept

> Zero is considered an even number because it can be divided by 2 without leaving a remainder.

> Zero is the «chameleon» of the number world — it fits in with the even crowd but still stands out as unique!

Theory and Practice

When dealing with numbers, we begin to notice the properties of even and odd numbers in operations between them. Based on the parity (even or odd) of the numbers involved in an operation, we can determine the parity of the resulting number without knowing the actual values.

In addition, the table looks like this:

$$\begin{aligned} \mathbf{E} + \mathbf{E} &= \mathbf{E}, \\ \mathbf{E} + \mathbf{O} &= \mathbf{O}, \\ \mathbf{O} + \mathbf{E} &= \mathbf{O}, \\ \mathbf{O} + \mathbf{O} &= \mathbf{E}. \end{aligned}$$

Interestingly, the same table applies for subtraction:

$$\begin{aligned} \mathbf{E} - \mathbf{E} &= \mathbf{E}, \\ \mathbf{E} - \mathbf{O} &= \mathbf{O}, \\ \mathbf{O} - \mathbf{E} &= \mathbf{O}, \\ \mathbf{O} - \mathbf{O} &= \mathbf{E}. \end{aligned}$$

Adding one to a number changes its parity, while adding two does not.

In multiplication, it is enough for at least one of the numbers to be even for the result to be even:

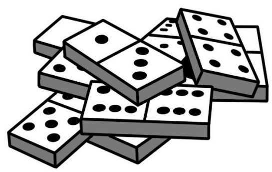
$$\begin{aligned} \mathbf{E} \cdot \mathbf{E} &= \mathbf{E}, \\ \mathbf{E} \cdot \mathbf{O} &= \mathbf{E}, \\ \mathbf{O} \cdot \mathbf{E} &= \mathbf{E}, \\ \mathbf{O} \cdot \mathbf{O} &= \mathbf{O}. \end{aligned}$$

Example 1.1. Let a and b be integers. Prove that the number $ab(a+b)$ is even.

Proof. When either a or b is even, according to the previously provided table, the product $ab(a+b)$ is also even. If both a and b are odd, then their sum $a+b$ is even, and therefore, the entire product is also even. $\square$

Example 1.2. Is it possible to exchange 25 «antidollars» into 10 banknotes with denominations of 1, 3, and 5 antidollars?

Solution: Let's refer back to the table of addition of even and odd numbers we created earlier. All banknotes have odd denominations. An even number of odd values in the sum will result in an even total; since 25 is odd, the answer is «impossible». □



Example 1.3. All dominoes with a «blank» (zero) have been lost from a domino set. Can the remaining dominoes be arranged in a row?

Solution: Tasks related to dominoes regularly appear in Olympiad problems, and it is beneficial to understand the fundamentals of this game. A domino set consists of 28 distinct tiles, each containing a pair of numbers from zero to six. There are seven tiles with identical numbers, known as «doubles». The game involves laying down tiles so that squares with matching numbers touch. In a complete domino set, each digit appears exactly eight times. Each tile placement either adds a double, which is already a pair or creates a new pair. If the entire set is laid out, each digit must be part of some pair in the sequence (unless we have closed a loop, in which case the ends should have identical numbers).

If we have lost all the tiles containing «blank», namely $(0,0)$, $(0,1)$, $(0,2)$, $(0,3)$, $(0,4)$, $(0,5)$, $(0,6)$, then each of the remaining six digits in the remaining set appears seven times. Each digit must form a pair if it is part of the sequence. Different digits may be at the ends of the sequence, but the remaining four cannot form pairs. □

Example 1.4. One day, the leadership of the KGB decided to provoke a group of Soviet hippies into participating in a protest against the Vietnam War near the US

Embassy in Moscow. A group of 100 hippies was selected for this purpose, with plans to persuade them to join the protest. However, before that, they needed to get acquainted with each other. On the one hand, it was dangerous to gather too many strangers at once; on the other hand, pairing off individuals would take too much time. The KGB agent assigned to the task decided to proceed as follows: introduce people in groups of three, ensuring that people who were already acquainted could not meet again at the second meeting. Will he be able to acquaint all the hippies with each other using this method?

This type of problem is widely disliked because it is formulated in many sentences when it could be written much shorter. This book contains many such tasks, teaching you to filter through only what is most important in a large block of text.

Solution: Suppose the agent was able to meet the required conditions. Consider any arbitrary hippie. During each meeting, they get acquainted with exactly 2 other hippies. However, they need to get acquainted with 99 hippies in total, and since 99 is an odd number, it is impossible to achieve this under the agent's conditions. □

In addition to the table presented above, it is useful to consider the following properties related to the parity of numbers:

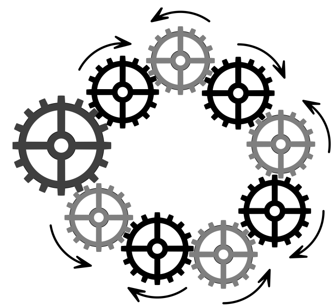
1. The difference between two numbers has the same parity as their sum: adding $(a-b)$ and $(a+b)$ gives $2a$ — an even number. As we know, both addends must be of the same parity to yield an even sum.
2. The algebraic sum (with $+$ or $-$ signs) of integers has the same parity as their sum (for example, $2-7+(-4)-(-3)=-6$ and $2+7+4+3=16$ are both even).

The principle of parity can manifest not only in numbers.

Example 1.5. On a flat surface, 9 gears are arranged and connected in a closed chain (the first with the second, the second with the third, and so on, with the ninth connecting back to the first). Can they all rotate simultaneously?

Solution: Assuming all gears can rotate simultaneously in a closed chain, the gears will have two directions to spin in: clockwise and counterclockwise. Since gears rotating in the same direction cannot be adjacent, the rotations of adjacent gears must alternate.

If there was an even number of gears, then the gears could spin all at the same time, as adjacent gears could spin in different directions, but if one more gear is added to the chain, then there will be an odd number of gears.



Wherever the new gear is placed, the two gears adjacent to the new gear will be spinning in different directions. As we have established, adjacent gears must spin in different directions, and gears can only spin clockwise or counterclockwise; hence, no matter which direction the new gear will spin in, it will spin in the same direction as an adjacent gear with an odd number of gears in a closed chain; all of the gears cannot spin simultaneously. In our case, there are 9 gears, which is an odd number of gears; hence, all of the gears cannot spin simultaneously. □

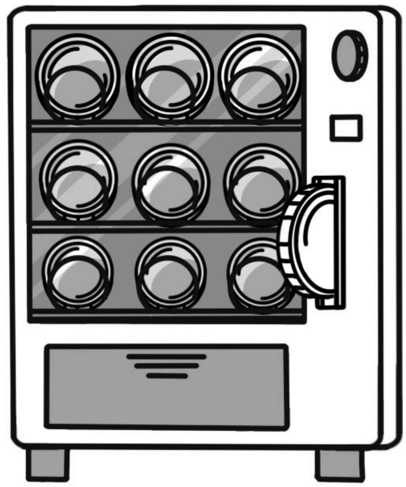
Example 1.6. On a square board of size $(2n+1)\times(2n+1)$, there are 2017 checkers, the layout of the checkers on the board is symmetrical across the main diagonals (if the checker lies on a diagonal, it is also reflected in itself). Prove that one of the checkers must be located in the central cell.

Proof. Let's prove that all checkers, except one, can be paired:

— If a checker does not lie on the main diagonal, there is a symmetrical checker with respect to the main diagonal.
— If a checker lies on the main diagonal but is not in the center of the square, there is a symmetrical checker with respect to the other diagonal.

Since there are a total of 2017 checkers, which is an odd number, one must be in the central cell. □

Problem Set



Problem 1.1. (TOT — 1986/1987.7-8.1): A broken vending machine dispenses five yellow coins when a green coin is inserted and five green coins when a yellow coin is inserted. Can Max, approaching the machine with one green coin, get an equal number of green coins and yellow coins after several insertions?

Problem 1.2. (TOT — 1992/1993.10-11.4): There are three piles of stones. It is allowed to add to any of them the sum of the number of stones in the other two piles or to remove this amount. For example: $(12, 3, 5) \to (12, 20, 5)$ (or $(4, 3, 5)$). Is it possible, starting with piles of 1993, 199, and 19 stones, to make one of the piles empty?

Problem 1.3. (TOT — 2010/2011.8-9.1): All integers from 1 to 2010 are written around a circle in such a way that when moving clockwise, the numbers alternatingly increase and decrease; thus, we never have 3 consecutive numbers that form an increasing or decreasing sequence. Prove that the difference between some of the two neighboring numbers is even.

Problem 1.4. (MF — 1990.5.1): In the parliament of a certain country, there are two chambers with an equal number of deputies. During the vote on an important issue, all deputies participated, and there were no abstentions. When the chairman announced that the decision was made with a margin of 23 votes, the leader of the opposition claimed that the voting results were falsified. How did he come to this conclusion?

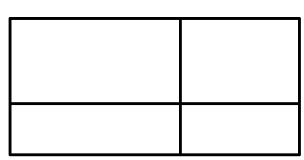
Problem 1.5. (AT — 2013.5.5): Alice, Beatrice, and Clarice once went from the post office to their home. Alice went first, and Beatrice went last. On the way home, Alice either overtook the others or was overtaken exactly 8 times. Beatrice either overtook the others or was overtaken exactly 6 times. It is known that Alice arrived home later than Clarice. In what order did the friends arrive home?

Problem 1.6. (MF — 2005.6.3): A fox and two bear cubs divide 100 chocolates. The fox distributes the chocolates into three piles, and the allocation is determined by a lottery. The fox knows that if the bear cubs receive different numbers of chocolates, they will ask her to equalize the piles, and then she will take the excess chocolates for herself. After that, everyone eats the chocolates they received.

a) Come up with a way for the fox to distribute the chocolates into piles so that she eats exactly 80 chocolates (no more, no less).

b) Can the fox arrange the piles so that she eats exactly 65 chocolates in the end?

Problem 1.7. (COM — 2010.6.3): In the figure, there are nine rectangles (some rectangles within other rectangles). It is known that each of the rectangles has sides of integer lengths. How many of these rectangles can have odd areas?



Problem 1.8. (MF — 2013.6.4): Thirteen children sat around a round table; the boys agreed to lie to the girls and tell the truth to each other, and the girls agreed to lie to the boys and tell the truth to each other. One of the children said to their right

neighbor, «The majority of us are boys». The neighbor said to their right neighbor, «The majority of us are girls», and so on until the last child said to the first one, «The majority of us are boys». How many boys were at the table?

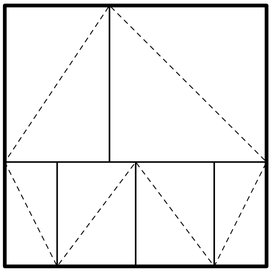
Problem 1.9. (MF — 1991.6.4;7.2): Underground millionaire Mrs. Owless came to the State Bank to exchange several old-style 50 and 100 ruble banknotes. She was given 1991 number of banknotes of smaller denominations[1], and among them, there were no 10 ruble banknotes. Prove that he was shortchanged.

Problem 1.10. (MF — 2002.7.3): Leo changed two digits in a multiplication equation written on the board. It turned out to be $4 \cdot 5 \cdot 4 \cdot 5 \cdot 4 = 2247$. Find the original equation and explain your reasoning.

Problem 1.11. (MF — 2000.7.4): Can the product of two consecutive natural numbers be equal to the product of two consecutive even numbers?

Problem 1.12. (COM — 2009.7.4): In a school, there are 450 students and 225 two-person desks. Exactly half of the girls sit at desks with boys. Is it possible to rearrange the students so that exactly half of the boys sit at a desk with the girls?

Problem 1.13. (MF — 2013.6.5): The Small and Big Islands have a rectangular shape and are divided into rectangular regions. In each region, a road is laid along one of the diagonals. On each island, these roads form a closed path that does not pass through any point twice and has no dead ends.



The figure shows the structure of the Small Island, which has only 6 regions (on the right side of the picture).

Draw at least one way how the Big Island could be arranged if it has an odd number of regions. How many regions did you get?

[1]At that time, banknotes of 1, 3, 5, 10, 25, 50, and 100 rubles (new design) were in circulation.

Problem 1.14. (MF — 2010.6.6): On the edge of a moving round table over equal intervals of distance stand 30 cups of tea. March Hare and Sonia sit at the table and start to drink tea from two different cups of tea (not necessarily two adjacent cups). When they both finished their cup of tea, the Hare turned the table in such a way that in front of both Sonia and the Hare were two new full cups of tea. When these cups were empty, the Hare turned the table again (not necessarily at the same angle as before) so that there were two new full cups of tea in front of both of them. This continued until all of the tea was drunk.

Prove that if the Hare turned the table at the same angle every time after the tea was finished (in other words his cup was 2 cups away from his previous cup), then they would always manage to drink all of the tea (making sure that with each turn his new cup and Sonia's were always full).

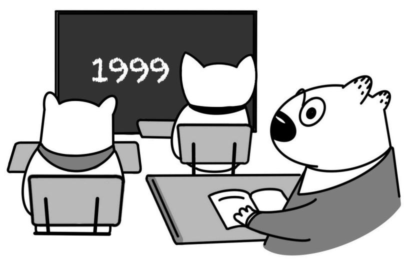
Problem 1.15. (COM — 2006.7.7): A game is played. A row of cells is drawn on the ground. Each cell contains numbers from 1 to 10 in order, as shown below. Esther jumped in from the outside into cell 1, then jumped in and out of the other cells (each jump can only be made to an adjacent cell). Esther jumped out of cell 10. It is known that Esther was in cell 1 once, in cell 2 twice, ..., and in cell 9 nine times. How many times did Esther visit cell 10?

1	4	5	8	9
2	3	6	7	10

Skill Assessment Problems

Skill Assessment Problem 1.1. There are 18 integers whose product is equal to 1. Prove that their sum is not equal to zero.

Skill Assessment Problem 1.2. On the board, 2017 integers from 1 to 2017 are written. Jean erases two numbers from the board and writes down the absolute value of their difference. Can the last remaining number on the board be equal to zero?



Skill Assessment Problem 1.3. While some children were taking a test, the teacher wrote all the natural numbers from 1 to 2018 in a circle on the class wall. Each number appeared only once, so that every number is a divisor of the sum of its two adjacent numbers. Jean saw the number 1999 written on the wall; a classmate's head blocked the view of one of the adjacent numbers to 1999. Jean bet another classmate that the number that Jean couldn't see was odd. What is the chance that Jean will win this bet?

Solutions to Skill Assessment Problems

Solution to Problem 1.1: For the product of integers to be equal to one, only the numbers 1 and -1 can be used as factors, and the number of -1 must be even. For the sum of a group of numbers 1 and -1 to be zero, it is necessary for the number of 1 to be 9 and the number of -1 also to be 9, which is an odd number. Thus, we have proven that the sum of these numbers is not equal to zero. $\square$

Solution to Problem 1.2: Calculate the sum of all the numbers written on the board:

$$1 + 2 + \ldots + 2017.$$

Rewrite the numbers in reverse order:

$$\underbrace{1 + 2017} + \underbrace{2 + 2016} + \ldots + 1009.$$

The sum of each pair of numbers is 2018 — an even number. The number 1009 is odd; so, the overall sum is also odd.

By considering tables of addition and subtraction of even and odd numbers, we observe that they match. Consequently, if we erase two numbers and write the absolute value of their difference, the parity of the sum of all the numbers written on the board does not change. Since 0 is an even number, it cannot be the last remaining number after a series of steps. $\square$

Solution to Problem 1.3: Suppose there are 2 consecutive even numbers a and b. Then the number c written to the left of them (such that the order of the numbers is $c\ a\ b$) is also even, as otherwise the number $c + b$ would be odd and could not be divisible by the even number a. Similarly, the number written to their right would also be even according to this principle. Continuing this chain of proofs (a number adjacent to two consecutive even numbers is even itself), we conclude that all written numbers must be even. However, this is impossible, as we also have odd numbers (given in question). There is an equal number of odd and even numbers, and adjacent even numbers do not appear; therefore, the numbers must alternate in parity (even, odd, even, odd, even, ...). Coming back to the question, 1999 is odd; as we know, the numbers alternate in parity, and the numbers on either side of 1999 are even. Jean has 0 chance to win his bet. $\square$

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Basic Properties of Divisibility

> Prime Numbers: A prime number is a natural number greater than 1 that is not a product of two smaller natural numbers. For example, 2, 3, 5, 7, 11, and 13 are prime numbers.

Mathematicians often refer to prime numbers as the «atoms» of arithmetic, meaning they are the building blocks of all numbers. Just like atoms, primes are basic and cannot be broken down into smaller components.

Theory and Practice

Divisibility is one of the fundamental concepts in number theory. Usually, we say that the number a is divisible by b if the remainder of dividing a by b is zero. But what is a remainder? We will discuss them in the next chapter, but for now, let's give a more rigorous definition of the divisibility of numbers.

Definition 1. An integer a is divisible by an integer b, not equal to zero if an integer c exists, such as $a = b \cdot c$.

Let's consider the *basic properties of divisibility.*

Claim 1. If an integer a is divisible by b, and b is divisible by c, then a is divisible by c.

Proof. Since the number a is a multiple of b, a can be represented as the product of integers, where one of the factors is equal to b: $a = b \cdot k$. Similarly, we can represent b as a product: $b = c \cdot q$. Then, substituting b into the first expression, we get $a = c \cdot kq$. We observe that one of the factors of the number a is c, which means $a \vdots c$ by definition. $\square$

Claim 2. If k is a common divisor of numbers a and b, then:
a) the numbers $a + b$ and $a - b$ are divisible by k;
b) the number ab is divisible by k^2.

Proof. As in the proof of the previous property, let's represent the numbers a and b as the respective products: $a = c \cdot k,\ b = c \cdot q$, where k and q are integers.

a) Now perform the operations: $a + b = c \cdot (k + q),\ a - b = c \cdot (k - q)$. One of the factors in each expression is c, which means the sum and the difference are multiples of c by the definition of divisibility.

b) Similarly to the previous part, substitute a and b and factor out the number c: $a \cdot b = c^2 \cdot (kq)$. By definition, $a \cdot b \,\vdots\, c^2$. □

Corollary 1. If one of the numbers a and b is divisible by k, and the other is not divisible by k, then the numbers $a + b$ and ab are not divisible by k.

Proof. As in the proofs of previous cases, let's express a as a product: $a = c \cdot k$. According to the definition of the remainder of the division, we can represent the number as $b = c \cdot q + m$, where $0 < m < c$. Then, by adding or subtracting the given numbers, we obtain $a + b = c \cdot (k \pm q) \pm m$. This implies that the sum and the difference are not multiples of c by the definition. □

Definition 2. A natural number p is called prime if it is not equal to one and is divisible only by natural numbers 1 and p.

Definition 3. Natural numbers p and q are called mutually prime if the only common natural divisor of these numbers is 1.

Claim 3. If the number $s = ab$ (a, b are integers) is divisible by a prime number p, then at least one of the numbers a and b is divisible by p.

In the context of mathematical logic, the statements A and B are said to be «equivalent» or «if and only if (iff)» denoted as $A \Leftrightarrow B$, if both $A \Rightarrow B$ and $B \Rightarrow A$ hold. This means that A is true whenever B is true, and vice versa.

These expressions are commonly used in mathematical reasoning to describe the logical relationships between different statements.

Let's recall the *divisibility rules*, which most of you learned in the 6th grade (and then promptly forgot). Remember that zero is a number that is divisible by anything!

A number divisible by 2 is called even, and its last digit should also be even, i.e., equal to 0, 2, 4, 6, or 8 (0 is even because it is divisible by anything!). Conversely, if the last digit is even, the number is even.

The divisibility rules for 3 and 9 are very similar: the sum of the number's digits is divisible by 3 (or 9) if and only if the number itself is divisible by 3 (or 9).

For a number to be divisible by 5, its last digit must be divisible by 5, i.e., equal to 0 or 5.

It is pretty evident that a number is divisible by 10 if and only if its last digit is 0.

The divisibility rule for 4 is a bit more complicated. It requires considering the number formed by its last two digits. The fact that this two-digit number is divisible by 4 is equivalent to the entire number being divisible by 4.

Finally, the divisibility rule for 11: for a number to be divisible by 11, it is necessary and sufficient for the alternating sum of its digits to be divisible by 11 (when summing, the first digit is taken with a plus sign, the next with a minus sign, and so on, until the last digit).

Let's go through an example.

Example 2.1. Is 435678232 divisible by 11?

Solution: Let's find the alternating sum of the digits of the given number:

$$4 - 3 + 5 - 6 + 7 - 8 + 2 - 3 + 2 = 0.$$

The sum is 0, which is divisible by 11. Therefore, the original number is divisible by 11. □

Example 2.2. Find and prove the divisibility rule for 9.

Proof. Let the number have the decimal representation $N = \overline{a_n a_{n-1} \ldots a_1 a_0}$. Then $N = a_0 + 10^1 a_1 + 10^2 a_2 + \ldots + 10^{n-1} a_{n-1} + 10^n a_n$. Note that the numbers $1, 10, 100, \ldots, 10^n$ all have a remainder of 1 when divided by 9 (since 0,9,$99, \ldots, 999 \ldots 99$ are divisible by 9). Thus, if the sum of the digits $a_0 + a_1 + \ldots + a_n$ is divisible by 9, then $N = a_0 + 10^1 a_1 + 10^2 a_2 + \ldots + 10^{n-1} a_{n-1} + 10^n a_n = a_0 + a_1 + \ldots + a_n + 9a_1 + 99a_2 + \ldots + 9 \ldots 9a_n$ is divisible by 9. □

There are no simple rules for divisibility by 7 and 13. Still, the checking process can be significantly simplified for any number greater than a thousand, reducing it to checking the divisibility of the sum or difference of numbers not exceeding a thousand. Divide the given number into groups of three digits, starting from the unit digit. For example, 211626363 is divided into 363, 626, and 211. The leftmost group may contain fewer than three digits. If the alternating sum of these groups is divisible by 7, 11, or 13, then the original number is divisible by 7, 11, or 13. In our case, we calculate the sum $211 + (-626) + 363 = -52$. It is not divisible by 7 or 11, but it is divisible by 13, which means the original number 211626363 is also divisible by 13. The proof is based on the fact that $1001 = 7 \cdot 11 \cdot 13$.

Considering that $999 = 27 \cdot 37$, a similar divisibility rule for 27 and 37 can be derived. To check divisibility, add all the triples. If the result is divisible by 27 or 37, then the original number has the same properties.

After reviewing the divisibility rules, it is crucial to pay attention to some considerations related to the practical factorization of a number into prime factors. Usually, it is most convenient to try all prime factors, starting from the smallest ones: 2, 3, 5, 7, 11, 13, 17, 23, 29, etc., except in cases where some factors can be determined immediately based on the appearance of the number. For example: $970097 = 97 \cdot 10001$ or $100020001 = 10000^2 + 2 \cdot 10000 \cdot 1 + 1^2 = (10000 + 1)^2$.

It is also essential to remember the following fact: if a number n is not divisible by any prime number less than or equal to some number p, and $p^2 \geqslant n$, then n is a prime number. Indeed, suppose n has at least one divisor $p_1 > p$, then $n = p_1 \cdot p_2$, where p_2 is an integer. If $p_2 \geqslant p$, we get $n = p_1 \cdot p_2 > p \cdot p > n$, which is a contradiction. This proves the statement since p_2 cannot be less than p according to the assumption.

Example 2.3. Can a number formed by 13 ones, 13 twos, and 13 threes be a perfect square?

Solution: Usually, when the problem provides only the digits forming the number without specifying their order, it involves testing divisibility rules for 3 and/or 9. Let's prove that the number formed by these digits cannot be a perfect square. The sum of the digits of this number is $13 \cdot (1 + 2 + 3) = 78$, which is divisible by 3. Thus, the original number is divisible by 3. It would also have to be divisible by 3^2 if it were a square, as 3 is prime. However, 78 is not divisible by 3^2, which means it cannot be a perfect square. □

Why did we choose to prove the impossibility of forming the number? If we assume for a moment that a square could be formed (a 39-digit number), proving that it is indeed a perfect square without a calculator (since calculators are not allowed in olympiads!) would be practically impossible.

Example 2.4. Find a four-digit number that is a perfect square, where the first 2 digits are equal to each other and the last 2 digits are also equal to each other.

Solution: Notice that this number must be written in the form $\overline{aabb}$. It can be observed that the alternating sum of the digits of this number is zero, which means the number is divisible by 11. Let $N^2 = \overline{aabb}$; then N is also divisible by 11, and N must be a two-digit number for the square to have four digits. All two-digit numbers divisible by 11 consist of identical digits, so $N = \overline{cc}$.

Write the equation: $\overline{cc}^2 = \overline{aabb}$, from which

$$(11c)^2 = 1000a + 100a + 10b + b = 11(100a + b) \Leftrightarrow 11c^2 = 100a + b.$$

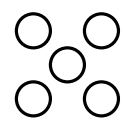
A simple check confirms that only $c = 8$ is suitable, and $88^2 = 7744$. □

Problem Set

Problem 2.1. (MF — 2016.6.1): Leo has five cards with digits: $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$, and $\boxed{5}$. Help him create two numbers from these cards — a three-digit and a two-digit one — such that the first number is divisible by the second.

Problem 2.2. (COM — 2013.7.1): An astrologer believes that the year 2013 (this year) is *lucky* because 2013 is divisible by $20 + 13$. In the future, will there ever be two lucky years in a row?

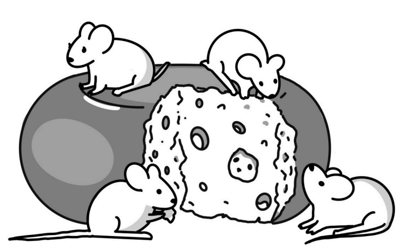
Problem 2.3. (MF — 2015.6.2): a) Fill in each circle with a non-zero digit so that the sum of digits in the two top circles is 7 times smaller than the sum of the other digits, and the sum of digits in the two left circles is 5 times smaller than the sum of the other digits.



b) Prove that the problem has a unique solution.

Problem 2.4. (COM — 2002.7.2): Are there such digits **G** and **U**, such that the 3 digit number **UGU** is divisible by 13, and the number **GUG** is not divisible by 13?

Problem 2.5. (MF — 2012.6.3): The inhabitants of the Unlucky Island, like us, divide the day into several hours, each hour into several minutes, and each minute into several seconds. But on their island, there are 77 minutes in a day and 91 seconds in an hour. How many seconds are there in a day on the Unlucky Island?



Problem 2.6. (MF — 2008.6.3): There were several whole heads of cheese in a warehouse. At night, some rats came and ate 10 heads of cheese; each rat ate the same amount of cheese. Some rats get stomachaches because they overate. The remaining 7 rats ate the remaining cheese the next night; however, each rat could only eat half as much cheese as it could the night before. How much cheese was there initially in the warehouse?

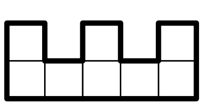
Problem 2.7. (MF — 2010.7.3): Little kids were eating candy. Each child ate the same number of pieces of candy, which is at least 1 piece of candy. Each child ate 7 pieces of candy, which is less than the total number of pieces of candy eaten by all children combined. How many candies were eaten in total?

Problem 2.8. (MF — 2006.7.4): The Mathematical Festival is held annually. The first Mathematical Festival was held in 1990. Hence, the number of this year's Mathematical Festival (2006) is number 17. The year of the current Mathematical Festival is divisible by its number: $2006/17 = 118$. Name:

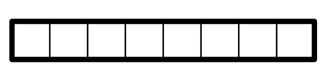
a) The first number of the Mathematical Festival for which this is also true (year of Mathematical Festival divisible by its number);
b) The last number of the Mathematical Festival for which this is also true (year of Mathematical Festival divisible by its number).

Problem 2.9. (COM — 2004.7.5): Among some 13 consecutive natural numbers 7 numbers are even, 5 are divisible by 3. How many of these numbers are divisible by 6?

Problem 2.10. (MF — 2017.7.5): Can the digits $1, 2, \ldots, 8$ be arranged in the cells a) of the letter **Ш**, b) of the strip (see figures below) in such a way that if the figure is split into two parts by one cut, that the sum of all the digits of one part is divisible by the sum of all the digits of the other part? (Cuts can only be done on the boundaries of a cell. Each cell contains only one digit, and each digit is only used once.)



a)



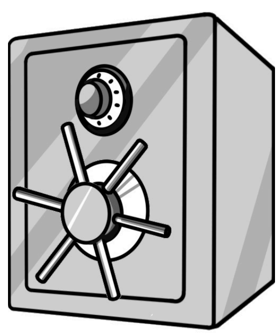
b)

Problem 2.11. (COM — 2005.7.7): Prove that the sum of digits of a number divisible by 7 can be equal to any natural number except one.

Problem 2.12. (COM — 2011.7.8): Consecutive natural numbers 2 and 3 are divisible by consecutive odd numbers 1 and 3 respectively; numbers 8, 9, and 10 are divisible by 1, 3, and 5 respectively. Are there 11 consecutive natural numbers that are divisible by 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, and 21 respectively?

Problem 2.13. (MF — 2004.7.1): Max thought of a prime three-digit number, all digits of which are different. What digit can it end with if its last digit is equal to the sum of the first two digits?

Problem 2.14. (MF — 2002.6.3): On the board, 10 consecutive natural numbers were written. When one of them was erased, the sum of the remaining nine turned out to be 2002. Which numbers remained on the board?



Problem 2.15. (MF — 2003.7.3): To open a safe, you must input a correct passcode. A correct passcode is made up of 7 digits. The digits used are 2 and 3 (they repeat). For a passcode to be correct, there must be more 2s than 3s in the code, and the code must be divisible by both 3 and 4. Come up with a code that can open the safe.

Problem 2.16. (MF — 1995.7.5): From a natural number, the sum of its digits was subtracted, then from the resulting number, the sum of its digits was subtracted again, and so on. After eleven such subtractions, zero was obtained. What was the initial number?

Problem 2.17. (Alabama ARML TST — 2005.4): For how many ordered pairs of digits (A,B) is $2AB8$ a multiple of 12?

Skill Assessment Problems

Skill Assessment Problem 2.1. Find the largest four-digit number divisible by 2, 5, 9, and 11, without repeating digits.

Skill Assessment Problem 2.2. The sum of the three smallest distinct divisors of a certain number A is 8. How many trailing zeros can A have? Specify all possibilities.

Solutions to Skill Assessment Problems

Solution to Problem 2.1: For the number to be divisible by 2, 5, 9, and 11, it must be divisible by the product of all these numbers, as they are mutually prime. The product of these numbers is 990. Note that $990 \cdot 9 = 8910$ satisfies the conditions of the problem, $990 \cdot 10 = 9900$ does not, and $990 \cdot 11$ is already a five-digit number. Therefore, the desired number is 8910. □

Solution to Problem 2.2: The smallest divisor of the number A is 1. Therefore, the sum of the two smallest divisors, not equal to 1, must equal 7. This is possible in two cases: $7 = 3+4$ or $7 = 2+5$. However, the first case is impossible, as it would imply that the number is divisible by 4, making it even, and the smallest divisor after 1 is 2.

In the second case, the smallest divisors are 1, 2, and 5, so the number is divisible by 10 and ends in 0. Let's prove that it cannot end in 00. Indeed, such a number would be divisible by 100 and, therefore, by 4, which is impossible. Therefore, our number can end in exactly one zero. □

Modular Arithmetic

> The Number Zero: The concept of zero as a number was developed independently by ancient civilizations like the Babylonians and the Mayans.

> The number zero is odd in a way that it's neither positive nor negative. It's kind of the «middle child» of numbers!

Theory and Practice

Almost every Olympiad includes problems where both coefficients and variables can only take integer values. These are known as problems in integer numbers. In this section, we will consider only such problems. One of the most powerful tools for solving such problems is modular arithmetic.

Definition 4. Numbers a and b are said to be congruent modulo x if their difference is divisible by x. This is denoted as $a \equiv b \pmod{x}$ or $a = b \pmod{x}$.

For definiteness, we will consider that the modulus x can only be positive integers.

For negative numbers, finding the remainder can have different options, each of which makes sense: for example, what is the remainder of dividing -17 by 10? The answer -7 seems convincing, as $-17 = 10 \cdot (-1) + (-7)$. But the answer 3 is just as valid since $-17 = 10 \cdot (-2) + 3$. Regarding modular arithmetic, we are interested in the second option, where if the divisor is positive, then the remainder is also positive.

It is easy to verify the following equivalent statement.

Claim 4. Numbers a and b are congruent modulo x if and only if they have the same remainder when divided by x.

Negative numbers can also be compared modulo.

For example, $13 \equiv 7 \pmod{3}$ or $11 \equiv -3 \pmod{7}$.

If $a \equiv 0 \pmod{x}$, then a is divisible by x.

With modular congruences, you can perform operations similar to regular equalities: add, subtract, multiply, and exponentiate them.

If $a \equiv b \pmod{x}$ and $c \equiv d \pmod{x}$, then

- $a + c \equiv b + d \pmod{x}$;
- $a \cdot c \equiv b \cdot d \pmod{x}$;
- $a^n \equiv b^n \pmod{x}$ for any natural n.

The division is trickier, and we will consider this operation later.

Sometimes, using modular congruences allows for a significant reduction in the complexity of calculations. Let's consider an example.

Example 3.1. Find the remainder of the division $2013 \cdot 2014 \cdot 2015 + 2016^3$ by 7.

Solution: Find the remainder of the division of 2013 by 7. Of course, this can be done using long division, but as Olympiad participants, we recall a beautiful fact: $1001 = 7 \cdot 11 \cdot 13$, from which it follows that the number 2002 is divisible by 7, i.e., $2002 \equiv 0 \pmod{7}$. However, $11 \equiv 4 \pmod{7}$, so $2002 + 11 \equiv 0 + 4 = 4 \pmod{7}$.

Therefore, $2014 \equiv 5 \pmod{7}$ and $2015 \equiv 6 \pmod{7}$, which implies $2013 \cdot 2014 \cdot 2015 \equiv 4 \cdot 5 \cdot 6 = 120 \equiv 1 \pmod{7}$. Since $2016 \equiv 0 \pmod{7}$, then $2016^3 \equiv 0^3 \pmod{7}$.

In conclusion, $2013 \cdot 2014 \cdot 2015 + 2016^3 \equiv 1 + 0 = 1 \pmod{7}$. □

In some cases, it is more convenient to switch to negative numbers. For example, $8^{100} \equiv (-1)^{100} = 1 \pmod{9}$.

When we use modular congruences, many well-known divisibility tests reveal themselves from a new perspective and can even provide more information. For example, the divisibility tests for 9 and 3 can be expressed as follows:

$$\overline{a_n a_{n-1} \ldots a_2 a_1 a_0} \equiv a_n + a_{n-1} + \ldots + a_2 + a_1 + a_0 \pmod{9}$$

and

$$\overline{a_n a_{n-1} \ldots a_2 a_1 a_0} \equiv a_n + a_{n-1} + \ldots + a_2 + a_1 + a_0 \pmod{3}.$$

One method for solving divisibility problems is to write down all possible remainders. In some problems, finding the modulus over which remainders are written down is straightforward; in others, it needs to be guessed.

Example 3.2. Prove that for any integer n, the number $n^2 + 3n + 4$ is not divisible by 9.

Proof. An integer can produce remainders $0, 1, 2, \dots, 8$ when divided by 9. Let's go through all these cases. For convenience, let's write down the remainder as a table.

$n \pmod 9$	0	1	2	3	4	5	6	7	8
$n^2 \pmod 9$	0	1	4	0	7	7	0	4	1
$3n \pmod 9$	0	3	6	0	3	6	0	3	6
$n^2 + 3n + 4 \pmod 9$	4	8	5	4	5	8	4	2	2

Consider, for example, the case when n gives a remainder of 5 when divided by 9. Then $n \equiv 5 \pmod 9$, from which

$$n^2 + 3n = 4 \equiv 5^2 + 3 \cdot 5 + 4 = 25 + 15 + 4 \equiv 7 + 6 + 4 = 17 \equiv 8 \pmod 9.$$

Other cases are handled similarly. We see that in no case did we get a remainder of 0, which means that the expression is not divisible by 9 for any integer n. □

Example 3.3. Can a number formed by 13 twos, 13 threes, 13 fours, and 13 fives be a perfect square?

Solution: We will prove that this is impossible. Since the problem refers to a number whose digits are known but their order is not, we will use the divisibility criterion for 3. Let N denote the number.

Then the sum of the digits of the number N is equal to $13(2 + 3 + 4 + 5)$. By the divisibility criterion, we get: $N \equiv 13 \cdot 14 \equiv 1 \cdot 2 = 2 \pmod 3$.

Let's see if the square of a natural number can yield a remainder of 2 modulo 3, as in the previous problem; consider cases.

$x \pmod 3$	0	1	2
$x^2 \pmod 3$	0	1	1

We conclude that the square of a natural number cannot yield a remainder of 2 modulo 3, which means that it is impossible to form a square from all the given digits.

□

Problem Set

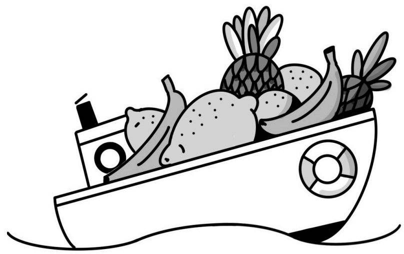
Problem 3.1. (MMO — 1940.7-8.4): How many pairs of integers x, y between 1 and 1000 inclusive exist, such that the sum $x^2 + y^2$ is divisible by 7?

Problem 3.2. (COM — 2013.6.1): There are 1000 candies laid out in a row. Initially, Max ate the ninth candy from the left, then Max ate every seventh candy, moving to the right. After Max left, Leo came. Initially, Leo ate the seventh candy from the left, from the remaining candies, and then Leo ate every ninth candy, moving to the right. How many candies remained after Max and Leo finished eating?

Problem 3.3. (MF — 1998.7.5): On the moon, there is a currency: coins with the denomination of 1, 15, 50 farthings are in circulation. Jean spent a number of coins on a purchase; then he received some change. The change came out to be one coin more than the number of coins that he had paid with. What is the smallest possible value of farthings that Jean could have spent?

Problem 3.4. (MMG — 2011/2012.8.3): How many natural numbers n (not exceeding 2012) exist, such that the last digit of the value of the sum of $1^n + 2^n + 3^n + 4^n$ is 0?

Problem 3.5. (Mos2ARSO — 2011.9.4): Jean claims that there are eight consecutive numbers such that in the prime factorization of each number, each prime factor appears in an odd power (for example, two consecutive numbers are $23 = 23^1$ and $24 = 2^3 \cdot 3^1$). Is Jean's claim correct?



Problem 3.6. (TOT — 2009/2010.10-11.1): From South America to Russia, 2010 ships carry bananas, lemons, and pineapples. The number of bananas on each ship is equal to the total number of lemons carried on all of the ships combined. The number of lemons carried on each ship is equal to the total number of pineapples carried on all the ships combined. Prove that the total number of fruits is divisible by 31.

Problem 3.7. (TOT — 1986/1987.9-10.1): Can the number 1986 be represented as the sum of six squares of odd numbers?

Problem 3.8. (MMG — 2012/2013.10.5): Given $b = 2013^{2013} + 2$. Are the numbers $b^3 + 1$ and $b^2 + 2$ coprime?

Problem 3.9. (MMG — 2012/2013.11.5): Do 4 consecutive natural numbers exist, each of which can be represented as the sum of the squares of two natural numbers?

Problem 3.10. (Mock AIME — 2007-2008.1.4): If x is an odd number, then find the largest integer that always divides the expression

$$(10x + 2)(10x + 6)(5x + 5).$$

Problem 3.11. (AIME — 2024.I.13): Let p be the least prime number for which there exists a positive integer n such that $n^4 + 1$ is divisible by p^2. Find the least positive integer m such that $m^4 + 1$ is divisible by p^2.

Problem 3.12. (AIME — 2023.I.7): Call a positive integer n extra-distinct if the remainders when n is divided by $2, 3, 4, 5$, and 6 are distinct. Find the number of extra-distinct positive integers less than 1000.

Problem 3.13. (UNCO Math Contest — 2011.II.6): What is the remainder when $1! + 2! + 3! + \ldots + 2011!$ is divided by 18?

Problem 3.14. (AMC — 2018.10B.16): Let $a_1, a_2, \ldots, a_{2018}$ be a strictly increasing sequence of positive integers such that

$$a_1 + a_2 + \ldots + a_{2018} = 2018^{2018}$$

What is the remainder when $a_1^3 + a_2^3 + \ldots + a_{2018}^3$ is divided by 6 ?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 3.15. (AMC — 2017.10A.13): Define a sequence recursively by $F_0 = 0, F_1 = 1$, and $F_n =$ the remainder when $F_{n-1} + F_{n-2}$ is divided by 3, for all $n \geqslant 2$. Thus the sequence starts $0,1,1,2,0,2,\ldots$ What is $F_{2017} + F_{2018} + F_{2019} + F_{2020} + F_{2021} + F_{2022} + F_{2023} + F_{2024}$?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Problem 3.16. (AMC — 2017.10B.14): An integer N is selected at random in the range $1 \leqslant N \leqslant 2020$. What is the probability that the remainder when N^{16} is divided by 5 is 1?

(A) $\frac{1}{5}$ (B) $\frac{2}{5}$ (C) $\frac{3}{5}$ (D) $\frac{4}{5}$ (E) 1

Problem 3.17. (AMC — 2015.10B.10): What are the sign and unit digits of the product of all the odd negative integers strictly greater than -2015?

(A) It is a negative number ending with a 1.
(B) It is a positive number ending with a 1.

(C) It is a negative number ending with a 5.
(D) It is a positive number ending with a 5.
(E) It is a negative number ending with a 0.

Problem 3.18. (AMC – 2011.10B.23): What is the hundreds digit of 2011^{2011} ?

(A) 1 (B) 4 (C) 5 (D) 6 (E) 9

Problem 3.19. (AMC – 2009.10A.21): What is the remainder when $3^0 + 3^1 + 3^2 + \ldots + 3^{2009}$ is divided by 8?

(A) 0 (B) 1 (C) 2 (D) 4 (E) 6

Problem 3.20. (Canadian MO – 1969.7): Show that there are no integers a,b,c for which $a^2 + b^2 - 8c = 6$.

Problem 3.21. (AIME – 2010.I.2) Find the remainder when $9 \times 99 \times 999 \times \ldots \times \underbrace{99\ldots9}_{\text{999 9's}}$ is divided by 1000.

Skill Assessment Problems

Skill Assessment Problem 3.1. Natural numbers x, y, z are such that $x^2 + y^2 = z^2$. They form a «Pythagorean triple». Prove that at least one of the three natural numbers is divisible by 3.

Skill Assessment Problem 3.2. There are seven natural numbers. It is known that the sum of any six of them is divisible by 5. Prove that all seven of the numbers are divisible by 5.

Skill Assessment Problem 3.3. How many natural numbers n are there, where $n < 2014$, for which $2^n - n^2$ is divisible by 7?

Solutions to Skill Assessment Problems

Solution to Problem 3.1: Let's consider the remainder that squares of natural numbers can give when divided by 3:

x	0	1	2
x^2	0	1	1

From this table, we can conclude that if a number is not divisible by 3, then its square must give 1 as the remainder. Suppose that none of the numbers divide by 3, then the equation $x^2 + y^2 \equiv z^2 \pmod 3$ is written as $1 + 1 = 1$, which is false; therefore, at least one of the numbers x, y, or z is divisible by 3. □

Solution to Problem 3.2: Let x_1, x_2, x_3, x_4, x_5, x_6, x_7 denote the numbers. Because the sum of any six of them is divisible by 5, the following simultaneous expressions are true:

$$\begin{cases} x_1 + x_2 + \ldots + x_6 \,\vdots\, 5, \\ x_2 + x_3 + \ldots + x_7 \,\vdots\, 5, \\ \ldots, \\ x_7 + x_1 + \ldots + x_5 \,\vdots\, 5. \end{cases}$$

When summing all of these expressions, each number x_i will appear exactly 6 times on the left:

$$6(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7) \,\vdots\, 5,$$

From this, we know that the following expression is true:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \,\vdots\, 5.$$

Subtracting each of the simultaneous expressions from our new expression shows us that each x_i is divisible by 5: $x_7 \,\vdots\, 5, x_1 \,\vdots\, 5, \ldots, x_6 \,\vdots\, 5$. □

Solution to Problem 3.3: Let's consider the remainders that the number n^2 can have when divided by 7. The remainder will have a period length of 7:

n	0	1	2	3	4	5	6
n^2	0	1	4	2	2	4	1

Let's also consider the remainders that the number 2^n can have when divided by 7. The remainder will have a period length of 3:

n	0	1	2	3	4	5
2^n	1	2	4	1	2	4

Therefore, for the number $n^2 - 2^n$, the remainders will have a period length of 21 $(3 \cdot 7)$:

n	0	1	2	3	4	5	6	7	8	9	10
n^2	0	1	4	2	2	4	1	0	1	4	2
2^n	1	2	4	1	2	4	1	2	4	1	2
$n^2 - 2^n$	-1	-1	0	1	0	0	0	-2	-3	3	0

n	11	12	13	14	15	16	17	18	19	20
n^2	2	4	1	0	1	4	2	2	4	1
2^n	4	1	2	4	1	2	4	1	2	4
$n^2 - 2^n$	-2	3	-1	-4	0	2	-2	1	2	-3

Now let's notice that $2014 = 21 \cdot 95 + 19$; therefore, the entire period will repeat 95 times.

Each period has 6 cases when $n^2 - 2^n \equiv 0 \pmod 7$:

$$n \equiv 2, 4, 5, 6, 10, 15 \pmod{21}.$$

There will also be 6 cases from the incomplete period; therefore, there will be $95 \cdot 6 + 6 = 96 \cdot 6 = 576$ such n for which $n^2 - 2^n$ is divisible by 7. $\square$

Fundamental Theorem of Arithmetic

> Perfect Numbers: A perfect number is a positive integer that is equal to the sum of its proper divisors (excluding itself). For example, 6 is a perfect number because its divisors (excluding 6 itself) are 1, 2, and 3, and $1 + 2 + 3 = 6$.

The ancient Greeks thought that perfect numbers had mystical properties. Some believed that the number 28 (perfect) had divine significance because there are 28 days in a lunar month.

Theory and Practice

We already know what a prime number is. Notice that all natural numbers can be divided into three disjoint sets — the set of prime numbers, the set of composite numbers, and one (yes, it is not a prime number, and we will soon understand why).

Let's take the number 360. What is the smallest prime number by which it is divided? Obviously, 2: $360 = 2 \cdot 180$. What is the smallest prime number by which 180 is divided? Again, 2: $180 = 2 \cdot 90$, so $360 = 2 \cdot 2 \cdot 90$. What is the smallest prime number by which 90 is divided? Again, 2: $90 = 2 \cdot 45$, so $360 = 2 \cdot 2 \cdot 2 \cdot 45$. What is the smallest prime number by which 45 is divided? It's 3: $45 = 3 \cdot 15$, so $360 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 15$. Finally, $15 = 3 \cdot 5$, and $360 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$. At this point, the process we started stops: all the obtained factors are prime. Such a product is called the prime factorization or canonical factorization.

Theorem 1. Any natural number (except one) can be represented as the product of prime factors, and this representation is unique (up to the order of factors).

This factorization would not be unique (as there would be multiple factorizations of the same number varying with different powers of one). In the 18th century, great mathematicians like Euler and Goldbach considered 1 as a prime number. For example, Goldbach's conjecture, one of the seven Millennium Prize Problems, states that every even integer greater than 2 can be expressed as the sum of two prime numbers. And $2 = 1 + 1$. Today, the conjecture is refined to «every even integer greater than 4». The last prominent mathematician to consider 1 as a prime number was Henri Lebesgue in 1899.

Above, we obtained the prime factorization of the number 360: $360 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$, or, as it is usually written, $360 = 2^3 \cdot 3^2 \cdot 5$.

Any number can be seen as composed of «bricks» — prime factors arising in its canonical decomposition. A prime number consists of one such «brick», itself.

Canonical decomposition is a powerful tool for solving a variety of problems. Thanks to it, all the divisors of a given number are revealed. Thus, for the number 360, we can now immediately say that it is divisible, for example, by $2^3 = 8$, by $2^2 \cdot 3 = 12$, by $2 \cdot 3^2 \cdot 5 = 90$ (since these numbers are «constructed» from individual elements of the canonical decomposition) and not divisible by 7 or $33 = 3 \cdot 11$ (since neither 7 nor 11 is present in the canonical decomposition).

It should be noted that equivalent formulations of the above Fundamental Theorem of Arithmetic can be found in the works of Euclid (3rd century BCE). In contrast, an exact formulation and proof were first provided by Carl Friedrich Gauss in his «Disquisitiones Arithmeticae» (1801).

Let's solve a problem from the school stage of the All-Russian Olympiad in Mathematics.

Example 4.1. (1ARSO — 2018.6.1): In Russia there are massive apartment buildings. For the sake of navigation in these buildings, the buildings are broken up into sectors known as «entrances». In each sector, there is an equal number of floors, and on each floor, there is an equal number of apartments (the number of apartments per floor must be greater than 1). It is known that the number of floors in each sector is greater than the number of apartments per floor but less than the total number of sectors in the building. How many floors are there in each sector if the total number of apartments in the building is 715?

Solution: Factorize the number 715 into its prime factors: $715 = 5 \cdot 11 \cdot 13$. Notice that the condition implies that the total number of apartments is equal to the number of sectors multiplied by the number of floors per sector multiplied by the number of apartments per floor, each of these quantities being greater than one. Since 715 can be expressed as the product of 3 distinct prime numbers (greater than 1), and considering the information provided in the condition about the relationship between the quantities in question, we arrive at the answer: 5 apartments per floor, 11 floors per sector, and 13 sectors. $\square$

Example 4.2. Find the number of natural numbers less than 100 such that the product of all its divisors (including 1 and itself) is equal to the cube of the number.

Solution: Have you ever noticed that if 120 is divisible by 6, then 120 is also divisible by $\frac{120}{6}$? This fact seems obvious, but it can be reformulated as follows: if c is divisible by a, then c is also divisible by $b = \frac{c}{a}$. Sometimes mathematicians say «a divides c» instead of «c is divisible by a». Thus, for any number c, all divisors can be paired up, the product of which is equal to c. However, if c is a perfect square, the last pair cannot be formed, and the number of divisors will be odd.

Now, let's return to the problem. For the product of the divisors to be a cube, the divisors must form 3 pairs (the number cannot be a perfect square as there are 6 divisors, which is an even number of divisors). Let's assume there are three prime numbers involved in the factorization of the number.

Hence, all of the divisors of c are $1, p_1, p_2, p_3, p_1 \cdot p_2, p_1 \cdot p_3, p_2 \cdot p_3, p_1 \cdot p_2 \cdot p_3$; all of these divisors, by the Fundamental Theory of Arithmetic, are distinct. Therefore, the number of divisors of c is greater than 6; similarly, if the number of prime divisors is greater than 3, the total number of divisors is greater than 6.

If the prime factorization of the number c includes exactly two prime divisors, then it must be in the form $p_1 \cdot p_2^2$.

In this case, all of the divisors of c are $1, p_1, p_2, p_1 \cdot p_2, p_1 \cdot p_2^2$.

Prove for yourself that other factorizations do not fit. Since $c < 100$, we can find all solutions by direct enumeration, checking the values of all numbers of the form $2 \cdot 3^2, 2 \cdot 5^2, 2 \cdot 7^2, 3 \cdot 2^2, 3 \cdot 5^2, 5 \cdot 2^2, 5 \cdot 3^2, 7 \cdot 2^2, 7 \cdot 3^2, 11 \cdot 2^2, 11 \cdot 3^2, 13 \cdot 2^2, 17 \cdot 2^2, 19 \cdot 2^2, 23 \cdot 2^2$.

We have not considered the case when the number of prime divisors is equal to one; thus, the number c is equal to p_1^n. In this case, all of the divisors of c are 1, p_1, p_1^2, $\ldots$, p_1^n. This gives us exactly one option — $c = p_1^5$, and, as c is less than 100, or $c = 2^5 = 32$.

Arranging all numbers in ascending order, we get the *answer*:

$$12, 18, 20, 28, 32, 44, 45, 50, 52, 63, 68, 75, 76, 92, 98, 99.$$

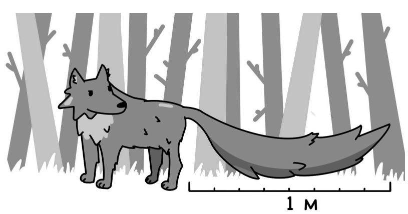
□

Problem Set

Problem 4.1. (MF — 1995.7.1): A natural number is multiplied by all of its digits successively. The result is 1995; find the original number.

Problem 4.2. (MF — 2008.7.1): A number is multiplied by the sum of its digits; this equals 2008. Find the original number.

Problem 4.3. (COM — 2009.7.1): Leo wrote down a four-digit number. Max added 1 to the first digit, 2 to the second digit, 3 to the third digit, and 4 to the fourth digit. He then multiplied the new digits together. The result Max obtained was 234. What number could Leo have written down?



Problem 4.4. (COM — 2015.6.2): A hunter told his friend that he saw a wolf in the forest with a one meter long tail. This friend then told another friend that the hunter had seen a wolf with a two meter long tail. Many people passed on the story; ordinary people doubled the length of the tail when they passed on the story, and creative people tripled the length. A while later, it was reported on TV that the hunter had found a wolf with a tail that was 864 meters long. Give two distinct answers: how many ordinary people that «grew» the tail and how many creative people «grew» the tail.

Problem 4.5. (MF — 1999.6.2): Find five positive integers whose sum is 20 and whose product is 420.

Problem 4.6. (MF — 2007.6.2;7.2): In Russia, there is a 5 point grade scheme in school, with 5 showing great performance and 2 showing terrible performance. Some schools have a «singing» class. At the end of the semester, Max wrote down his current grades for «singing» in a line. Between some numbers, he would place multiplication signs. The result of multiplying these numbers is equal to 2007. Max's overall grade in singing for the semester is the mean of all his scores. Deduce Max's grade for singing for the semester.

For example, if he had received 4 grades: $a\ b\ c\ d$, he could have made the equation $\overline{ab} \cdot c \cdot d = e$ where e is some natural number; his grade for the semester would be $(a + b + c + d)/4$.

Problem 4.7. (COM — 2016.6.2): This year is 2016. There are 4 cards with digits: 2, 0, 1, 6. These cards can be rearranged to form many 4 digit numbers (0 cannot come first). For each number from 1 to 9, the cards can be arranged to make a 4 digit number that is divisible by the number; for example, 2016 is divisible by 7 and 2160 is divisible by 5. When is next year that we will share this property?

Problem 4.8. (MF — 2009.7.6): You have access to tiles with the numbers 1, 2, 5, 10. You're allowed to use all 4 arithmetic operations and brackets. Compose an expression with the value 2009 where the sum of the numbers on all the tiles used is the lowest. For example, if the expression is $a \cdot b/(c + d)$, then the sum of the numbers on the tiles is $a + b + c + d$.

Problem 4.9. (MF — 1996.7.6): The factorial of n is the consecutive product of all natural numbers from 1 to n, the n-factorial is denoted as $n!$ ($1 \cdot 2 \cdot 3 \cdot \ldots \cdot n = n!$).

We have the following expression

$$1! \cdot 2! \cdot 3! \cdot \ldots \cdot 100!$$

Is it possible to cross out one of the factorials from the expression so that the product of all the remaining factorials is a perfect square?

Problem 4.10. (University of South Carolina High School Math Contests — 1993.15): If we express the sum $\frac{1}{3\cdot5\cdot7\cdot11}+\frac{1}{3\cdot5\cdot7\cdot13}+\frac{1}{3\cdot5\cdot11\cdot13}+\frac{1}{3\cdot7\cdot11\cdot13}+\frac{1}{5\cdot7\cdot11\cdot13}$

as a rational number in reduced form, then the denominator will be

(A) 15015 (B) 5005 (C) 455 (D) 385 (E) 91

Problem 4.11. (Mock AIME — 2005.2.1): Compute the largest integer k such that 2004^k divides $2004!$.

Problem 4.12. (Mock AIME — 2006-2007.2.3): Let S be the sum of all positive integers n such that $n^2+12n-2007$ is a perfect square. Find the remainder when S is divided by 1000.

Problem 4.13. (AMC — 2023.12A.22): Let f be the unique function defined on the positive integers such that

$$\sum_{d|n} d\cdot f\left(\frac{n}{d}\right)=1$$

for all positive integers n. What is $f(2023)$?

(A) -1536 (B) 96 (C)108 (D) 116 (E) 144

Problem 4.14. (AIME — 2023.I.4): The sum of all positive integers m such that $\frac{13!}{m}$ is a perfect square can be written as $2^a3^b5^c7^d11^e13^f$, where a,b,c,d,e, and f are positive integers. Find $a+b+c+d+e+f$.

Problem 4.15. (UNCO Math Contest — 2018.II.4): How many positive integer factors of $36{,}000{,}000$ are not perfect squares?

Problem 4.16. (UNCO Math Contest — 2012.II.6): How many 5-digit positive integers have the property that the product of their digits is 600?

Problem 4.17. (AIME — 2010.II.3): Let K be the product of all factors $(b-a)$ (not necessarily distinct) where a and b are integers satisfying $1 \leqslant a < b \leqslant 20$. Find the greatest positive integer n such that 2^n divides K.

Problem 4.18. (AMC — 2020.10B.25): Let $D(n)$ denote the number of ways of writing the positive integer n as a product $n = f_1 \cdot f_2 \ldots f_k$ where $k \geqslant 1$, the f_i are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as $6 = 2 \cdot 3$, and $3 \cdot 2$, so $D(6) = 2$. What is $D(96)$?

(A) 112 (B) 128 (C) 144 (D) 172 (E) 184

Problem 4.19. (AMC — 2019.10B.19): Let S be the set of all positive integer divisors of 100,000. How many numbers are the product of two distinct elements of S?

(A) 98 (B) 100 (C) 117 (D) 119 (E) 121

Problem 4.20. (AMC — 2018.10A.7): For how many (not necessarily positive) integer values of n is the value of $4000 \cdot (\frac{2}{5})^n$ an integer?

(A) 3 (B) 4 (C) 6 (D) 8 (E) 9

Problem 4.21. (AMC — 2016.10A.22): For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $81n^4$ have?

(A) 110 (B) 191 (C) 261 (D) 325 (E) 425

Problem 4.22. (AMC — 2015.10B.23): Let n be a positive integer greater than 4 such that the decimal representation of $n!$ ends in k zeros and the decimal representation of $(2n)!$ ends in $3k$ zeros. Let s denote the sum of the four least possible values of n. What is the sum of the digits of s ?

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

Problem 4.23. (AMC — 2014.10B.17): What is the greatest power of 2 that is a factor of $10^{1002} - 4^{501}$?

(A) 2^{1002} (B) 2^{1003} (C) 2^{1004} (D) 2^{1005} (E) 2^{1006}

Problem 4.24. (AMC — 2013.10B.9): Three positive integers are each greater than 1, have a product of 27000, and are pairwise relatively prime. What is their sum?

(A) 100 (B) 137 (C) 156 (D) 160 (E) 165

Problem 4.25. (AMC — 2014.10B.12): The largest divisor of $2{,}014{,}000{,}000$ is itself. What is its fifth-largest divisor?

(A) $125{,}875{,}000$ (B) $201{,}400{,}000$ (C) $251{,}750{,}000$
(D) $402{,}800{,}000$ (E) $503{,}500{,}000$

Problem 4.26. (iTest — 2007.24): Let N be the smallest positive integer such that $2008N$ is a perfect square and $2007N$ is a perfect cube. Find the remainder when N is divided by 25.

Problem 4.27. (iTest — 2007.20): Find the largest integer n such that $2007^{1024} - 1$ is divisible by 2^n

Skill Assessment Problems

Skill Assessment Problem 4.1. Does such an integer exist that the product of its digits is equal to a) 1980, b) 1990, c) 2000, d) 2018 ?

Skill Assessment Problem 4.2. Does such a natural number greater than 100 exist that the product of all its divisors (including 1 and itself) is equal to the original number raised to the eight power?

Skill Assessment Problem 4.3. (Lomonosov — 2017.7–8.6, 9.4): For natural numbers m and n, it is known that $3n^3 = 5m^2$. Find the smallest possible value of $m + n$.

Skill Assessment Problem 4.4. (2ARSO — 2009.8.1) There are two natural numbers that are not divisible by ten. Their product is equal to 1000. Find their sum.

Solutions to Skill Assessment Problems

Solution to Problem 4.1: a) $1980 = 2^2 \cdot 3^2 \cdot 5 \cdot 11$, 11 cannot be a digit of the number;

b) $1990 = 2 \cdot 5 \cdot 199$ — similar to item a), the presence of the prime number 199 among the divisors also indicates that such a number does not exist;

c) $2000 = 2^4 \cdot 5^3$, from the given factorization, it follows that such a number exists — it can be composed of powers of 2 (up to the third power) and powers of 5, for example, 2222555, 44555, 28555;

d) $2018 = 2 \cdot 1009$ — similar to the first two items, such a number does not exist.

Answer: a) no; b) no; c) yes; d) no. □

Solution to Problem 4.2: Yes, for example, the number 2^{15} has exactly 16 divisors: $2^0, 2^1, \ldots, 2^{15}$, which makes $(15+1) : 2 = 8$ pairs of divisors, each of which contributes the number itself to the product, so it fits the criteria. □

Solution to Problem 4.3: Since the left side of this equation is divisible by 3, the right side must also be divisible by 3. Thus, since 5 is not divisible by 3, in the prime factorization of m^2, there must be at least one 3, which means that the factorization of m itself must also contain 3. So, the number m can be written as $m = 3a$; similarly, $n = 5b$, where a and b are natural numbers. Substituting these values into the original equation, we get $3(5b)^3 = 5(3a)^2$. Hence:

$$3 \cdot 5^3 \cdot b^3 = 5 \cdot 3^2 \cdot a^2,$$

$$5^2 \cdot b^3 = 3 \cdot a^2.$$

Repeating similar reasoning as above, we get that a must also be divisible by 5 and b must be divisible by 3; therefore, a and b can be rewritten as $a = 5x$, $b = 3y$. So $n = 15y$, $m = 15x$. After substituting these values into the equation, we get:

$$5^2 \cdot (3y)^3 = 3 \cdot (5x)^2$$

$$5^2 \cdot 3^3 \cdot y^3 = 3 \cdot 5^2 \cdot x^2,$$

which leads to $3^2 \cdot y^3 = x^2$, and substituting x for $3t$, we get, $3^2 \cdot y^3 = (3t)^2$, $y^3 = t^2$. We must find the smallest possible value for the number $m + n$, which is $15y + 45t$. The smallest possible value is achieved when $y = t = 1$. Don't forget that in this problem, guessing the answer is not enough; you must also explain how you got the answer.

Answer: $15 + 45 = 60$. □

Solution to Problem 4.4: We can see that $1000 = 5^3 \cdot 2^3$. 2 and 5 cannot be present simultaneously in the prime factorizations of either number, as then the number would be divisible by 10. Therefore, we can see that the first number is equal to 2^3, and the second number is equal to 5^3. The first is 8; the second is 125. Thus, $125 + 8 = 133$.

Answer: $5^3 + 2^3 = 125 + 8 = 133$. □

GCD and LCM, Euclidean Algorithm

> The Chicken McNugget Theorem: This theorem is a fun way to find the largest number of McNuggets that cannot be bought using only boxes of n and m nuggets.

Also known as the Frobenious coin problem, it states that for any relatively prime integers m and n, the largest integer that cannot be expressed as $am + bn$ for non-negative integers a and b is $mn - m - n$. In other words, if I have boxes of 6 and 11 chicken nuggets, the largest number of chicken nuggets that is «non-purchasable» with a combination of these boxes is $6 \cdot 11 - 6 - 11 = 49$.

It is the «fast food fantasy» of number theory — it turns a snack into a mathematical quest.

Theory and Practice

Let's recall the concepts of $\gcd$ (greatest common divisor) and lcm (least common multiple).

Suppose, with the help of the fundamental theorem of arithmetic, the prime factorization of numbers M and N looks as follows:

$$N = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n} \text{ and } M = p_1^{\beta_1} p_2^{\beta_2} \dots p_n^{\beta_n}.$$

We will show that their greatest common divisor is given by

$$\gcd(M, N) = p_1^{\min(\alpha_1,\beta_1)} p_2^{\min(\alpha_2,\beta_2)} \dots p_n^{\min(\alpha_n,\beta_n)}.$$

Here, we used the concept of the minimum function $\min(x,y)$, indicating the smaller of the two values x and y.

First, let's demonstrate that the mentioned number is a divisor of M and N. For convenience, let our number be denoted as a:

$$\frac{N}{a} = \frac{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}}{p_1^{\min(\alpha_1,\beta_1)} p_2^{\min(\alpha_2,\beta_2)} \dots p_n^{\min(\alpha_n,\beta_n)}} =$$

$$= p_1^{\alpha_1-\min(\alpha_1,\beta_1)} p_2^{\alpha_2-\min(\alpha_2,\beta_2)} \dots p_n^{\alpha_n-\min(\alpha_n,\beta_n)}.$$

Since the numbers of the form $\alpha_i - \min(\alpha_i,\beta_i)$ are non-negative integers, the numbers $p^{\alpha_i-\min(\alpha_i,\beta_i)}$ are natural numbers, as is their product. Therefore, N is divisible by a. The reasoning for M is entirely analogous. Thus, a is a common divisor of M and N. It remains only to show that it is the greatest one.

Suppose there is a number b greater than a that is also a common divisor of M and N. Consider its prime factorization and compare it with the factorization of a:

$$b = p_1^{\gamma_1} p_2^{\gamma_2} \dots p_n^{\gamma_n}.$$

Since $b > a$, one of the prime factors of b enters into b to a greater power than it does in a. Then, there must be some i such that $\gamma_i > \min(\alpha_i,\beta_i)$. Consequently, b will not divide N if $\alpha_i \leqslant \beta_i$, and it will not divide M otherwise. Therefore, $a = \gcd(M, N)$.

By similar reasoning, it can be shown that the least common multiple of M and N is given by

$$\mathrm{lcm}(M, N) = p_1^{\max(\alpha_1,\beta_1)} p_2^{\max(\alpha_2,\beta_2)} \dots p_n^{\max(\alpha_n,\beta_n)}.$$

Here, $\max(x,y)$ represents the greater of the two values x and y. A very useful relationship connects the minimum and maximum values of two variables: $\min(x,y) + \max(x,y) = x + y$. Indeed, if the numbers are equal, then the minimum and maximum values coincide with the numbers. If they are not equal, then one of the numbers will be the maximum value, and the other will be the minimum value, and the sum remains unchanged regardless of the order of the terms.

Hence, in particular, the formula holds: $\mathrm{lcm}(M, N) \cdot \gcd(M, N) = M \cdot N$.

Numbers are called *relatively prime* or *coprime* if their greatest common divisor is 1.

The concept of Bézout's identity connects relatively prime numbers and $\gcd$. Let a and b be integers, both not equal to zero; then, integers u and v exist such that the equality $\gcd(a, b) = a \cdot u + b \cdot v$ holds.

From this, it follows that if a and b are relatively prime, then u and v exist such that $a \cdot u + b \cdot v = 1$.

Let's solve a few simple problems for practice.

Example 5.1. The number $5A$ is divisible by 3. Must the number A be divisible by 3?

Solution: Suppose that the number A is not divisible by 3. Then, in its prime factorization, there is no 3. However, the factorization of the number $5A$ differs from that of A only by the presence of an additional power of 5. Hence, 3 is not present in it, and it is not divisible by 3. From this contradiction, we conclude that A must be divisible by 3. □

Example 5.2. The number $5A$ is divisible by 15. Must the number A be divisible by 15?

Solution: No, it does not follow. For example, the condition of the problem is not satisfied for $A = 3$. □

However, it would be nice to identify some regularity for such logical transitions, and as it turns out, there is indeed such a regularity.

Lemma 1. If ab is divisible by c, and b is relatively prime to c, then a is divisible by c.

Proof. Firstly, note that for relatively prime numbers, their prime factorizations have no common factors; otherwise, their $\gcd$ would be divisible by that common factor and would not be equal to 1. Assume the contrary, i.e., a is not divisible by c. In this case, at least one prime number exists such that in the prime factorization of c, it occurs to a greater power than in the factorization of a (possibly, it does not occur in the factorization of a at all). Also, in the factorization of b, this prime number is absent due to the relative primality of b and c. Thus, in the factorization of c, this prime number occurs to a greater power than in the factorization of ab, i.e., ab is not divisible by c. By arriving at a contradiction, we establish the statement of this lemma. □

Remark 1. If two numbers, b and c, are not relatively prime, then a number a is guaranteed to exist such that ab is divisible by c, and a is not divisible by it. It can be easily verified that the number

$$a = \frac{c}{\gcd(b, c)}$$

fits.

Claim 5. If an integer s is divisible by numbers a and b (relatively prime), then s is divisible by ab.

Proof. Despite its simple formulation, this intuitively clear statement relies on the fundamental theorem of arithmetic in its proof. Two relatively prime numbers in their factorization have different prime numbers; otherwise, they would not be relatively prime. In this case, in the factorization of s, all these prime numbers occur to powers not less than those in the numbers a and b. Thus, s is divisible by ab. □

Let's also recall the *Euclidean algorithm* for finding the $\gcd$ of two numbers: $\gcd(a,b) = \gcd(a-b,b)$ if $a > b$ (at each step, the larger number is replaced by its difference from the smaller one). Its natural improvement is the *extended Euclidean algorithm*, where numbers are not subtracted but divided with a remainder. For example, $\gcd(315, 100) = \gcd(100, 15) = \gcd(15, 10) = \gcd(5, 10) = \gcd(0, 5) = 5$.

Example 5.3. What is the value of

$$\frac{(a+b)(b+c)(c+a)}{abc},$$

if it is known that the expression is equal to an integer and a,b, and c are pairwise coprime natural numbers?

Solution: Let's check if there are equal numbers among a, b, c. Two scenarios are possible:

1) Among them, there are two equal numbers. Without the loss of generality (due to the symmetry of the conditions), let a and b be equal. Since they must be coprime, this is possible only when $a = b = 1$. Therefore,

$$\frac{(a+b)(b+c)(c+a)}{abc} = \frac{2(c+1)^2}{c}.$$

However, $c+1$ and c are always coprime, so both $(c+1)^2$ and c are coprime. However, $2(c+1)^2$ is divisible by c (as the fraction must equal an integer), meaning that 2 is divisible by c. Hence, $c = 1$ or $c = 2$. In the first case, the value

$$\frac{(a+b)(b+c)(c+a)}{abc}$$

is equal to 8; in the second case, it is 9.

2) Assume that numbers a, b, and c are different. Without the loss of generality, we can assume that $a < b < c$. Since a, b, and c are pairwise coprime, neither $a + c$

nor $b+c$ can be divisible by c. The fraction must equal an integer, and $a+b$ is the only remaining multiplier in the numerator after $a+c$ and $b+c$; therefore, $a+b$ is divisible by $c \Leftrightarrow a+b=kc$ for some integer k. Using the same logic, we can make the statement that $a+c=l\cdot b$ for some integer l.

Since $a+b<2c$, and $a+b=kc$, then $a+b=c$, implying that $a=c-b$. From this,

$$a+c=(c-b)+c=lb \Leftrightarrow 2c=b(l+1).$$

The right side of the final expression is divisible by b, so the left side must also be divisible by b, so the left side of the expression must also be divisible by b. Since c and b are coprime, 2 must be divisible by b; thus $b=1$ or $b=2$. However $1\leqslant a<b$, so b can only be equal to 2. $a<b$ therefore $a=1$. Then, $c=a+b=3$. In this case,

$$\frac{(a+b)(b+c)(c+a)}{abc}=\frac{3\cdot 5\cdot 4}{1\cdot 2\cdot 3}=10.$$

□

Example 5.4. Find the $\gcd$ of the numbers $\underbrace{11\ldots11}_{m}$ and $\underbrace{11\ldots11}_{n}$, where $m>n$.

Solution: Let's use the Euclidean algorithm with modification. First, subtract the second number from the first one $(\underbrace{11\ldots11}_{n},\underbrace{11\ldots11}_{m}) = (\underbrace{11\ldots11}_{m-n}\underbrace{00\ldots00}_{n},\underbrace{11\ldots11}_{n})$. Now, notice that the $\gcd$ of these two numbers cannot be divisible by 2 or 5. Therefore,

$$(\underbrace{11\ldots11}_{m-n}\underbrace{00\ldots00}_{n},\underbrace{11\ldots11}_{n})=(\underbrace{11\ldots11}_{m-n},\underbrace{11\ldots11}_{n}).$$

We see that after one step of the modified algorithm, the number of ones in our numbers has changed the same way as it would have in the normal Euclidean algorithm. One can say that one step of the modified algorithm is analogous to one step of the Euclidean algorithm applied to the number of ones in the numbers. Since applying the normal Euclidean algorithm results in the $\gcd$ of two numbers, m and n, applying the modified algorithm will produce a number consisting of $\gcd(m,n)$ ones. □

Problem Set

Problem 5.1. (COM — 2003.7.3): There are two villages, Marino and Roschino. Every km between them, there is a sign on the side of the road. On one side of the pole, the distance to Marino is written, and on the other side, the distance to Roschino is written. While walking on the road, Bobik calculated the greatest common divisor for each sign. The numbers Bobik obtained were only 1, 3, and 5 (each occurring at least once. Find the distance between the two villages).

Problem 5.2. (COM — 2012.7.4): Let's call two natural numbers a and b *friends* if their product is a perfect square. Prove that if a is a friend of b, then a is also a friend of $\gcd(a, b)$.

Problem 5.3. (TOT — 1996/1997.10-11.2): There are two natural numbers a and b. It is known that $a^2 + b^2$ is divisible by ab. Prove that $a = b$.

Problem 5.4. (MMO — 1964.9.2): Prove that the product of two consecutive natural numbers cannot be equal to an integer raised to a power.

Problem 5.5. (TOT — 2014/2015.8-9.2): Are there ten pairwise distinct natural numbers such that their arithmetic mean is greater than their greatest common divisor:

a) exactly six times;
b) exactly five times?

Problem 5.6. (MMO — 1935.8.1): Prove that

$$\operatorname{lcm}(a, b, c) \cdot \gcd(a, b) \cdot \gcd(b, c) \cdot \gcd(c, a) = \gcd(a, b, c) \cdot abc.$$

Problem 5.7. (TOT — 1983/1984.7-8.2): Find all natural numbers k that can be represented as the sum of two coprime numbers different from 1.

Problem 5.8. (TOT — 1998/1998.8-9.1): a) Prove that if $\gcd(a, a+5) = \gcd(b, b+5)$ (a, b are natural), then $a = b$.

b) Can $\gcd(a, b)$ and $\gcd(a + c, b + c)$ be equal (a, b, c are natural)?

Problem 5.9. (AIME — 2023.II.5): Let S be the set of all positive rational numbers r such that when the two numbers r and $55r$ are written as fractions in lowest terms, the sum of the numerator and denominator of one fraction is the same as the sum of the numerator and denominator of the other fraction. The sum of all the elements of S can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Problem 5.10. (CIME — 2020.I.3): In a math competition, all teams must consist of between 12 and 15 members, inclusive. Mr. Beluhov has $n > 0$ students, and he realizes that he cannot form teams, so each of his students is on exactly one team. Find the sum of all possible values of n.

Problem 5.11. (AMC — 2020.10A.24): Let n be the least positive integer greater than 1000 for which $\gcd(63, n + 120) = 21$ and $\gcd(n + 63, 120) = 60$. What is the sum of the digits of n?

(A) 12 (B) 15 (C) 18 (D) 21 (E) 24

Problem 5.12. (AMC — 2018.10A.22): Let a, b, c, and d be positive integers such that $\gcd(a,b) = 24$, $\gcd(b,c) = 36$, $\gcd(c,d) = 54$, and $70 < \gcd(d,a) < 100$. Which of the following must be a divisor of a?

(A) 5 (B) 7 (C) 11 (D) 13 (E) 17

Problem 5.13. (AMC — 2018.10B.23): How many ordered pairs (a,b) of positive integers satisfy the equation

$$ab + 63 = 20\,\mathrm{lcm}(a,b) + 12\gcd(a,b)$$

where $\gcd(a,b)$ denotes the greatest common divisor of a and b, and $\text{lcm}(a,b)$ denotes their least common multiple?

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

Problem 5.14. (AMC — 2016.10A.25): How many ordered triples (x, y, z) of positive integers satisfy $\text{lcm}(x, y) = 72$, $\text{lcm}(x, z) = 600$ and $\text{lcm}(y, z) = 900$?

(A) 15 (B) 16 (C) 24 (D) 27 (E) 64

Problem 5.15. (AMC — 2015.10A.25): Let S be a square of side length 1. Two points are chosen independently at random on the sides of S. The probability that the straight-line distance between the points is at least $\frac{1}{2}$ is $\frac{a-b\pi}{c}$, where a, b, and c are positive integers with $gcd(a, b, c) = 1$. What is $a + b + c$?

(A) 59 (B) 60 (C) 61 (D) 62 (E) 63

Problem 5.16. (UNCO Math Contest — 2009.II.3): An army of ants is organizing a march to the Obama inauguration. If they form columns of 10 ants, there are 8 left over. If they form columns of $7, 11$, or 13 ants, there are 2 left over. What is the smallest number of ants that could be in the army?

Problem 5.17. (AMC — 2002.12B.12): For how many integers n is $\dfrac{n}{20-n}$ the square of an integer?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 10

Skill Assessment Problems

Skill Assessment Problem 5.1. A different natural number is written in each of the vertices of a cube. On each edge of the cube, the greatest common divisor of the two numbers written on either end of the edge is recorded. Could the sum of all the numbers written on the vertices be equal to the sum of all the numbers written on the edges?

Skill Assessment Problem 5.2. Find the greatest common divisor of the numbers $2^n - 1$ and $2^m - 1$.

Solutions to Skill Assessment Problems

Solution to Problem 5.1: Let's prove that the inequality $a + b \geqslant 3\gcd(a, b)$ holds for any two different numbers. Let $\gcd(a, b) = d$, then $a = a_1 d$, $b = b_1 d$, where a_1, b_1 are natural numbers and $a_1 \neq b_1$. Then

$$a + b = (a_1 + b_1)d \geqslant 3d \Leftrightarrow a_1 + b_1 \geqslant 3,$$

which is true by definition. Moreover, equality can only be achieved if $a = 2b$ or $b = 2a$.

Therefore, the sum of any two numbers in neighboring vertices is at least three times the number on the edge between them. Summing up all these inequalities (writing an inequality on each edge), each number in the vertex will be counted exactly 3 times. Therefore, the sum of the numbers in the vertices is greater than or equal to the sum of the numbers on the edges, and equality can only be achieved when all inequalities being summed are equalities $a + b = 3\gcd(a,b)$. Therefore, any two numbers in neighboring vertices differ by a factor of 2.

Let a be a number in some vertex. There are 3 neighboring vertices with each vertex; the neighboring numbers can only be $a/2$ or $2a$, and the third number must repeat of one of the other two for the inequality to be an equality. We have 8 different numbers on the 8 vertices; hence, neighboring vertices cannot all differ by a factor of 2; therefore, the sum of all the numbers on the vertices cannot be equal to the sum of all the numbers on the edges. □

Solution to Problem 5.2: Let, without loss of generality, $n > m$. Let's use the modified Euclidean algorithm:

$$\gcd(2^n - 1, 2^m - 1) = \gcd(2^n - 2^m, 2^m - 1) = \gcd(2^{n-m} - 1, 2^m - 1).$$

Here, we used that the second of the obtained numbers is odd, so the first number can be divided by 2^m without changing the gcd.

One step of such a modified Euclidean algorithm performs the same operations with powers as the regular Euclidean algorithm — with numbers. Therefore, as a result of the work of such an algorithm, the number $2^{\gcd(n,m)} - 1$ will be obtained, which is what we needed to find. □

Prime Numbers

> Mersenne Primes: These are prime numbers that can be written in the form $2^p - 1$, where p is also a prime number. For example, 3, 7, and 31 are Mersenne primes.

The largest known prime number is a Mersenne prime with 24,862,048 digits! Mathematicians love hunting for these «giant» primes.

Theory and Practice

This chapter combines mainly problems devoted to examples and counterexamples in number theory.

Example 6.1. If we substitute numbers $n = 1, 2, 3, 4, 5$ into the expression $n^2 + n + 41$, prime numbers $43, 47, 53, 61, 71$ are obtained. Is it true that substituting any natural number n into this expression will result in a prime number?

Solution: Encountering such a problem in a competition, you might be horrified because there are no explicit signs that a number is prime (while proving that a number is not prime is sometimes very straightforward). So, this problem statement is an obvious hint — you don't need to prove that all numbers of this form are prime; just provide an example of a non-prime number.

Notice that the first two terms are divisible by n, and the third term is 41. If we take n divisible by 41, the sum of the numbers will be divisible by 41, forming a counterexample, which completes the solution to the problem. □

Example 6.2. Find all prime numbers p such that among the numbers from 1 to p inclusive, the number of composite numbers is exactly twice the number of primes.

Solution: Write down all prime numbers not exceeding 59. There are 17 of them:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59.$$

It can be found that the only suitable p is 37.

Consider remainders modulo 6 for any six consecutive numbers, the first of which is at least 4. These remainders will be $0, 1, 2, 3, 4, 5$. Numbers that give remainders $0, 2, 3$, and 4 when divided by 6 cannot be prime since they are divisible by 2 or 3. Thus, among any 6 consecutive numbers, the first of which is at least 4, there are at most 2 primes.

Consider some number $p_1 > 59$. There will be 17 prime numbers from 1 to 59 inclusive. From 60 to p_1 inclusive, there will be $p_1 - 60 + 1$ numbers, that is, at most $\frac{p_1-60+1}{6}$ blocks of six numbers plus the remaining numbers after the last block (in which a number can be prime only if it leaves a remainder of 1 when divided by 6, since if it is a remainder of 5 when divided by 6, then it is in a complete block). Thus, there are at most $2 \cdot \frac{p_1-60+1}{6} + 1$ prime numbers. Therefore, from 1 to p_1 inclusive, there will be at most

$$17 + 2 \cdot \frac{p_1 - 60 + 1}{6} + 1 = \frac{p_1 - 5}{3}$$

prime numbers. Then, there will be at least $p_1 - 1 - \frac{p_1-5}{3} = \frac{2p_1+2}{3}$ composite numbers (since 1 is neither prime nor composite). Thus, the number of composite numbers will be more than twice the number of primes. Hence, there are no other solutions besides the found $p = 37$. □

The solution to this problem is thanks to the observation that prime numbers are initially quite frequent, and then they become less and less frequent, which raises a reasonable question — could it be that at some point, they just end? In other words, is the set of prime numbers infinite, or is there a largest prime number? It turns out that answering this question is not that difficult. Suppose the set of prime numbers is finite. Then consider a number equal to the product of all prime numbers, plus one: $p_1 p_2 \ldots p_n + 1$. It cannot be prime since it is greater than any of them. However, when divided by any prime number, this number gives a remainder of 1, so it must be prime. This contradiction proves the infinity of the set of prime numbers.

Example 6.3. Euclid's proof of the infinitude of the set of prime numbers suggests defining the Euclidean numbers recursively:

$$e_1 = 2, e_n = e_1 e_2 \ldots e_{n-1} + 1 \ (n \geqslant 2).$$

Are all numbers e_n prime?

Solution: No, because $e_5 = 1807 = 13 \cdot 139$. □

Example 6.4. Prove that $p^2 - 1$ is divisible by 24 if p is a prime number and $p > 3$.

Solution: Apply the difference of squares formula: $p^2 - 1 = (p-1)(p+1)$. Since p is prime and $p > 3$, p is odd.

Possible remainders of p modulo 4: 1 and 3. Then the remainders of $p-1$ and $p+1$ modulo 4 will be 2 and 0, i.e., $(p-1)(p+1)$ is guaranteed to be divisible by 8.

Similarly, with remainders modulo 3, the remainder of p divided by 3 is 1 or 2, since it is prime and greater than 3, but then one of the numbers $p-1$ or $p+1$ is guaranteed to be divisible by 3. That is, $(p-1)(p+1)$ is divisible by both 3 and 8, and therefore divisible by 24. □

Example 6.5. Esther wrote down several prime numbers, using each digit from 1 to 9 exactly once. The sum of these prime numbers turned out to be 225. Using exactly the same digits once, can you write down several prime numbers so that their sum is less?

Solution: All even digits, except 2, must be in the tens place (otherwise, the corresponding number will not be prime). The sum will be minimal if all other numbers are in the unit's place. For example,

$$207 = 2 + 3 + 5 + 41 + 67 + 89,$$

$$207 = 2 + 3 + 5 + 47 + 61 + 89,$$

$$207 = 2 + 5 + 7 + 43 + 61 + 89.$$

□

Problem Set

Problem 6.1. (MF — 2013.6.1): Esther multiplied a certain number by 10 and got a prime number. Jean also multiplied the same number by 15, but still got a prime number. Can it be that neither of them made a mistake?

Problem 6.2. (MF — 2001.7.1): In the Guinness Book of Records, it is written that the largest known prime number is

$$23021^{377} - 1.$$

Is it a typo?

Problem 6.3. (MF — 2017.6.2): There are two cards, 4 digits are written on the cards, one digit on both sides of each card. Is it possible that all two-digit numbers that can be formed by placing the cards side by side are prime? It is not allowed to flip the digits upside down, i.e., turn a 6 into a 9 and vice versa.

Problem 6.4. (COM — 2008.7.3): Mrs. Owless asked Leo to write down all nine-digit numbers composed of different digits. Leo forgot how to write the digit 7, so she only wrote down those nine-digit numbers in which this digit is absent. Then Mrs. Owless suggested that she cross out six digits from each number in such a way that the remaining three-digit number is prime. Leo immediately claimed that this was not possible for all the numbers she wrote. Is he correct?

Problem 6.5. (COM — 2015.7.4): Jean wants to write 2015 numbers around in a circle in such a way that for any two neighboring numbers, the result of dividing the larger number by the smaller number is a prime number. Jean's friend Esther claims that this is impossible. Is Esther correct?

Problem 6.6. (COM — 2005.6.6): Is the number 1111112111111 a prime number?

Problem 6.7. (LT — 1989.7.13): Find all prime numbers that cannot be written as the sum of two composite numbers.

Problem 6.8. (Mock AIME — 2005-2006.5.10): Find the smallest positive integer n such that $\binom{2n}{n}$ is divisible by all the primes between 10 and 30.

Problem 6.9. (AMC — 2018.10B.11): Which of the following expressions is never a prime number when p is a prime number?

(A) $p^2 + 16$ (B) $p^2 + 24$ (C) $p^2 + 26$ (D) $p^2 + 46$ (E) $p^2 + 96$

Problem 6.10. (AIME — 2015.I.3): There is a prime number p such that $16p + 1$ is the cube of a positive integer. Find p.

Problem 6.11. (UNCO Math Contest — 2006.II.8): Find all positive integers n such that $n^3 - 12n^2 + 40n - 29$ is a prime number. For each of your values of n compute this cubic polynomial, showing that it is, in fact, a prime.

Skill Assessment Problems

Skill Assessment Problem 6.1. Find all natural numbers p such that both p and $5p+1$ are prime.

Skill Assessment Problem 6.2. Prove that any prime number greater than 3 can be expressed in one of two forms: $6n+1$ or $6n-1$, where n is a natural number.

Skill Assessment Problem 6.3. Prove that $p^2 - q^2$ is divisible by 24 if p and q are prime numbers greater than 3.

Skill Assessment Problem 6.4. Does a number n exist so that the numbers
a) $n-96$, n, $n+96$;
b) $n-1996$, n, $n+1996$
are all prime simultaneously? (Consider all prime numbers as positive)

Solutions of Skill Assessment Problems

Solution to Problem 6.1: If p is prime and $p > 2$, then p must be odd, and then $5p+1$ must be even and greater than 2. If $p > 2$, then $5p+1$ can never be prime. Therefore, the only prime number that p can be is 2. $2 \cdot 5 + 1 = 11$. 11 is a prime number, and 2 is a prime number, so the only solution is $p = 2$. □

Solution to Problem 6.2: In the theoretical part of the chapter, we emphasized the observation that any prime number greater than 3 can have only remainders 1 or 5 when divided by 6. This directly implies the statement of the problem. □

Solution to Problem 6.3: Applying the difference of squares formula: $p^2 - q^2 = (p-q)(p+q)$. Since p and q are primes and $p, q > 3$, both p and q are odd.

Possible remainders when p and q are divided by 4, due to the odd nature of primes greater than 3, can be 1 and 3. Then, the remainders of $p-q$ and $p+q$, when divided by 4, can be the following pairs:

$$\begin{aligned} 1-3 &\equiv 2 \pmod 4 \text{ and } 1+3 \equiv 0 \pmod 4, \text{ or} \\ 1-1 &\equiv 0 \pmod 4 \text{ and } 1+1 \equiv 2 \pmod 4, \text{ or} \\ 3-3 &\equiv 0 \pmod 4 \text{ and } 3+3 \equiv 2 \pmod 4, \text{ or} \\ 3-1 &\equiv 2 \pmod 4 \text{ and } 3+1 \equiv 0 \pmod 4, \text{ i.e.,} \end{aligned}$$

always 2 or 0. Therefore, $(p-q)(p+q)$ is guaranteed to be divisible by 8.

Similarly, considering remainders modulo 3, where p and q remainders are 1 or 2 since these numbers are primes greater than 3, one of $p-q$ or $p+q$ is guaranteed to be divisible by 3. Thus, $(p-q)(p+q)$ is divisible by both 3 and 8, and due to the mutual primality of 3 and 8, it is divisible by 24. □

Solution to Problem 6.4: (a) Yes, for example, $n = 101$ works, as $5, 101, 197$ are primes.

(b) As we found in the previous problem, among the numbers $p-q$, p, $p+q$, at least one must be divisible by 3 if q is not divisible by 3. Therefore, for all the numbers among $n-1996$, n, $n+1996$ to be primes, $n-1996$ must be divisible by 3; therefore, $n-1996=3$. However, this leads to $n+1996=3995$, a composite number. $\square$

Fractions

> Collatz Conjecture: This is a conjecture which states that no matter what positive integer you start with, if you follow this rule:
> If the number is even, divide it by 2.
> If the number is odd, multiply it by 3 and add 1.
> You will eventually reach 1.

The Collatz Conjecture is like the «hot potato» of number theory! Many mathematicians pass it around, hoping someone will find a way to prove it true or false.

Theory and Practice

What is a fraction? You studied fractions back in primary school. Some word problems related to fractions have already been solved in previous chapters of this book. In this chapter, we will move to a more formal mathematical language and explore how fractions can be used in number theory.

Let's recall some definitions in a more formal language.

Definition 5. *Irreducible fraction* is a fraction represented as $\frac{m}{n}$, where m is an integer, n is a natural number, and $\gcd(m, n) = 1$. For an integer m, the representation of an irreducible fraction is $m = \frac{m}{1}$.

Definition 6. *Proper irreducible fraction* is a fraction represented as $\frac{m}{n}$, where m and n are natural numbers, $\gcd(m, n) = 1$, and $m < n$.

Note the following important observation: if $m, n, a > 0$ and $m < n$, then

$$\frac{m}{n} < \frac{m+a}{n+a}.$$

Indeed:

$$\frac{m+a}{n+a} - \frac{m}{n} = \frac{(m+a)n - (n+a)m}{n(n+a)} = \frac{a(n-m)}{n(n+a)} > 0.$$

Since the sign changes when positive fractions are reversed in the inequality, the following is also true:

$$\frac{n}{m} > \frac{n+a}{m+a}.$$

Example 7.1. (MMO — 2016.8.1) Can the number $\frac{1}{10}$ be represented as the product of ten positive proper fractions? (That is, expressions of the form $\frac{p}{q}$, where p and q are natural numbers, and $p < q$.)

Solution: Intuitively, we would like the numerator of each fraction to be either one or to be canceled with the denominator of the previous fraction. Then, the product takes the form:

$$\frac{1}{10} = \frac{n_1}{n_2} \cdot \frac{n_2}{n_3} \cdot \ldots \cdot \frac{n_{10}}{n_{11}} = \frac{n_1}{n_{11}}, \quad n_1 < n_2 < \ldots < n_{11}.$$

From this, we conclude that $n_{11} = 10n_1$. We need 11 distinct natural numbers, so we take $n_1 = 2$, which gives $n_{11} = 20$. We obtain the example

$$\frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{7}{8} \cdot \frac{8}{9} \cdot \frac{9}{10} \cdot \frac{10}{11} \cdot \frac{11}{20} = \frac{1}{10}.$$

Therefore, the answer to this problem is «yes». □

Example 7.2. (MMO — 1999.8.1): Arrange the following fractions in ascending order: 111110/111111, 222221/222223, 333331/333334.

Solution. Compare the fractions:

$$\frac{111110}{111111} = \frac{333330}{333333} < \frac{333330+1}{333333+1},$$

$$\frac{333331}{333334} = \frac{666662}{666668} < \frac{666662+1}{666668+1} = \frac{222221}{222223}.$$

Therefore,

$$\frac{111110}{111111} < \frac{333331}{333334} < \frac{222221}{222223}.$$

□

Example 7.3. (PVG 2016.5-6.3): Find all positive irreducible fractions where if the numerator and denominator are both increased by 12, then the value of the fraction triples.

Solution: Let's denote the numerator and denominator of our fraction as a and b, respectively. According to the condition:

$$3\frac{a}{b} = \frac{a+12}{b+12}, \Rightarrow 3ab + 36a = ab + 12b, \Rightarrow$$

$$a(b+18) = 6b, \Rightarrow \frac{a}{b} = \frac{6}{b+18}.$$

Suppose that b is not divisible by 2 or 3. In this case, the last fraction is irreducible, and $a = 6, b = b + 18$, which is impossible.

Let b be divisible by 2 but not by 3. Then $b = 2k$, where k is an integer not divisible by 3:

$$\frac{a}{2k} = \frac{6}{2k + 18} = \frac{3}{k + 9}.$$

The last fraction is irreducible, so $a = 3$, $2k = k + 9$, $k = 9$, $b = 2k = 18$, but since a and b must be coprime, these numbers do not fit.

Let b be divisible by 3 but not by 2. Then $b = 3k$, where k is an integer not divisible by 2:

$$\frac{a}{3k} = \frac{6}{3k + 18} = \frac{2}{k + 6}.$$

The last fraction is irreducible, so $a = 2$, $3k = k + 6$, $k = 3$, $b = 3k = 9$, i.e., $a = 2$, $b = 9$ is a solution.

Let b be divisible by both 2 and 3. Then $b = 6k$, where k is an integer:

$$\frac{a}{6k} = \frac{6}{6k + 18} = \frac{1}{k + 3}.$$

The last fraction is irreducible, so $a = 1$, $6k = k+9$, $k = \frac{9}{5}$, which is also not suitable.

So, the only fraction satisfying the condition is $\frac{2}{9}$. □

Example 7.4. After the lesson, Jean argued with Esther, claiming that he knows a natural number m such that the number $\frac{m}{3} + \frac{m^2}{2} + \frac{m^3}{6}$ is not an integer. Is Jean correct? Moreover, if he is, what could that number be?

Solution: Let's express the given expression as follows:

$$\frac{m}{3} + \frac{m^2}{2} + \frac{m^3}{6} = \frac{m}{3} + \frac{m^2 - m}{2} + \frac{m}{2} + \frac{m^3 - m}{6} + \frac{m}{6} =$$

$$= m + \frac{m(m-1)}{2} + \frac{m(m^2-1)}{6} =$$

$$= m + \frac{m(m-1)}{2} + \frac{m(m-1)(m+1)}{6}.$$

Among the numbers m and $m-1$, at least one is guaranteed to be divisible by 2, and among m, $m-1$, and $m+1$, at least one is guaranteed to be divisible by 2 and one by 3. Thus, each term in this sum is an integer, and their sum. So, sly Jean was wrong. $\square$

Problem Set

Problem 7.1. (MF — 1999.7.1): What is the largest possible fraction such that the sum of the numerator and the denominator (both natural numbers) is 101 and the fraction does not exceed $1/3$.

Problem 7.2. (COM — 2012.7.1): Six positive irreducible fractions are written. Some are proper fractions, and some are improper. The sum of all numerators is equal to the sum of all denominators. Is it guaranteed that there will be two numbers whose integer parts or fractional parts are the same?

Problem 7.3. (MF — 1997.6.2): In the Rinda papyrus (Ancient Egypt), among other information, there are decompositions of fractions into the sum of fractions with a numerator of 1, for example,

$$\frac{2}{73} = \frac{1}{60} + \frac{1}{219} + \frac{1}{292} + \frac{1}{x}.$$

One of the denominators here is replaced by the letter x. Find this denominator.

Problem 7.4. (MF — 2000.7.2): Jean wrote the fraction $10/97$. Jean can: 1) add any natural number to both the numerator and denominator simultaneously; 2) multiply the numerator and denominator by the same natural number simultaneously. Can Jean obtain a fraction equal to a) $1/2$; b) equal to 1?

Problem 7.5. (MF — 2004.6.3): Write down three proper irreducible fractions, the sum of which is an integer, and if each of these fractions is «reversed» (i.e., replaced with its reciprocal), then the sum of the resulting fractions will also be an integer.

b) Repeat this but with the added condition that all numbers are different non-repeating natural numbers, i.e., no numerator or denominator is equal to another numerator or denominator.

Problem 7.6. (COM — 2016.7.3): Mrs. Owless wrote down in order 2016 ordinary proper fractions: $1/2$, $1/3$, $2/3$, $1/4$, $2/4$, $3/4$, ...(including reducible ones). She colored red those fractions whose values were less than $1/2$, and the rest of the fractions were colored blue. By how much is the number of red fractions less than the number of blue fractions?

Problem 7.7. (MF — 2016.7.4): Fill in six different digits instead of asterisks so that all fractions are irreducible, and the equality holds:

$$\frac{*}{*}+\frac{*}{*}=\frac{*}{*}.$$

Problem 7.8. (COM — 2017.6.5): Can we replace the asterisks with digits from 1 to 9, taken once each, in the equality

$$\frac{*}{*}+\frac{*}{*}+\frac{*}{*}+\frac{*}{*}=*$$

so that equality becomes true?

Problem 7.9. (Mock AIME — 2007-2008.1.9): Let n represent the smallest integer that satisfies the following conditions:

1. $\frac{n}{2}$ is a perfect square. 2. $\frac{n}{3}$ is a perfect cube. 3.$\frac{n}{5}$ is a perfect fifth.

How many divisors does n have that are not multiples of 10?

Problem 7.10. (AMC — 2023.12A.17): Flora, the frog, starts at 0 on the number line and makes a sequence of jumps to the right. In any one jump, independent of previous jumps, Flora leaps a positive integer distance m with probability $\frac{1}{2^m}$.

What is the probability that Flora will eventually land at 10?

(A) $\frac{5}{512}$ (B) $\frac{45}{1024}$ (C) $\frac{127}{1024}$ (D) $\frac{511}{1024}$ (E) $\frac{1}{2}$

Problem 7.11. (AMC – 2018.10A.14): What is the greatest integer less than or equal to

$$\frac{3^{100}+2^{100}}{3^{96}+2^{96}}?$$

(A) 80 (B) 81 (C) 96 (D) 97 (E) 625

Problem 7.12. (AMC – 2012.10B.10): How many ordered pairs of positive integers (M, N) satisfy the equation $\frac{M}{6} = \frac{6}{N}$?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Problem 7.13. (AMC – 2011.10B.10): Consider the set of numbers $1,10,10^2, 10^3, \ldots, 10^{10}$. The ratio of the largest element of the set to the sum of the other ten elements of the set is closest to which integer?

(A) 1 (B) 9 (C) 10 (D) 11 (E) 101

Problem 7.14. (AHSME – 1960.8): The number $2.5252525\ldots$ can be written as a fraction. When reduced to the lowest terms, the sum of the numerator and denominator of this fraction is:

(A) 7 **(B)** 29 **(C)** 141 **(D)** 349 **(E)** none of these

Skill Assessment Problems

Skill Assessment Problem 7.1. Prove that the fraction $\frac{1}{n}$ can be represented as a finite decimal fraction only if the prime factorization of n contains only twos and fives.

Skill Assessment Problem 7.2. A new fraction is made by subtracting a number from the numerator and adding it to the denominator in the fraction $\frac{537}{463}$. The new fraction simplifies to $\frac{1}{9}$. Find the number subtracted from the numerator and added to the denominator.

Solutions to Skill Assessment Problems

Solution to Problem 7.1: Let's assume the given fraction can be represented as a finite decimal fraction. Equate the decimal and fractional forms:

$$\overline{0,a_1a_2\ldots a_k} = \frac{1}{n},$$

after cross-multiplication, we get $\overline{a_1a_2\ldots a_k} \cdot n = 10^k$. In the prime factorization of 10^k, there are only 2 and 5, so by the fundamental theorem of arithmetic, the prime factorization of n can only have these prime factors, which is what needs to be proved. □

Solution to Problem 7.2: Let's denote this number as n, then:

$$\frac{537-n}{463+n} = \frac{1}{9},$$
$$(537-n)\cdot 9 = 463+n, \Rightarrow\ 4833-9n = 463+n,$$
$$n = 437.$$

□

Numeral Systems: Basic Properties

> The base-60 system used by the ancient Babylonians still influences our modern timekeeping and geometry.

> Base-60 is the «legacy software» of number systems — old but still running in the background!

Theory and Practice

In the modern world, we use the decimal number system for counting. This means that if we can break down a number into «tens», «hundreds», «thousands», and so on, i.e., powers of ten. For example, $2018 = 2 \cdot 10^3 + 0 \cdot 10^2 + 1 \cdot 10^1 + 8 \cdot 10^0$.

In the decimal numeral system, each digit can be from 0 to 9, meaning any non-negative integer less than the base raised to the power of the position.

And what about, for example, the binary number system? This is the numbering system in which a computer «thinks». It does so because it is most straightforward to physically represent numbers in the binary system. In this system, there are only 2 possible digits. The digit 1 can correspond to the presence of current in a certain segment of a circuit (or a lit bulb in a vacuum tube computer), and the digit 0 represents its absence. The numerical representation of a binary number is interpreted as $\overline{a_n \ldots a_1 a_0}_2 = a_n 2^n + \ldots + a_1 2^1 + a_0 2^0$.

A generalization for a numeral system with a base of s is given by the formula: $\overline{a_n \ldots a_1 a_0}_s = a_n s^n + \ldots + a_1 s^1 + a_0 s^0$, where digits range from 0 to $s - 1$. If the base of the numbering system is greater than 10, usually Latin letters $A, B, \ldots$ are used for digits like «10», «11», and so on.

Definition 7. A *positional numeral system*, also known as *place-value notation* is a numeral system in which the value of each numerical sign (digit) in the representation of a number depends on its position (order).

The first mentions of positional systems date back to Sumerian and Babylonian works. The sexagesimal system, invented by the Sumerians in the 3rd millennium BCE, is a positional system with a base of 60. This system became widespread due to ancient and medieval astronomers, who used it primarily to represent fractions. Therefore, medieval scholars often referred to sexagesimal fractions as «astronomical». These fractions were used to record astronomical coordinates — angles, and this tradition has survived to this day. There are 60 minutes in one degree and 60 seconds in one minute.

The emergence of the decimal system is associated with counting on fingers. In medieval Europe, it spread through Italian merchants, who borrowed it from the inhabitants of Central Asia.

However, there are also non-positional numeral systems. For example, the Roman numeral system. We constantly encounter this system in life, as the numbers of centuries, millennia, etc., are always written in the Roman numeral system.

Let's remind ourselves of its structure.

The meanings of its «digits» are as follows:

I 1,
V 5,
X 10,
L 50,
C 100,
D 500,
M 1000.

The conversion from the decimal number system to the Roman numeral system is carried out as follows: First, the number needs to be decomposed into decimal digits, and then each digit is represented as follows:

I 1,
II 2,
III 3,
IV 4,
V 5,
VI 6,
VII 7,
VIII 8,
IX 9.

Similarly (but with different letters), other digits are represented. Then, all the symbols of each digit are written in a row, from left to right, starting from the larger ones.

For example, the number $1917 = 1000 + 900 + 10 + 7$ is written as MCMXVII.

Let's return to positional number systems. In general, it is clear that the above definition, through decomposition into digits, you can convert numbers from the decimal system to another and vice versa. This is done as follows: suppose we have the number 4231 that we want to convert to the ternary number system. First, write down all the powers of three that do not exceed this number: $3^0 = 1$, $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$, $3^6 = 729$, $3^7 = 2187$. Then start subtracting these powers from the original number, starting with the larger ones, as many times as possible: $4231 - 2187 = 2044$, $2044 - 729 \cdot 2 = 586$, $586 - 243 \cdot 2 = 100$, $100 - 81 = 19$, $19 - 9 \cdot 2 = 1$. Therefore,

$$4231 = 2187 + 729 \cdot 2 + 243 \cdot 2 + 81 + 9 \cdot 2 + 1 =$$
$$= 1 \cdot 3^7 + 2 \cdot 3^6 + 2 \cdot 3^5 + 1 \cdot 3^4 + 0 \cdot 3^3 + 2 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0 = 12210201_3.$$

But there is also another way, based on division with remainder: the number is successively divided by the base power, until it becomes 0, then all the obtained remainders, written in reverse order, form the number in the required number system. Let's illustrate this method using the same number:

$$\begin{aligned}
4231 : 3 &= 1410 \text{ (remainder } 1),\\
1410 : 3 &= 470 \text{ (remainder } 0),\\
470 : 3 &= 156 \text{ (remainder } 2),\\
156 : 3 &= 52 \text{ (remainder } 0),\\
52 : 3 &= 17 \text{ (remainder } 1),\\
17 : 3 &= 5 \text{ (remainder } 2),\\
5 : 3 &= 1 \text{ (remainder } 2),\\
1 : 3 &= 0 \text{ (remainder } 1),
\end{aligned}$$

Hence, $4231 = 12210201_3$.

To convert from a non-decimal number system to a decimal one, you can use these two methods applied in reverse. The first one hardly needs comments, and the second one is done as follows for a system with base s:

$$\overline{a_n \dots a_1 a_0}_s = \Big(\dots (a_n \cdot s + a_{n-1}) \cdot s + \dots + a_1 \Big) \cdot s + a_0.$$

For our example:

$$12210201_3 = \Big(((((1 \cdot 3 + 2) \cdot 3 + 2) \cdot 3 + 1) \cdot 3 + 0) \cdot 3 + 2) \cdot 3 + 0\Big) \cdot 3 + 1.$$

This gives 4231.

It is also worth noting the possibility of quickly converting from a system with base s to a system with base s^k, done as follows. First of all, the original number is divided into groups of k digits; starting from the right, each group is converted to the new number system, and the result should consist of 1 digit; the resulting digits are written in the order in which the groups of digits of the original number were written. Conversion from a base s^k system to a base s system is done in a similar way, in reverse order. Let's provide an example.

Example 8.1. Convert the number $37ba7af8_{16}$ from hexadecimal to octal.

Solution: It could take a lot of time and effort to solve this problem by first converting it to the decimal system and then to the octal system. Therefore, let's use the method described above. First, convert the given number to the binary system by representing each digit in binary, adding leading zeros to make each group have 4 digits:

$$3_{16} = 0011_2,\ 7_{16} = 0111_2,\ b_{16} = 1011_2,\ a_{16} = 1010_2,$$
$$7_{16} = 0111_2,\ a_{16} = 1010_2,\ f_{16} = 1111_2,\ 8_{16} = 1000_2.$$

Therefore, $37ba7af8_{16} = 110111101110100111101011111000_2$.

Now, convert the resulting binary number to the octal system by splitting it, starting from the end, into groups of 3 digits and converting them to octal digits:

$$110_2 = 6_8,\ 111_2 = 7_8,\ 101_2 = 5_8,\ 110_2 = 6_8,\ 100_2 = 4_8,$$
$$111_2 = 7_8,\ 101_2 = 5_8,\ 011_2 = 3_8,\ 111_2 = 7_8,\ 000_2 = 0_8.$$

Therefore, the number is 6756475370_8, which is the answer. □

Problem Set

Problem 8.1. (MF — 2008.6.1): Today is February 17, 2008. Esther noticed that in the notation of the date, the sum of the first four numbers is equal to the sum of the last four numbers. When will such a coincidence happen again for the last time this year?

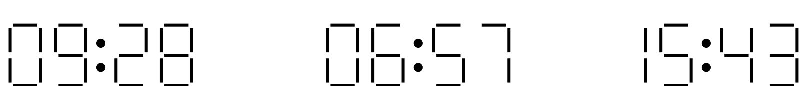
Problem 8.2. (MF — 2009.6.1): The year 2009 has a special property; by swapping around the digits of the number 2009, you cannot get a smaller four-digit number (numbers do not start with zero). When is the next year that will have the same property?

Problem 8.3. (MF — 2002.7.1): This year is 2002. 2002 is a palindrome, meaning the number reads the same left to right and right to left. The previous palindrome was 11 years ago (1991). What is the maximum number of non-palindrome years in a row (between 1000 and 9999)?

Problem 8.4. (MF — 1993.6.4): The sum of the digits of the number x is equal to a. The sum of the digits of the number a is equal to b. The sum of the digits of the number b is equal to c. Find the smallest value of x for which $x \neq a \neq b \neq c$ and $c = 2$.

Problem 8.5. (MF — 1991.6.5): Find all numbers which are equal to twice the sum of their digits.

Problem 8.6. (COM — 2013.7.6): In the figure, three examples of readings of working digital clocks are given. Each digit can be made up of maximum 7 «sticks», hence there are 28 «sticks» in the reading. What is the maximum number of «sticks» that can be removed so that the time can be determined unambiguously, i.e., no two different digits will have the same representation in «sticks»?



Problem 8.7. (COM — 2002.7.6): Some numbers can be represented as the sum $\overline{abc}+\overline{ab}+a$, while others cannot. (For example, the number 1101 is representable because $1101 = 993 + 99 + 9$. Numbers 220 and 1514 are not representable.) How many three-digit numbers can be represented as the sum $\overline{abc} + \overline{ab} + a$?

Problem 8.8. (COM — 2017.7.7): Alice thought of a two-digit number and told Max the product of the digits in the notation of this number and Leo the sum of these digits. The boys had the following dialogue.

Max: «I will guess the intended number in three attempts, but two may not be enough for me.»

Leo: «If so, then I will need four attempts for this, but three may not be enough.»

What number was told to Leo?

Problem 8.9. (AMC — 2021.10A.11): For which of the following integers b is the base- b number $2021_b - 221_b$ not divisible by 3 ?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 7 **(E)** 8

Problem 8.10. Find the number of positive integers less than or equal to 2017 whose base-three representation contains no digit equal to 0.

Problem 8.11. (UNCO Math Contest — 2017.II.4): Harold writes an integer; its rightmost digit is 4. When Curious George moves that digit to the far left, the new number is four times the integer that Harold wrote. What is the smallest possible positive integer that Harold could have written?

Problem 8.12. (AIME — 2009.I.1): Call a 3-digit number geometric if it has 3 distinct digits that, when read from left to right, form a geometric sequence. Find the difference between the largest and smallest geometric numbers.

Problem 8.13. (AMC — 2021.10A.11): For which of the following integers b is the base-b number $2021_b - 221_b$ not divisible by 3 ?

(A) 3 (B) 4 (C) 6 (D) 7 (E) 8

Problem 8.14. (AMC — 2021.10B.13): Let n be a positive integer and d be a digit such that the value of the numeral $32d$ in base n equals 263, and the value of the numeral 324 in base n equals the value of the numeral $11d1$ in base six. What is $n + d$?

(A) 10 (B) 11 (C) 13 (D) 15 (E) 16

Problem 8.15. (AMC — 2019.10B.12): What is the greatest possible sum of the digits in the base-seven representation of a positive integer less than 2019?

(A) 11 (B) 14 (C) 22 (D) 23 (E) 27

Problem 8.16. (AMC — 2018.10A.18): How many nonnegative integers can be written in the form

$$a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 + a_1 \cdot 3^1 + a_0 \cdot 3^0,$$

where $a_i \in \{-1,0,1\}$ for $0 \leqslant i \leqslant 7$?

(A) 512 (B) 729 (C) 1094 (D) 3281 (E) 59,048

Problem 8.17. (AMC — 2018.10A.25): For a positive integer n and nonzero digits a, b, and c, let A_n be the n -digit integer each of whose digits is equal to a; let B_n be the n-digit integer each of whose digits is equal to b, and let C_n be the $2n$-digit (not n -digit) integer each of whose digits is equal to c. What is the greatest possible value of $a + b + c$ for which there are at least two values of n such that $C_n - B_n = A_n^2$?

(A) 12 (B) 144 (C) 16 (D) 18 (E) 20

Problem 8.18. (AMC – 2018.10B.21): Mary chose an even 4-digit number n. She wrote down all the divisors of n in increasing order from left to right: $1,2,\ldots,\frac{n}{2},n$. At some moment, Mary wrote 323 as a divisor of n. What is the smallest possible value of the next divisor written to the right of 323 ?

(A) 324 (B) 330 (C) 340 (D) 361 (E) 646

Problem 8.19. (AMC – 2017.10B.23): Let $N = 123456789101112\ldots4344$ be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

(A) 1 (B) 4 (C) 9 (D) 18 (E) 44

Problem 8.20. (AMC – 2016.10B.6): Laura added two three-digit positive integers. All six digits in these numbers are different. Laura's sum is a three-digit number S. What is the smallest possible value for the sum of the digits of S?

(A) 1 (B) 4 (C) 5 (D) 15 (E) 20

Problem 8.21. (AMC – 2016.10B.24): How many four-digit integers $abcd$, with $a \neq 0$, have the property that the three two-digit integers $ab < bc < cd$ form an increasing arithmetic sequence? One such number is 4692, where $a = 4, b = 6, c = 9$, and $d = 2$.

(A) 9 (B) 15 (C) 16 (D) 17 (E) 20

Problem 8.22. (AMC – 2015.10A.18): Hexadecimal (base-16) numbers are written using numeric digits 0 through 9 as well as the letters A through F to represent 10 through 15. Among the first 1000 positive integers, there are n whose hexadecimal representation contains only numeric digits. What is the sum of the digits of n?

(A) 17 (B) 18 (C) 19 (D) 20 (E) 21

Problem 8.23. (AMC — 2014.10A.20): The product $(8)(888\ldots8)$, where the second factor has k digits, is an integer whose digits have a sum of 1000. What is k ?

(A) 901 (B) 911 (C) 919 (D) 991 (E) 999

Problem 8.24. (AMC — 2013.10A.13): How many three-digit numbers are not divisible by 5, have digits that sum to less than 20, and have the first digit equal to the third digit?

(A) 52 (B) 60 (C) 66 (D) 68 (E) 70

Problem 8.25. (AMC — 2013.10A.19): In base 10, the number 2013 ends in the digit 3. In base 9, on the other hand, the same number is written as $(2676)_9$ and ends in the digit 6. For how many positive integers b does the base-b -representation of 2013 end in the digit 3?

(A) 6 (B) 9 (C) 13 (D) 16 (E) 18

Problem 8.26. (AMC — 2013.10B.18): The number 2013 has the property that its unit digit is the sum of its other digits, that is $2+0+1=3$. How many integers less than 2013 but greater than 1000 have this property?

(A) 33 (B) 34 (C) 45 (D) 46 (E) 58

Problem 8.27. (AMC — 2013.10B.25): Bernardo chooses a three-digit positive integer N and writes both its base-5 and base-6 representations on a blackboard. Later, LeRoy sees the two numbers Bernardo has written. Treating the two numbers as base-10 integers, he adds them to obtain an integer S. For example, if $N=749$, Bernardo writes the numbers $10{,}444$ and $3{,}245$, and LeRoy obtains the sum $S=13{,}689$. For how many choices of N are the two rightmost digits of S, in order, the same as those of $2N$?

(A) 5 (B) 10 (C) 15 (D) 20 (E) 25

Problem 8.28. (AMC — 2011.10A.13): How many even integers are there between 200 and 700 whose digits are all different and come from the set 1,2,5,7,8,9?

(A) 12 (B) 20 (C) 72 (D) 120 (E) 200

Problem 8.29. (AMC — 2009.10A.5): What is the sum of the digits of the square of 111111111?

(A) 18 (B) 27 (C) 45 (D) 63 (E) 81

Problem 8.30. (AIME — 2006.I.3): Find the least positive integer such that when its leftmost digit is deleted, the resulting integer is $\frac{1}{29}$ of the original integer.

Problem 8.31. (UNM-PNM — 2017.II.2): Suppose A, R, S, and T all denote distinct digits from 1 to 9. If $\sqrt{STARS} = SAT$, what are A, R, S, and T?

Problem 8.32. (Pan African MO — 2003.3): Does there exist a base in which the numbers of the form: $10101, 101010101, 1010101010101, \ldots$ are all prime numbers?

Problem 8.33. (AHSME — 1973.6): If 554 is the base b representation of the square of the number whose base b representation is 24, then b, when written in base 10, equals

(A) 6 (B) 8 (C) 12 (D) 14 (E) 16

Skill Assessment Problems

Skill Assessment Problem 8.1. Convert the number 321 from decimal to quinary.

Skill Assessment Problem 8.2. Convert the number 3210_4 to the decimal system.

Skill Assessment Problem 8.3. Convert the number jpg_{27} to the base-9 numeral system.

Solutions to Skill Assessment Problems

Solution to Problem 8.1: Write: $321 - 125 \cdot 2 = 71$, $71 - 25 \cdot 2 = 21$, $21 - 5 \cdot 4 = 1$. Then:

$$321 = 5^3 \cdot 2 + 5^2 \cdot 2 + 5^1 \cdot 4 + 5^0 \cdot 1 = 2241_5.$$

□

Solution to Problem 8.2: Write: $3210_4 = 4^3 \cdot 3 + 4^2 \cdot 2 + 4^1 \cdot 1 + 4^0 \cdot 0 = 228_{10}$. □

Solution to Problem 8.3: Use the method described in the theoretical part of this chapter. First, convert the number to the ternary system:

$$j_{27} = 201_3,\ p_{27} = 221_3,\ g_{27} = 121_3,\ \Rightarrow\ jpg_{27} = 201221121_3.$$

Then, convert it to the base-9 numeral system:

$$2_3 = 2_9,\ 01_3 = 1_9,\ 22_3 = 8_9,\ 11_3 = 4_9,\ 21_3 = 7_9,$$

$$\Rightarrow\ 201221121_3 = 21847_9.$$

□

Divisibility: Concepts, Theorems, and Proofs

> Ramanujan's Taxi Cab Number: The number 1729 is known as the Hardy-Ramanujan number or the «taxi cab number» because it is the smallest number expressible as the sum of two cubes in two different ways: $1^3 + 12^3 = 9^3 + 10^3 = 1729$.

1729 is like the «multi-tasker» of numbers — it can do more than one thing at a time and still be unique!

Theory and Practice

Let us recall Bezout's Identity, which we will still leave without proof — it will be given in the chapter devoted to the Euclidian algorithm.

Theorem 2 (Bézout's Identity)**.** For any a, b such that $a \neq 0$ or $b \neq 0$, there exist numbers u, v such that

$$au + bv = \gcd(a, b).$$

Property 1. If $\gcd(a, b) = 1$ and $bc \div a$, then $c \div a$.

Proof. Using Bézout's Identity, if $\gcd(a, b) = 1$, then there exist integers u and v such that $au + bv = 1$. Multiplying it by c, we get $acu + bcv = c$, where a divides the left side. Thus, a also divides the right side, implying $c \div a$. □

Property 2. If the integer a is divisible by coprime integers m and n, then a is divisible by mn.

Proof using fundamental theorem of arithmetic. Factorize m and n into prime numbers. Assuming knowledge of the fundamental theorem of arithmetic (though we do not prove it here), as a is divisible by m and n, it is also divisible by all their divisors, including the product of all the divisors (which is equal to mn), since m and n have no common divisors other than 1. □

Proof without using fundamental theorem of arithmetic. We use Property 1 to prove Property 2.

If $a \vdots m$, then $a = mx$, where x is an integer. Thus, $mx \vdots n$, and $\gcd(m, n) = 1$. According to Property 2, $x \vdots n$, implying $x = ny$, where y is an integer. Therefore, $a = mny$, which means $a \vdots mn$. □

Property 3 (Euclid's Lemma)**.** If the number $m = ab$ (a, b are integers) is divisible by a prime number p, then at least one of the numbers a and b is divisible by p.

Proof using fundamental theorem of arithmetic. Assume that neither a nor b is divisible by p: $a = pk + l$, $b = pq + s$, where $0 < l, s < p$. Then $m = ab = p(ks + lq + pkq) + ls$.

If m is divisible by p, then ls is also divisible by p. However, by the fundamental theorem of arithmetic, the prime divisor of both l and s cannot be p, leading to a contradiction.

It is crucial in this property that p is a prime number. For example, for $p = 6$, the lemma is not true, as the product of 2 and 3 is divisible by 6, but neither factor is divisible by 6. ☐

Proof using Bézout's Identity. Let $m = ab$ be divisible by p, but a is not divisible by p. Then a and p are coprime. Using Property 1, if $\gcd(a, p) = 1$ and $ba \vdots a$, then $b \vdots p$. ☐

Example 9.1. Prove that if n is divisible by 2 and n is divisible by 3, then n is divisible by 6.

Proof. Let's try to prove this without using properties of divisibility. If $n \vdots 2$, then n can be represented as $n = 2k$. Thus, $3n = 6k \vdots 6$.

Similarly, $n = 3m$. Therefore, $2n = 6m \vdots 6$.

$n = 3n - 2n$, so it is divisible by 6 as the difference of numbers divisible by 6. ☐

For example, how can you now prove divisibility by 6 of a number that is represented in letter form?

Example 9.2. Prove that $n(n + 1)(n + 2) \vdots 6$.

Proof. It is obvious that among three consecutive numbers, at least one is divisible by 2, and exactly one is divisible by 3 (indeed, when divided by 3, the remainders can be $0, 1, 2$, so among three consecutive numbers, all three remainders are different, meaning one of them is divisible by 3).
Then the product of these numbers is divisible by 2 and 3, and hence, divisible by 6.

□

Example 9.3. Prove that $n^4 + 4$, where n is a natural number with $n \geqslant 2$, is never a prime number.

Proof. The statement suggests that we need to either factorize the given polynomial or prove its divisibility by some prime number. Simple methods of checking divisibility, like parity or divisibility by 3, do not yield results. Let's try factorizing the polynomial, utilizing the difference of squares (classical algebra comes in handy here):

$$n^4 + 4 = n^4 + 4n^2 + 4 - 4n^2 = (n^2 + 2)^2 - (2n)^2.$$

Thus, $n^4 + 4 = (n^2 + 2 - 2n)(n^2 + 2 + 2n)$.

Have we solved the problem? No, we forgot to check one more thing: what if one of the factors is 1, and the other is a prime number?

Let's check if this is possible. Since n is a natural number, the first factor is smaller than the second. Therefore, if one of the factors is 1, then $n^2 + 2 - 2n = 1$, implying $n = 1$, which contradicts the problem's condition. □

Example 9.4. Prove that $333^{555} + 555^{333}$ is divisible by 37.

Solution: We need to prove that the sum of the two terms is divisible by 37. What would be simpler? If both terms were divisible by 37. Factorizing 333 and 555, we find that both numbers are divisible by 37. Therefore, any natural power of these numbers is also divisible by 37, and the sum of numbers divisible by 37 is also divisible by 37. □

Example 9.5. The sequence is given by the formula: $a_n = 2^n + 5^n$, where n is a natural number. What is the maximum number of consecutive prime numbers in this sequence?

Solution: The keyword «maximum» indicates that this is an estimation + example problem.

Estimation: We can use the formula for factoring the sum of odd powers:

$$a^{2n+1} + b^{2n+1} = (a+b)(a^{2n} - a^{2n-1}b + \ldots - ab^{2n-1} + b^{2n}).$$

Using this formula, it is evident that every second element of the sequence is divisible by 7.

Notice that the numbers in our sequence increase, and there is only one prime number divisible by 7 — which is 7 itself. Therefore, starting from $n = 3$, every second number will be composite, as it will be divisible by 7 and also greater than 7. Hence, starting from $n = 3$, we won't find even two consecutive prime numbers. There is one more interval of length two left, $n = 1, n = 2$. Therefore, the estimation is proven: there cannot be more than 2 consecutive prime numbers.

Example: It is constructed straightforwardly: there is only one interval of length two, and we take it. Checking $n = 1, 2$, we see that the sequence values are 7 and 29, which are prime numbers. □

Example 9.6. (MMO — 1946.9.3): Prove that $n^2 + 3n + 5$ is never divisible by 121.

Solution: First, let's prove that for most values of n, the given expression is not divisible by 11. Indeed,

$$n^2 + 3n + 5 = (n+7)(n-4) + 33.$$

Both brackets either divide or do not divide by 11 simultaneously because their difference is 11. If the number $(n+7)(n-4)$ is not divisible by 11, then $(n+7)(n-4)+33$ is also not divisible by 11. If $(n+7)(n-4)$ is divisible by 11, then must also be divisible by 121. But in that case, $(n+7)(n-4)+33$ is not divisible by 121. □

Example 9.7. Find the 23rd digit from the end of the number $100!$

Solution: Intuitively, we can understand that the number $100!$ ends with a sufficiently large number of zeros. The following numbers give us zeros at the end of the factorial contribution:

1. Zeros from numbers $10, 20, 30, 40, 60, 70, 80, 90$, giving a total of 8 zeros.

2. Pairs of numbers 4 and 25, 2 and 50, 8 and 75, and the number 100 contribute 2 zeros each, resulting in another 8 zeros.

3. Multiples of 5 paired with even numbers contribute one zero each. There are a total of 20 multiples of 5, from which we've used 12. So, 8 numbers are left, contributing another 8 zeros.

In total, the number $100!$ should end with 24 zeros, meaning the 23rd digit from the end is zero. □

Problem Set

Problem 9.1. (LEO — 2012.R.1) A four-digit number x is defined to be *funny* if each of its digits (simultaneously) can be increased or decreased by 1 (with the condition that 9 can only be decreased and 0 can only be increased) so that the resulting number is divisible by x.
a) Find two funny numbers.
b) Find three funny numbers.
c) Can there be four or more funny numbers?

Problem 9.2. (3ARSO — 1998.8.1): Do there exist n-digit numbers M and N such that all digits of M are even, all digits of N are odd, each digit from 0 to 9 appears at least once in the decimal representation of M or N, and M is divisible by N?

Problem 9.3. (MMO — 2011.8.2): Max was born in the 19th century, and his brother Leo was born in the 20th century. They were both born on the same calendar day. One day, the brothers met to celebrate their common birthday. Max said, «My age is equal to the sum of the digits of the year of my birth». «Mine too», Leo replied. How many years younger is Leo than Max?

Problem 9.4. (LEO — 2017.2): Provide an example of six distinct natural numbers such that the product of any two of them is not divisible by the sum of all six numbers, but the product of any three of them is divisible by the sum of all six numbers.

Problem 9.5. (TOT — 2012.8.1): Leo writes a sequence of natural numbers on the board. The first number N (where $N > 1$) is written in advance. Each next number written down is the previous number with any of its divisors greater than 1 added to it or subtracted from it. Can Leo get to the number 2011 beginning from any natural number N (where $N > 1$)?

Problem 9.6. (MMO — 2014.6.1): The sum of the three smallest distinct divisors of a certain number A is 8. How many trailing zeros can the number A have?

Problem 9.7. (MMG — 2016.8.3): Find the smallest natural number n for which $(n+1)(n+2)(n+3)(n+4)$ is divisible by 1000.

Problem 9.8. (MMO — 1968.8): Given the numbers: $4, 14, 24, \dots, 94, 104$. Prove that it is impossible to cross out one number first, then two more from the remaining, then three more, and finally four more numbers in such a way that after each erasure, the sum of the remaining numbers is divisible by 11.

Problem 9.9. (TOT — 1991.9.3): Find 10 distinct natural numbers such that the sum of the ten numbers is divisible by each of them individually.

Problem 9.10. (TOT — 1991.9.1): There are n integers (where $n > 1$). It is known that each of them differs from the product of all the other numbers by a value divisible by n. Prove that the sum of the squares of all the numbers is divisible by n.

Problem 9.11. (TOT — 2001.11.2): Natural numbers a, b, c, d are such that $ad - bc > 1$. Prove that at least one of the numbers a, b, c, d is not divisible by $ad - bc$.

Problem 9.12. (3ARSO — 2011.9.5): Find all numbers a such that for any natural number n, the number $an(n+2)(n+4)$ is an integer.

Problem 9.13. (3ARSO — 2011.10.5): Find all numbers a such that for any natural number n, the number $an(n+2)(n+3)(n+4)$ is an integer.

Problem 9.14. (COM — 2012.6.7): We define a five-digit number to be irreducible if it cannot be factored into the product of two three-digit numbers. What is the maximum number of consecutive irreducible five-digit numbers?

Problem 9.15. (ARSO — 2014.10.1): A natural number is called «good» if it has exactly two prime divisors. Can there be 18 consecutive «good» numbers?

Problem 9.16. (AIME — 2017.II.6): Find the sum of all positive integers n such that $\sqrt{n^2 + 85n + 2017}$ is an integer.

Problem 9.17. (Mock AIME — 2006-2007.2.8): The positive integers $x_1, x_2, \ldots, x_7$ satisfy $x_6 = 144$ and $x_{n+3} = x_{n+2}(x_{n+1} + x_n)$ for $n = 1, 2, 3, 4$. Find the last three digits of x_7.

Problem 9.18. (CIME — 2020.II.5): A positive integer n is said to be k-consecutive if it can be written as the sum of k consecutive positive integers. Find the number of positive integers less than 1000 that are either 9-consecutive or 11-consecutive (or both), but not 10-consecutive.

Problem 9.19. (AIME — 2019.II.9): Call a positive integer n k-pretty if n has exactly k positive divisors, and n is divisible by k. For example, 18 is 6-pretty. Let S be the sum of positive integers less than 2019 that are 20-pretty. Find $\frac{S}{20}$.

Problem 9.20. (AIME — 2019.I.9): Let $\tau(n)$ denote the number of positive integer divisors of n. Find the sum of the six least positive integers n that are solutions to $\tau(n) + \tau(n+1) = 7$.

Problem 9.21. (AIME — 2009.II.4): A group of children held a grape-eating contest. When the contest was over, the winner had eaten n grapes, and the child in k-th place had eaten $n + 2 - 2k$ grapes. The total number of grapes eaten in the contest was 2009. Find the smallest possible value of n.

Problem 9.22. (AMC — 2020.10A.9): A single bench section at a school event can hold either 7 adults or 11 children. When N bench sections are connected end to end, an equal number of adults and children seated together will occupy all the bench space. What is the least possible positive integer value of N?

(A) 9 (B) 18 (C) 27 (D) 36 (E) 77

Problem 9.23. (AMC — 2020.10A.22): For how many positive integers $n \leq 1000$ is $\left\lfloor \frac{998}{n} \right\rfloor + \left\lfloor \frac{999}{n} \right\rfloor + \left\lfloor \frac{1000}{n} \right\rfloor$ not divisible by 3? (Recall that $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

(A) 22 (B) 23 (C) 24 (D) 25 (E) 26

Problem 9.24. (AMC — 2019.10A.9): What is the greatest three-digit positive integer n for which the sum of the first n positive integers is not a divisor of the product of the first n positive integers?

(A) 995 (B) 996 (C) 997 (D) 998 (E) 999

Problem 9.25. (AMC — 2019.10A.18): For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is $0.\overline{23}_k = 0.232323\ldots_k$. What is k?

(A) 13 (B) 14 (C) 15 (D) 16 (E) 17

Problem 9.26. (AMC — 2019.10B.6): There is a positive integer n such that $(n+1)! + (n+2)! = n! \cdot 440$. What is the sum of the digits of n?

(A) 3 (B) 8 (C) 10 (D) 11 (E) 12

Problem 9.27. (AMC — 2018.10B.13): How many of the first 2018 numbers in the sequence $101, 1001, 10001, 100001 \ldots$ are divisible by 101?

(A) 253 (B) 504 (C) 505 (D) 506 (E) 1009

Problem 9.28. (AMC — 2013.10B.24): A positive integer n is nice if there is a positive integer m with exactly four positive divisors (including 1 and m) such that the sum of the four divisors is equal to n. How many numbers in the set 2010,2011,2012, $\ldots$, 2019 are nice?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 9.29. (AIME — 1995.10): What is the largest positive integer that is not the sum of a positive integral multiple of 42 and a positive composite integer?

Problem 9.30. (iTest — 2007.11): Consider the «tower of power» $2^{2^{2^{\cdot^{\cdot^{2}}}}}$, where there are 2007 twos including the base. What is the last (unit digit) of this number?

Problem 9.31. (AHSME — 1967.22): For natural numbers, when P is divided by D, the quotient is Q, and the remainder is R. When Q is divided by D', the quotient is Q', and the remainder is R'. Then, when P is divided by DD', the remainder is:

(A) $R + R'D$ (B) $R' + RD$ (C) RR' (D) R (E) R'

Problem 9.32. (AHSME — 1964.3): When a positive integer x is divided by a positive integer y, the quotient is u, and the remainder is v, where u and v are integers. What is the remainder when $x + 2uy$ is divided by y?

(A) 0 (B) $2u$ (C) $3u$ (D) v (E) $2v$

Skill Assessment Problems

Skill Assessment Problem 9.1. (MIPT − 2015.10.3): How many natural numbers k (where $k \leqslant 291000$) exist such that $k^2 - 1 \,\vdots\, 291$?

Skill Assessment Problem 9.2. Prove that for all integers a, the expression $a^5 - a$ is divisible by 30.

Skill Assessment Problem 9.3. The sum of two natural numbers is 201. Prove that the product of these two numbers cannot be divisible by 201.

Solutions to Skill Assessment Problems

Solution to Problem 9.1: Decomposing the dividend and divisor into factors, we get the condition $(k+1)(k-1)\,\vdots\,(3\cdot 97)$. Thus, one of the numbers $(k+1)$ or $(k-1)$ is divisible by 97. Let's consider two cases.

a) $(k+1)\,\vdots\,97$, i.e., $k = 97p+96$, $p \in \mathbb{Z}$. Then we get $(97p+95)(97p+97)\,\vdots\,(3\cdot 97) \Leftrightarrow (97p+95)(p+1)\,\vdots\,3$. The first factor is divisible by 3 when $p = 3q+1$, $q \in \mathbb{Z}$, and the second one when $p = 3q+2$, $q \in \mathbb{Z}$. Therefore, $k = 291q+193$, $k = 291q+290$, $q \in \mathbb{Z}$.

b) $(k-1)\,\vdots\,97$, i.e., $k = 97p+1$, $p \in \mathbb{Z}$. Then we get $97p(97p+2)\,\vdots\,(3\cdot 97) \Leftrightarrow (97p+2)p\,\vdots\,3$. The first factor is divisible by 3 when $p = 3q+1$, $q \in \mathbb{Z}$, and the second one when $p = 3q$, $q \in \mathbb{Z}$. Therefore, $k = 291q+98$, $k = 291q+1$, $q \in \mathbb{Z}$. Thus, the numbers satisfying the conditions of the problem are those giving remainders $193, 290, 98, 1$ when divided by 291, i.e., every 4 out of 291 consecutive numbers. Since $291{,}000 = 291 \cdot 1000$, we obtain $4 \cdot 1000 = 4000$ numbers. □

Solution to Problem 9.2:

$$a^5 - a = a(a^2+1)(a-1)(a+1).$$

As $(a-1)$, a, $(a+1)$ are three consecutive integers, one of them must be divisible by 3, and one must be divisible by 2. Thus, their product must be divisible by 6. If a is in the form $5k+2$ or $5k+3$, then neither of these three integers are divisible by 5. In other cases, one of these factors will be divisible by 5, making the entire expression divisible by 30.

1) $a = 5k+2$: $a^2+1 = (5k+2)^2+1 = 25k^2+20k+4+1 = 5(5k^2+4k+1)$ is divisible by 5, making the entire expression divisible by 30.

2) $a = 5k+3$: $a^2+1 = (5k+3)^2+1 = 25k^2+30k+9+1 = 25k^2+30k+10 = 5(5k^2+6k+2)$ is divisible by 5, and the product is divisible by 30.

By the way, from the divisibility by 2, 3, and 5, we can conclude the divisibility by 30 without using the fundamental theorem of arithmetic.

In the lesson, we already proved that if a number is divisible by 2 and 3, then it is divisible by 6. Similarly, we can prove that if a number is divisible by 5 and 6, then it is divisible by 30. If $n \vdots 6$, then n can be written as $n = 6k$, therefore, $5n = 30k \vdots 30$. Similarly, if $n = 5m$, then $6n = 30m \vdots 30$. Thus, $n = 6n - 5n$, indicating that n is divisible by 30 as n is the difference of two numbers that are both divisible by 30 ($6n$ and $5n$). □

Solution to Problem 9.3: Let the given numbers be a and b. Then $a + b = 201$. Consider the product ab. Suppose ab is divisible by $201 = 3 \cdot 67$. Then either a or b is divisible by 3. But if a is divisible by 3, then $b = 201 - a$; therefore, b is also divisible by 3, and vice versa. Thus, both numbers a and b are multiples of 3. Similarly, we are able to prove that both a and b are multiples of 67. Hence, each of the numbers a, b is divisible by $3 \cdot 67 = 201$. However, this implies that each of them is not less than 201. The sum of two natural numbers greater than or equal to 201 cannot be equal to 201. □

Divisibility Criteria

> Goldbach's Conjecture: This is an unproven theory that states that every even integer greater than 2 can be expressed as the sum of two prime numbers.

In 1742, Christian Goldbach wrote a letter to Leonhard Euler proposing this conjecture as a fun mathematical puzzle. It's been a fun challenge for mathematicians ever since!

Theory and Practice

At the beginning of this book, we already reminded you of the criteria of divisibility by a set of numbers. But, in fact, to be a real mathematical competitor, of course, it is not enough just to know the criteria of divisibility; you also need to be able to prove them (and sometimes also derive your own!).

Lemma 2 (Divisibility by 2 Criterion)**.** A number is divisible by 2 if and only if its last digit is divisible by 2 (Note: 0 is divisible by any number except zero!).

Proof. Consider the number $\overline{x_n x_{n-1} \ldots x_0} = \overline{x_n x_{n-1} \ldots x_1} \cdot 10 + x_0$. It is evident that the first term is divisible by 10, and hence, by 2. Therefore, the divisibility depends on the last digit. If it is divisible by 2, then the entire number is divisible by 2; otherwise, it is not. □

Lemma 3 (Divisibility by 3 Criterion)**.** A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

Proof. Let the number have a decimal representation $N = \overline{a_n a_{n-1} \ldots a_1 a_0}$. Then, $N = a_0 + 10^1 a_1 + a_2 10^2 + \ldots + a_{n-1} 10^{n-1} + a_n 10^n$. Notice that the numbers $1, 10, 100, \ldots, 10^n$ all leave a remainder of 1 when divided by 3 (since $0, 9, 99, \ldots, 999 \ldots 99$ are divisible by 3). Therefore, if the sum of digits $a_0 + a_1 + \ldots + a_n$ is divisible by 3, then $N = a_0 + a_1 + \ldots + a_n + 9a_1 + 99a_2 + \ldots + 9 \ldots 9a_n$ is also divisible by 3.

Moreover, the remainder when dividing the number by 3 is the same as the remainder when dividing the sum of its digits by 3. □

Lemma 4 (Divisibility by 11 Criterion)**.** If the alternating sum of digits of a number is divisible by 11, then the number itself is divisible by 11.

Proof. Recall the basic principles of congruence. Consider the number

$$\overline{x_n x_{n-1} \ldots x_0} = x_n 10^n + x_{n-1} 10^{n-1} + \ldots + x_0.$$

Note that

$$10 \equiv -1 \pmod{11},$$
$$100 \equiv (-1)^2 \equiv 1 \pmod{11},$$
$$1000 \equiv (-1)^3 \equiv -1 \pmod{11}.$$

That is, an odd power of ten is congruent to -1 modulo 11, while an even power is congruent to 1 modulo 11. Therefore, the number has the same remainder when divided by 11 as the alternating sum of its digits. □

The divisibility by composite numbers is based on divisibility by coprime factors. For example, the divisibility rule for 6 can be derived from the divisibility rules for 2 and 3 and the materials from the previous chapter.

The divisibility by powers of prime numbers, such as 2^n and 5^n, is also based on the decimal representation of the number. The number is divisible by 2^n or 5^n if the number formed by its last n digits is divisible by 2^n or 5^n.

Attention! Divisibility by powers of three cannot be formulated similarly to the divisibility rules for 3 and 9 for powers greater than 2. For example, the number 27 is divisible by 27, but its digit sum, which is 9, is not divisible by 27.

Example 10.1. a) For the number 100!, we calculated the sum of digits, then calculated the sum of digits again, and so on until only one digit remained. What is this digit?

b) The same question for the number 2^{2018}.

Solution: The remainder of the original number divided by 9 and the remainder of the resulting number divided by 9 are the same. Thus, the last digit is essentially the remainder of the number divided by 9, with the only difference being that instead of the remainder 0, the digit 9 will be obtained (since obtaining zero through such additions is impossible).

a) 100! includes a factor of 9. Thus, it is divisible by 9. Therefore, the remaining digit will be 9.

b) Since the remainders when dividing the original and final numbers by 9 are the same, we only need to find the remainder of 2^{2018} divided by 9. Notice that powers of two have remainders of $2, 4, 8, 7, 5, 1, 2, \dots$ when divided by 9. These remainders repeat with a period of 6 (i.e., $2^1 \equiv 2 \pmod 9$ and $2^7 \equiv 2 \pmod 9$). Therefore, we only need to find the remainder of dividing 2018 by 6. $2018 \equiv 2 \pmod 6$, so $2^{2018} \equiv 4 \pmod 9$, and thus the remaining digit is 4. □

When we know how to prove the basic divisibility rules, we can prove something original.

Example 10.2. Prove that the number $19202122\dots7980$ is divisible by 1980.

Proof. First, let's factorize the number 1980 into prime factors. To be divisible by 1980, the number must be a multiple of all prime factors of 1980 raised to the required powers. Factorizing, we get $1980 = 2^2 \cdot 3^2 \cdot 5 \cdot 11$. Therefore, we need to prove that our number is divisible by $4, 5, 9$, and 11. It's easy to see that the number is divisible by 4 and 5 by looking at two last digits.

Consider the number $8079\dots2019$. If we add it to the original number, the resulting sum will be a number consisting of an even number of nines.

At the same time, the original number, when divided by 9 or 11, has the same remainder as the number $8079\dots2019$. But the sum of these numbers is divisible by 99. Therefore, each term must be divisible by 99. □

Example 10.3. In a k-digit number divisible by 495, two zeros were inserted between two of the k digits. Prove that the resulting number is also divisible by 495.

Proof. Let's denote the number of digits to the right of the insertion by n. The group of digits to the right of the insertion forms the number B, and the group of digits to the left forms the number A (for example, when transforming 2014 into 201004, $A = 201$, $B = 4$, $n = 1$). Then the original number was $10^n A + B$, and the new number became $10^{n+2}A + B$. Thus, the number was increased by $(10^{n+2} - 10^n)A = 99A \cdot 10^n$. Since $n \geqslant 1$, the added number is divisible by 990, and therefore, it is divisible by 495. The new number consists of the original number and the obtained difference, and since each term is divisible by 495, the new number is also divisible by 495. □

Example 10.4. Let $S(n)$ denote the sum of digits of the number n. Solve the equations:

a) $n + S(n) + S(S(n)) = 2020$;

b) $n + S(n) + S(S(n)) + S(S(S(n))) = 2020$.

Proof. a) According to the divisibility rule for 3, the numbers n and $S(n)$ have the same remainders when divided by 3. The same remainder will also be for the number $S(S(n))$. Thus, the sum $n + S(n) + S(S(n))$ is divisible by 3 (since it is the sum of three numbers with the same remainders when divided by 3). However, 2020 is not divisible by 3, so there are no solutions to this equation.

b) Clearly, $n < 2020$. It is easy to notice that among the numbers less than 2020, the number 1999 has the largest digit sum, which is 28. Therefore, $S(n) \leqslant 28$. Furthermore, $S(S(n)) \leqslant S(19) = S(28) = 10$. Finally, $S(S(S(n))) \leqslant 9$. From the equation, we have

$$n = 2020 - S(n) - S(S(n)) - S(S(S(n))) \geqslant$$
$$\geqslant 2020 - 28 - 10 - 9 = 1973.$$

Similar to part a), all numbers $n, S(n), S(S(n)), S(S(S(n)))$ have the same remainders when divided by 9. 2020 has a remainder of 4 when divided by 9, so $n \equiv 1 \pmod 9$. Among the numbers from 1973 to 2020, the numbers $1981, 1990, 1999, 2008, 2017$ have a remainder of 1 when divided by 9. Checking these numbers, we find that 1990 and 2008 are solutions to the equation. □

Problem Set

Problem 10.1. (Lomonosov — 2014.10.2): Max wrote down the numbers from 1 to 2014, then for each number, he calculated the sum of the digits, and so on until there was only one digit left for each. What is the sum of all the remaining single-digit numbers on the board?

Problem 10.2. (3ARSO — 2007.10.5): In the natural number A, the digits were rearranged to obtain the number B. It turned out that $A - B = 11\ldots1$ (a total of k ones). Find the smallest possible value of k.

Problem 10.3. (MMO — 1995.8.2): Prove that all numbers $10017, 100117, 1001117, \ldots$ are divisible by 53.

Problem 10.4. (Lomonosov — 2010.11.3): Find all two-digit numbers $\overline{xy}$ such that the number $\overline{64x72y}$ is divisible by 72.

Problem 10.5. (AU): Find the smallest natural number consisting only of zeros and ones that are divisible by 225.

Problem 10.6. (Mos2ARSO — 2016.11.2): In a rectangular 4 column $\times$ 18 row table, numbers from 1 to 72 were arranged in some order, then the products of the numbers in each column were calculated, and the sum of digits for each of the products was found. Could there be 18 identical sums?

Problem 10.7. (MMO — 1964.7.3): Prove that the sum of digits of a number that is a perfect square cannot be equal to 5.

Problem 10.8. (AU): Prove that the numbers from 1 to 2001 inclusive cannot be written in sequence in some order to obtain a perfect cube. For example, $\overline{123\ldots2001}$ is not a perfect cube.

Problem 10.9. (MMO — 1964.8.5): Consider the sum of digits of all numbers from 1 to $1\,000\,000$ inclusive. Among the obtained numbers, consider the sum of digits again, and so on, until a million single-digit numbers are obtained. Which numbers are more common among them — ones or twos?

Problem 10.10. (Mock AIME — 2006-2007.4.7): Find the remainder when $3^{3^{3^3}}$ is divided by 1000.

Problem 10.11. (AMC — 2021.10B.16): Call a positive integer an uphill integer if every digit is strictly greater than the previous digit. For example, 1357, 89, and 5 are all uphill integers, but 32, 1240, and 466 are not. How many uphill integers are divisible by 15?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Problem 10.12. (AMC — 2020.10A.6): How many 4-digit positive integers (that is, integers between 1000 and 9999, inclusive) having only even digits are divisible by 5?

(A) 80 (B) 100 (C) 125 (D) 200 (E) 500

Problem 10.13. (AMC — 2017.10A.20): Let $S(n)$ equal the sum of the digits of positive integer n. For example, $S(1507) = 13$. For a particular positive integer n, $S(n) = 1274$. Which of the following could be the value of $S(n+1)$?

(A) 1 (B) 3 (C) 12 (D) 1239 (E) 1265

Problem 10.14. (AMC — 2017.10A.25): How many integers between 100 and 999, inclusive, have the property that some permutation of its digits is a multiple of 11 between 100 and 999? For example, both 121 and 211 have this property.

(A) 226 (B) 243 (C) 270 (D) 469 (E) 486

Problem 10.15. (iTest — 2007.TB1): The sum of the digits of an integer is equal to the sum of the digits of three times that integer. Prove that the integer is a multiple of 9.

Problem 10.16. (iTest — 2008.47): Find $a+b+c$, where a, b, and c are the hundreds, tens, and units digits of the six-digit integer $123abc$, which is a multiple of 990.

Skill Assessment Problems

Skill Assessment Problem 10.1. (MMO — 1995.9.1): Prove that if you insert any number of threes between the zeros in the number 12008, you get a number divisible by 19.

Skill Assessment Problem 10.2. (AU): Prove that in the decimal representation of 2^{30}, there are at least two identical digits without calculating 2^{30}.

Skill Assessment Problem 10.3. Let $S(n)$ denote the sum of digits of the number n. Solve the equation:

$$n + S(n) + S(S(n)) + S(S(S(n))) + S(S(S(S(n)))) = 2020.$$

Solutions to Skill Assessment Problems

Solution to Problem 10.1: Let's use the method of mathematical induction. Base case: 12008 is divisible by 19 since $12008 = 19 \cdot 632$. Inductive step: We will show that if a number of the given form is divisible by 19, then the next one is also divisible by 19. To prove this, it is enough to show that the difference between two consecutive numbers is divisible by 19. Indeed:

$$120\underbrace{3\ldots3}_{k}08 - 120\underbrace{3\ldots3}_{k-1}08 = 1083 \cdot 10^k$$

This difference is divisible by 19 since $1083 = 19 \cdot 57$. This completes the induction step. □

Solution to Problem 10.2: Since $2^{30} = 1024^3 > 10^9$ and $1024^3 < 2000^3 = 8 \cdot 10^9$, the number 2^{30} has more than nine digits. Therefore, it is a ten-digit number. Suppose, for the sake of contradiction, that no digit repeats in the decimal representation of 2^{30}. In that case, 2^{30} contains all the digits from 0 to 9. However, the sum of the digits of this number is 45, which means 2^{30} is divisible by 9. This leads to a contradiction. Thus, there must be at least one repeated digit in the decimal representation of 2^{30}. □

Solution to Problem 10.3: Clearly, $n < 2020$. From the proof of the last problem from the lesson, we know that all five summands have the same remainder when divided by 9. Since $2020 \equiv 4 \pmod 9$, if the equation has a solution, the sum of the remainders of the summands modulo nine must be congruent to 4. It is easy to see that the congruence $x \cdot 5 \equiv 4 \pmod 9$, where x is the remainder of n when divided by 9, has a unique solution: $x = 8$. Thus, $n \equiv 8 \pmod 9$. Notice that among the numbers less than 2020, the one with the greatest sum of digits is 1999. Therefore, $S(n) \leqslant 28$. Furthermore, $S(S(n)) \leqslant S(19) = S(28) = 10$, $S(S(S(n))) \leqslant 9$, $S(S(S(S(n)))) \leqslant 9$. Consequently, $n = 2020 - S(n) - S(S(n)) - S(S(S(n))) - S(S(S(S(n)))) \geqslant 2020 - 28 - 10 - 9 - 9 = 1964$. Now, consider the numbers satisfying our conditions: $1970, 1979, 1988, 1997, 2006, 2015$. However, substituting these values, we find no valid solutions. Therefore, the given equation has no solutions. □

Parity: Advance Topics

> In quantum mechanics, particles can be classified as fermions or bosons based on their spin, which is related to parity.

Parity in physics is the «spin cycle» of quantum mechanics — it's all about the spin and the symmetry!

Theory and Practice

One of the most extensive areas in number theory is the set of problems related to divisibility by the «simplest prime number», namely, two. The theme of problems related to «parity» may seem suitable only for elementary-level Olympiads, but that is far from the truth. We have already solved elementary problems on parity at the beginning of the book.

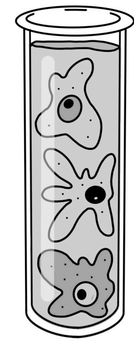
Example 11.1. (AU): *Crows on Trees.* Along the street, there is a line of 6 trees. There is a crow on each tree. Once an hour, two of them fly off, and each lands on one of the neighboring trees. Is it possible that all crows will gather on one tree?

Solution: There is a simple invariant in this problem. Let the first, third, and fifth trees be oaks and the rest birches. Notice that the parity of the number of crows on oaks remains the same. Initially, this number is 3, so it can never become even; therefore, all crows cannot gather on one tree. □

Example 11.2. (AU): *The Termite and the 27 Cubes.* Imagine a large cube assembled from 27 smaller cubes. A termite sits in the center of one of the outer cubes and starts gnawing through it. After being inside a cube, the termite does not return to it. When moving from one small cube to another, the termite moves parallel to one edge of the large cube, i.e., the termite cannot move diagonally. Can the termite gnaw through all 26 outer cubes and finish its move in the central cube? If possible, show the path of the termite.

Solution: The idea of this problem is related to the concept of «coloring». Color the cubes in a chessboard pattern so that the corner cubes are black. This results in 14 black and 13 white cubes. As the termite moves, the colors alternate. Since the last cube is white according to the condition, the termite, on its way to it, could visit at most 13 black cubes. Therefore, it must have missed at least one black cube. □

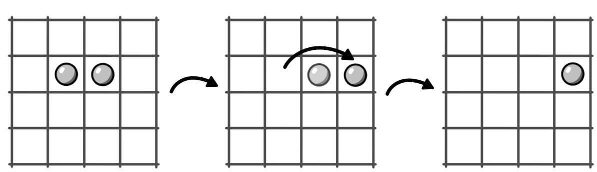
Example 11.3. (AU): *Martian Amoebas.* In a test tube, there are Martian amoebas of three types **A**, **B**, and **C**. Two amoebas of different types can merge into an amoeba of the third type. After several such mergers, only one amoeba remains in the test tube. What is the type of the remaining amoeba if initially there were 20 of type **A**, 21 of type **B**, and 22 of type **C**?



Solution: We can use the fact that in any merging, the parity of the sum total number of amoebas of types **A** and **B** does not change. The same can be said for types **A**, **C**, and **B**, **C**.

Therefore, after all mergers, there will be an amoeba of type **B** remaining (as there was an odd number of them initially, and the number of amoeba of other types was even). □

Example 11.4. (AU): *Yoga Game.* There are 32 chips on the game board (see Figure 11.1a)). When making a move, one chip jumps over another, and another one is to be taken out of the board. It is almost like in checkers, but not diagonally, but rather horizontally or vertically.



Suppose at the end of the game, only 1 chip remains. Explain why there are only 5 possible positions for it, as shown in Figure 11.1b).

Solution: Perhaps you've already grown fond of Martian amoebas after solving the previous problem. Let's seat them on the game board. For example, let amoebas occupy cells **A** and **B**, with cell **C** being empty. Then amoeba **A**, jumping over amoeba **B**, transforms into amoeba **C**, and amoeba **B** disappears.

a) initial arrangement b) final positions

Figure 11.1: Yoga Game.

Now, let's consider a specific arrangement of amoebas. Place them on the board as shown in Figure 11.2a).

a) first coloring b) second coloring

Figure 11.2: Yoga Game: arrangement of Martian amoebas.

If, at the beginning of the game, a chip is removed from the central cell (marked with **A**), then reasoning as in the previous problem, we conclude that the last chip can only remain in the cell marked with the letter **A**. However, amoebas could be initially arranged differently (see Figure 11.2b). There are only 5 cells that are marked with the letter **A** in both arrangements; they are exactly the cells mentioned in the problem statement.

By the way, if you were asked to determine the exact number of cells where the last remaining chip could be found, then for each of them it would be necessary to provide an example of a corresponding game. □

Example 11.5. (AU): *Error Fixing Code.* Suppose you need to transmit a message consisting of n^2 zeros and ones. Write it in the form of an $n \times n$ square table. Add one more column of height n to the end of the table. Write the sum of all elements in each row modulo 2 in the last column. Similarly, add a row to the bottom of the table of length $n + 1$. Write the sum of all elements in each column modulo 2 in the bottom row; this includes the added column. For example, if you need to transmit the message 0111, then the 2×2 table will be expanded to a 3×3 table:

$$\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \quad \rightarrow \quad \begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 0 \\ \hline 1 & 0 & 1 \end{array}$$

Prove that if one error occurs while transmitting the extended table $(n+1) \times (n+1)$, then this error can be found and corrected. What is the smallest number of errors that must occur before the errors become undetectable?

Solution: In the extended table of size $(n + 1) \times (n + 1)$, the sum of elements in any column or row is even. If one element is changed, the sums for one row and one column will become odd. To correct such an error, it will be necessary to change the element that is at the intersection of the row and column with odd sums. The minimum number of errors that cannot be detected is 4. For example, you can change all four digits in the message 0111. In this case, the sums in all rows and columns in the 3×3 table will remain even. □

Problem Set

Problem 11.1. (3ARSO — 2010.9.5): Max wrote down 11 natural numbers in a circle. For each pair of neighboring numbers, he calculated their difference (subtracted the smaller one from the larger one). The 11 differences found, as a result, were four ones, four twos, and three threes. Prove that Max made a mistake somewhere.

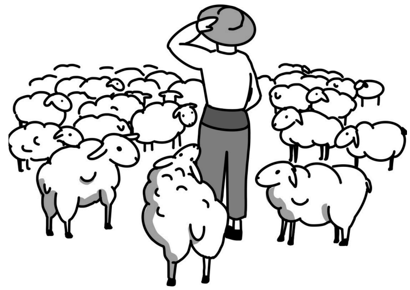
Problem 11.2. (3ARSO — 2006.8.5;9.5): On the board, the product $a_1 \cdot a_2 \cdot \ldots \cdot a_{100}$ is written, where $a_1, a_2, \ldots, a_{100}$ are natural numbers. A new expression is made by replacing one multiplication sign with an addition sign. 99 expressions are made; it is known that exactly 32 of the expressions are even. What is the maximum number of even numbers among $a_1, a_2, \ldots, a_{100}$?

Problem 11.3. (MMO — 2018.8.2): 39 non-zero numbers are written in a row. The sum of any two neighboring numbers is positive, and the sum of all numbers is negative. What could be the sign of the product of all the numbers? (Specify all options and prove that there are no other options)

Problem 11.4. (MMO — 2018.9.1): 81 non-zero numbers are written in a row. The sum of any two neighboring numbers is positive, and the sum of all numbers is negative. What could be the sign of the product of all numbers?

Problem 11.5. (MMO — 2004.9.1): The stock price of the company «Horns and Hooves» increases or decreases by 17% every day at $12:00$ (the price is not rounded). Can the stock price have the exact same value on two different days?

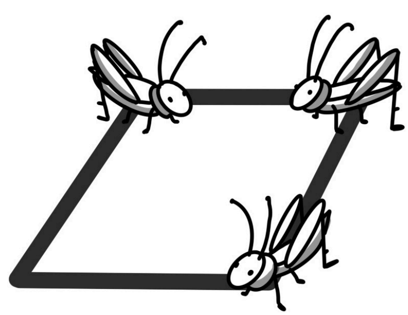
Problem 11.6. (3ARSO — 2013.9.1;10.1): Natural numbers M and N are given, both are greater than ten. Both consist of the same number of digits, such that $M = 3N$. To obtain the number M, 2 must be added to one of the digits in the number N, and an odd digit must be added to the rest of the digits. What digit can the number N end on?



Problem 11.7. (3ARSO — 1998.8.6): Combined, several peasants have 128 sheep. If one peasant has at least half of all the sheep, the rest get together and banish him; each peasant takes as many sheep as he already has. If two have 64 sheep each, then one of the two is banished. Seven banishments occurred. Prove one peasant remained with all the 128 sheep.

Problem 11.8. (MMO — 1993.8.3): Two chips lie on a straight line. The left chip is a red one, and the right chip is a blue one. It is allowed to perform any of two operations: place two new chips of the same color adjacently anywhere on the line or remove any two adjacent chips of the same color. Is it possible to leave exactly two chips on the line after a finite number of operations, where a red one is on the right, and a blue one is on the left?

Problem 11.9. (MMO — 2005.8.4): There are 2005 natural numbers are arranged in a circle. Prove that there are two neighboring numbers, such that after removing them, the remaining numbers cannot be divided into two groups of equal sums.



Problem 11.10. (MMO — 1973.8.5): Three grasshoppers are in three vertices of a square. They play leapfrog; that is, they jump over each other. For example, if the grasshopper A jumps over the grasshopper B, then after the jump, it is at the same distance from B as before the jump and, of course, on the same straight line. Can one of them reach the fourth vertex of the square?

Problem 11.11. (LEO — 2013.7): Let a, b, c be three natural numbers. Three three products ab, ac, bc were written on the board. All digits except for the last two were erased from each product. Could it happen that the result is three consecutive two-digit numbers $(n, n+1, n+2)$?

Problem 11.12. (LEO — 2015.5): 40 bandits crossed the river with the help of a two-seat boat from the left bank to the right (some trips may have been made alone). Could it happen that each pair of bandits crossed the river together exactly once (from the left bank to the right or from the right to the left)?

Problem 11.13. (LEO — 2012.5): Is it possible to place 12 natural numbers on the edges of a cube, such that the sums of numbers on the edges on any two opposite faces differ by exactly one?

Problem 11.14. (LEO — 2011.2): 40 people are sitting around a round table. Could it happen that any two of them, between whom an even number of people sit, have a common acquaintance at the table, and any two, between whom an odd number of people sit, do not have a common acquaintance?

Problem 11.15. (LEO — 2017.7): Given a circle of circumference 90. Is it possible to mark 10 points on the circumference such that any arc so that among the arcs with endpoints at these points, there are arcs with all integer lengths from 1 to 89?

Problem 11.16. (ARSO — 2007.9.2): On the board, 100 fractions were written, with each numerator being one of the numbers from 1 to 100 exactly once, and each denominator being one of the numbers from 1 to 100 exactly once. It turned out that the sum of these fractions is an irreducible fraction with a denominator of 2. Prove that it is possible to swap the numerators of two fractions so that the sum becomes an irreducible fraction with an odd denominator.

Problem 11.17. (AU): On the hockey field, three pucks **A**, **B**, and **C** are lying. The hockey player hits one of them in such a way that it passes between the other two. He does this 25 times. Is it possible that the pucks end up in their original positions after this?

Problem 11.18. (AU): All dominoes are laid out in a chain. On one end, there are 5 points. How many points are there on the other end?

Problem 11.19. (AU): Can the set of all natural numbers greater than 1 be divided into two non-empty subsets such that for any two numbers a and b from the same subset, the number $ab1$ belongs to the other subset?

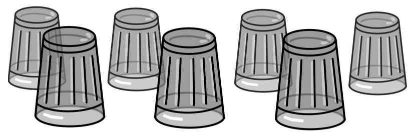
Problem 11.20. (AU): A convex $2n$-gon with the vertices $A_1, \dots, A_{2n}$ is given. Inside it, a point P is taken that does not lie on any of the diagonals. Prove that joining P with vertices at points $A_1, \dots, A_{2n}$ will create an even number of triangles with P on it.

Problem 11.21. (AU): In a row, the numbers $1, 2, \dots, 10$ are written. Is it possible to place the signs «$+$» and «$-$» between them so that the value of the resulting expression is equal to zero?

Problem 11.22. (AU): To a 17-digit number, the number written with the same digits in reverse order was added. Prove that at least one digit of the value of the resulting sum is even.

Problem 11.23. (AU): A snail crawls on a plane at a constant speed, turning at a right angle every 15 minutes. Prove that it can only return to the starting point after an integer number of hours.

Problem 11.24. (AU): There are 101 coins, of which 50 are fake. The fakes differ in weight by 1 gram from the real ones. Jean takes a coin and places it on the scales, then places another coin on the other side. He wants to determine whether the coin is fake from only one weighing. Can he do it?



Problem 11.25. (AU): 7 *glasses.* There are 7 glasses on the table; they all stand upside down. In one move, it is allowed to turn any 4 glasses. Is it possible to have all the glasses standing upright at the same time?

Problem 11.26. (AU): In the cells of a square 4×4 table, signs $+$ and $-$ are arranged as shown in the figure

$+$	$-$	$+$	$+$
$+$	$+$	$+$	$+$
$+$	$+$	$+$	$+$
$+$	$+$	$+$	$+$

In one move, it is allowed to simultaneously change the sign in all cells located in one row, in one column, or on a line parallel to any diagonal (in particular, you can change the sign in any corner cell). Prove that no matter how many such moves we make, we will not be able to get a table with all plus signs.

Problem 11.27. (AIME – 2010.I.12): Let $m \geq 3$ be an integer and let $S = \{3,4,5,\ldots,m\}$. Find the smallest value of m such that for every partition of S into two subsets, at least one of the subsets contains integers a, b, and c (not necessarily distinct) such that $ab = c$.

Note: A partition of S is a pair of sets A, B such that $A \cap B = \emptyset$, $A \cup B = S$.

Problem 11.28. (AIME — 2007.II.6): An integer is called parity-monotonic if its decimal representation $a_1a_2a_3 \cdots a_k$ satisfies $a_i < a_{i+1}$ if a_i is odd, and $a_i > a_{i+1}$ if a_i is even. How many four-digit parity-monotonic integers are there?

Problem 11.29. (AIME — 2000.II.4): What is the smallest positive integer with six positive odd integer divisors and twelve positive even integer divisors?

Problem 11.30. (AMC — 2002.12B.11): The positive integers A, B, $A-B$, and $A+B$ are all prime numbers. The sum of these four primes is

(A) even (B) divisible by 3 (C) divisible by 5
(D) divisible by 7 (E) prime

Skill Assessment Problems

Skill Assessment Problem 11.1. Max participated in several school olympiads. In mathematics, he scored 17 out of 28, in physics, he scored 14 out of 28, in informatics, he scored 23 out of 40, and in history, he scored 1 out of 36. Unfortunately, his school is not that great, and to qualify for the municipal stage, he needs to score full marks in one of the olympiads. He has a good relationship with the school director, who happens to teach his class for mathematics, so he makes a deal with her. At his request, she can decrease the score for three olympiads by 1 point each and increase the score for the remaining olympiad by 5 points. Max can make any number of requests. Does Max have a chance to qualify for the municipal stage in all four olympiads?

Skill Assessment Problem 11.2. A beaver left his lodge for a walk in an open field. Every 30 minutes, he turns 120 degrees in any direction. Prove that he can return home only after walking for some whole number of hours.

Solutions to Skill Assessment Problems

Solution to Problem 11.1: Each of the director's actions changes the parity of all scores simultaneously. It is necessary to score an even number of points in each subject because the qualifying scores are even. Initially, one score is even, and the rest are odd; hence, all scores can never be even. This contradiction completes the solution to the problem. □

Solution to Problem 11.2: Simple reasoning shows that the beaver will move along the edges of a hexagonal lattice («honeycomb»). Thus, the beaver's route is essentially a traversal of a path along the grid of the hexagonal lattice. Each addition of a hexagon or isolated edge to the path adds an even number of segments to the path, resulting in an even number of time intervals. □

Advance Topics in Divisibility

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The Ulam Spiral: This is a graphical depiction of the set of prime numbers where the numbers are arranged in a spiral pattern.

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The Ulam Spiral is the «snail mail» of number theory — primes take a scenic route around the spiral to show off their patterns!

Theory and Practice

Now, we will delve even deeper into the topic of divisibility. We have recalled the divisibility criteria and the fundamental properties of divisibility, which will allow us to see the practical applicability of these skills. In this chapter, you will not find a new theory but rather a new perspective on classic Olympiad problems.

Example 12.1. Prove that $2^{70} + 3^{70} \,\vdots\, 13$.

Solution: Here, we will use the classical expansion of the sum of odd powers. But aren't our powers even? Well, not really! $2^{70} + 3^{70} = 4^{35} + 9^{35}$.

The expansion of the sum of odd powers looks like $a^{2n+1} + b^{2n+1} = (a + b)(a^{2n} - a^{2n-1}b^1 + \ldots + b^{2n})$.

Thus, $4^{35} + 9^{35} \,\vdots\, (4 + 9)$, which is what we needed to prove. □

Example 12.2. Prove that if $14x + 13y \,\vdots\, 11$, then $19x + 9y \,\vdots\, 11$.

Solution: Let's try to obtain the required expression from what we have, as well as terms that are multiples of 11.

Indeed, $19x + 9y = 22x - 3x + 11y - 2y = 22x + 11y - (3x + 2y) = 22x + 11y - (14x + 13y) + (11x + 11y) = 11(3x + 2y) - (14x + 13y)$.

Since both terms of the obtained expression are multiples of 11, the required expression is also a multiple of 11, as required. □

Example 12.3. What is the highest power of two that divides the number $10^{20} - 2^{20}$?

Solution: $10^{20} - 2^{20} = 2^{20}(5^{20} - 1)$. There are already twenty twos in the factorization.

Now let's determine how many twos are left in the factorization of $5^{20} - 1$. Of course, such a number can be calculated manually and factored (four hours of olympiad time should be enough for you), but we will take a less energy-consuming path.

$5^{20} - 1 = (5^{10} - 1)(5^{10} + 1)$. First, consider the second bracket. $5^{10} + 1 = 25^5 + 1 = (25 + 1)(25^4 - 25^3 + 25^2 - 25 + 1)$. Here, the second bracket is composed of five odd numbers, so it is odd, and the first one has a power of two as a factor.

Move on to the bracket $5^{10} - 1$. By factoring, we get $5^{10} - 1 = (5^5 - 1)(5^5 + 1)$. Here, $5^5 - 1 = (5 - 1)(5^4 + 5^3 + 5^2 + 5 + 1)$, which, similar to the previous case, gives a power of two as 2. Similarly, $5^5 + 1 = (5 + 1)(5^4 - 5^3 + 5^2 - 5 + 1)$ gives another 2^1.

In total, this number has a power of two in its factorization equal to $20 + 1 + 2 + 1 = 24$. □

Example 12.4. Prove that there exists a natural number n such that all numbers $n + 1, n^n + 1, n^{n^n} + 1, \ldots$ are divisible by 2020.

Proof. If n is odd, then all the specified numbers are divisible by $n + 1$. Therefore, any odd number n for which $n + 1$ is a multiple of 2020 is suitable. For example, $n = 2019$. □

Example 12.5. Do 10 distinct natural numbers exist such that when any 9 of them are added together, the result obtained is a square of some natural number (this natural number does not need to be one of the 10)?

Solution: Such numbers exist, for example, 4, 203, 400, 595, 788, 979, 1168, 1355, 1904, 2608. We can calculate all possible sums of 9 numbers and show that they are squares. However, the question arises — how exactly did we come up with this example? After all, it is not expected that an Olympiad participant, without the help of a computer or calculator, will be able to find these numbers. Of course, there is a logical justification for how we can find the desired numbers.

Let's denote our numbers as $a_1, \ldots, a_{10} > 0$ and also let $S = a_1 + \ldots + a_{10}$. Then, we want to find a solution to the following system in natural numbers:

$$\begin{cases} S - a_1 & = & x_1^2, \\ & \cdots & \\ S - a_{10} & = & x_{10}^2, \end{cases}$$

where, by adding all the equations, we get:

$$10S - (a_1 + \ldots + a_{10}) = 9S = x_1^2 + \ldots + x_{10}^2 \Rightarrow S = \frac{x_1^2 + \ldots + x_{10}^2}{9},$$

substituting the value for S into the previous system of equations, we get an equivalent system:

$$\begin{cases} a_1 & = & \dfrac{x_1^2 + \ldots + x_{10}^2}{9} - x_1^2 > 0, \\ & \cdots & \\ a_{10} & = & \dfrac{x_1^2 + \ldots + x_{10}^2}{9} - x_{10}^2 > 0. \end{cases}$$

Now, it is enough to just find 10 natural numbers, such that the sum of their squares is divisible by 9, and so that all numbers $a_1, \ldots, a_{10}$ are positive. Notice also that if the numbers are approximately equal, then the smallest of the numbers $a_1, \ldots, a_{10}$: a_{10} will be equal to $\frac{x_1^2+\ldots+x_{10}^2}{9} - x_{10}^2 \approx \frac{10}{9}x_{10}^2 - x_{10}^2 = \frac{1}{9}x_{10}^2 > 0$.

This remark suggests the idea to look for numbers in the following form:

$$x_1 = 3N, x_2 = 3N + 3, \ldots, x_{10} = 3N + 27,$$

for some large natural N, then we can be sure that the sum of the squares of such numbers (x_1, x_2, …, x_{10}) will be divisible by 9. Notice also that $a_{10} > 0$ for a sufficiently large N, because a_{10} is a quadratic function of N, where the coefficient at N^2 equal to 1. If desired, you can calculate all the numbers $a_1, \ldots, a_{10}$ for $N = 1000$, for example, but this is not necessary: in the problem statement, it is asked whether such numbers exist, and we have proved that such numbers exist. □

Example 12.6. Find 10 natural numbers such that each of them is not divisible by any of the others, but the square of each number is divisible by all other 9 numbers.

Solution: In this problem, the construction is built more logically. It can be quickly understood that one prime divisor is not enough; however, two prime divisors is definitely enough. Thus, we can compile the numbers $2^9 \cdot 3^{18}, 2^{10} \cdot 3^{17}, \ldots, 2^{18} \cdot 3^9$. $\square$

Problem Set

Problem 12.1. Prove that $11^{10} - 1$ is divisible by 100.

Problem 12.2. Prove that $1^{2017} + 2^{2017} + 3^{2017} + \ldots + 16^{2017}$ is divisible by 17.

Problem 12.3. If $6x + 11y$ is divisible by 31, prove that $x + 7y$ is also divisible by 31.

Problem 12.4. If $a^2 + 9ab + b^2$ is divisible by 11, prove that $a^2 - b^2$ is also divisible by 11.

Problem 12.5. Prove that $n^2 + 5n + 16$ is not divisible by 169 for any integer n.

Problem 12.6. In a six-digit number divisible by 7, the last digit is moved to the beginning. Prove that the resulting number is also divisible by 7.

Problem 12.7. A number in the range from 000000 to 999999 is called «happy» if the sum of its first three digits is equal to the sum of its last three digits. Prove that the sum of all happy numbers is divisible by 13.

Problem 12.8. (3ARSO — 2018.9.2): Five natural numbers are written on the board. It turns out that the sum of any three of them is divisible by each of the remaining two. Is it necessarily true that among these numbers, four numbers are equal?

Problem 12.9. (3ARSO — 2000.9.2): Do there exist distinct pairwise coprime natural numbers a, b, and c, greater than 1, such that $2^a + 1$ is divisible by b, $2^b + 1$ is divisible by c, and $2^c + 1$ is divisible by a?

Problem 12.10. (LEO — 2015.3): Let's call a divisor of a natural number *proper* if it is less than the natural number but greater than 1. For a natural number n, all the proper divisors (there are at least three) were found, and all possible pairwise sums of them were written down (repeated sums were not written down). Prove that the obtained set could not be the set of all proper divisors of any natural number.

Problem 12.11. (MMO — 1997.8.4): a) Prove that there exists a natural number that remains composite after replacing any triplet of neighboring digits with an arbitrary triplet.

b) Does there exist a 1997-digit number with this property?

Problem 12.12. (MMO — 1998.8.2): Can you find eight natural numbers such that none of them divide any other, but the square of any of these numbers is divisible by all of the others?

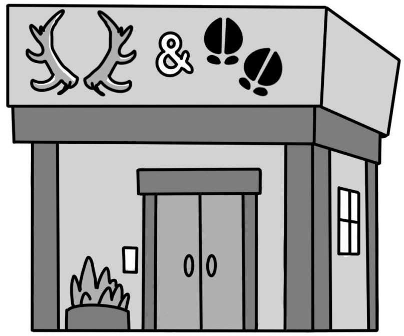
Problem 12.13. (MMO — 2006.9.1): Jean received 777 candies. Jean wants to eat all the candies in n days, so that each of these days (not including the first, but including the last), he eats one more candy than on the previous day. What is the largest value of n for which this is possible?

Problem 12.14. (3ARSO — 2018.10.6;11.6): Esther chose a natural number n and wrote down the following n fractions on the board:

$$\frac{0}{n}, \frac{1}{n-1}, \frac{2}{n-2}, \frac{3}{n-3}, \ldots, \frac{n-1}{n-(n-1)}.$$

Let the number n be divisible by the natural number d. Prove that among the written fractions, there is a fraction equal to the number $d-1$.

Problem 12.15. (3ARSO — 1997.8.6;9.6): Numbers from 1 to 37 were written in a row in such a way that each next number divides the sum of all consecutive numbers before it. What number is in third place if the first number is 37 and the second is 1?



Problem 12.16. (MMO — 2004.8.4): The stock price of the company «Horns and Hooves» increases or decreases by $n\%$ each day at $12:00$, where n is a fixed natural number less than 100 (the price is not rounded). Does there exist a value of n for which the stock price is identical on two different days?

Problem 12.17. (MMO — 2016.8.4): Find the smallest natural number divisible by 99, where this number is only composed of even digits in its decimal notation.

Problem 12.18. (MMO — 2015.9.2): All natural numbers from 1 to 1000 are arranged in some order around a circle in such a way that each number is a divisor of the sum of its two neighbors. It is known that two odd numbers are next to the number k. What parity can the number k be?

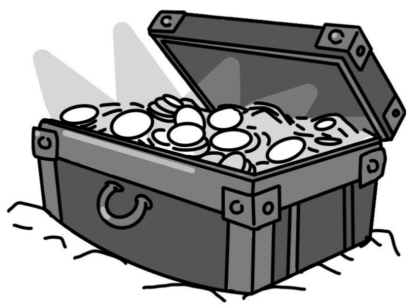
Problem 12.19. (MMO — 1976.8.2): A square room is partitioned by partitions into several smaller square rooms. The length of the side of each smaller room is an integer. Prove that the sum of the lengths of all partitions is divisible by 4.

Problem 12.20. (MMO — 1995.9.3): Natural numbers a, b, c, and d are such that $ab = cd$. Can the number $a + b + c + d$ be prime?

Problem 12.21. (3ARSO — 2011.9.5): Find all numbers a such that for any natural number n the number $an(n + 2)(n + 4)$ is an integer.

Problem 12.22. (MMO — 2012.8.5): Rational numbers x, y, and z are such that all numbers $x+y^2+z^2$, x^2+y+z^2, and x^2+y^2+z are integers. Prove that the number $2x$ is an integer.

Problem 12.23. (LEO — 2015.6): A natural number is called *perfect* if it is equal to half of the sum of all its natural divisors: for example, 6 is perfect because $2 \cdot 6 = 1+2+3+6$. Can the sum of all pairwise products of the natural divisors of a perfect number n be divisible by n^2?



Problem 12.24. (LEO — 2014.3): In the hundredth year of the reign of the Immortal Treasurer, the Treasurer decided to start issuing new coins. In the hundredth year, he released an unlimited supply of coins with a denomination of $2^{100}-1$; the next year, he released coins with a denomination of $2^{101}-1$, and so on. As soon as a newly released coin can be exchanged for previously issued coins (without change), Treasurer will be removed. In what year of his reign will this happen?

Problem 12.25. (ARSO — 2017.10.5): All proper divisors of a certain composite natural number n increased by 1 were written on the board. Find all such numbers n for which the numbers on the board turn out to be all proper divisors of some natural number m. (The *proper divisors* of a natural number $a > 1$ are all its natural divisors, except for a and 1.)

Problem 12.26. (AIME — 2022.II.8): Find the number of positive integers $n \leqslant 600$ whose value can be uniquely determined when the values of $\left\lfloor \frac{n}{4} \right\rfloor$, $\left\lfloor \frac{n}{5} \right\rfloor$, and $\left\lfloor \frac{n}{6} \right\rfloor$ are given, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to the real number x.

Problem 12.27. (AIME — 2022.II.14): For positive integers a, b, and c with $a < b < c$, consider collections of postage stamps in denominations a, b, and c cents that contain at least one stamp of each denomination. If there exists such a collection that contains sub-collections worth every whole number of cents up to 1000 cents, let $f(a, b, c)$ be the minimum number of stamps in such a collection. Find the sum of the three least values of c such that $f(a, b, c) = 97$ for some choice of a and b.

Problem 12.28. (CIME — 2020.I.10): Let $1 = d_1 < d_2 < \ldots < d_k = n$ be the divisors of a positive integer n. Let S be the sum of all positive integers n satisfying

$$n = d_1^1 + d_2^2 + d_3^3 + d_4^4.$$

Find the remainder when S is divided by 1000.

Problem 12.29. (UMO — 2016.5): Let $a_0, a_1, a_2, \ldots$ be a sequence of integers (positive, negative, or zero) such that for all nonnegative integers n and k, $a_{n+k}^2 - (2k + 1)a_n a_{n+k} + (k^2 + k)a_n^2 = k^2 - k$. Find all possible sequences (a_n).

Problem 12.30. (Indonesia MO — 2009.1): Find all positive integers $n \in \{1, 2, 3, \ldots, 2009\}$ such that

$$4n^6 + n^3 + 5$$

is divisible by 7.

Skill Assessment Problems

Skill Assessment Problem 12.1. Prove that $2^9 + 2^{99}$ is divisible by 100.

Skill Assessment Problem 12.2. Prove that for any integers a, b, c, and d, the product $(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$ is divisible by 12.

Skill Assessment Problem 12.3. Given that for natural numbers x and y, it holds that $x^2 + y^2 + 1$ is divisible by xy, prove that $\frac{x^2+y^2+1}{xy} = 3$.

Solutions to Skill Assessment Problems

Solution to Problem 12.1: Let's try to factorize our expression into $2^9 + 2^{99} = 2^9(1024^9+1) = 2^9(1024^3+1)(1024^6+1024^3+1) = 2^9(1024+1)(1024^2+1024+1)(1024^6+1024^3+1)$. Notice that the first factor is divisible by 4, and the second factor, 1025, is divisible by 25. Therefore, the entire product is divisible by 100. □

*[**Solution to Problem 12.2:*** We know that there are only three possible remainders when divided by 3. Thus, four integers a, b, c, d cannot have 4 distinct remainders. This implies that two numbers exist with the same remainder, and their difference is divisible by 3. Therefore, the proposed product is divisible by 3.

Now, we need to show that the product is also divisible by 4. There are two cases: 1) all numbers have distinct remainders when divided by 4; 2) at least two numbers have the same remainder when divided by 4. In the second case, we can apply a similar argument as in the first task to show that one of the factors is divisible by 4.

Now consider the case when remainders are distinct. Then, we can pair the four numbers in such a way that the first pair has remainders 0 and 2, and the second pair has remainders 1 and 3. The difference of these remainders is 2, so there are two terms with remainder 2 when divided by 4. Since the product of two even numbers is divisible by 4, the entire expression is divisible by 4. As it is divisible by both 3 and 4, it is also divisible by 12. □

Solution to Problem 12.3: Let $\frac{x^2+y^2+1}{xy} = k$, where k is a natural number. Then,

$$x^2 + y^2 + 1 = kxy$$

$$x^2 + y^2 + 1 - 2xy = (k-2)xy$$

The left side of the equation, being equal to a sum of squares $(x-y)^2+1$, is positive, implying that $k \geqslant 3$.

Now consider the equation

$$x^2 + y^2 - kxy + 1 = 0.$$

Assume that $k > 3$. Let a and b be a solution to this equation with $a + b$ being minimal, and $a \geqslant b > 0$.

Now consider the function $p(w) = w^2 + b^2 - kbw + 1$. Since a is the root of this equation, let c be the second root. Then, these roots satisfy Vieta's formulas. Thus, $a + c = kb$, and $ac = b^2 + 1$. Therefore, the root c is also an integer. If c is negative, then $0 < a^2 + ac + 1 - 3ab = a^2 + ac - \frac{3c}{k}(a + c) < 0$, leading to a contradiction. This also implies $c \neq 0$, as otherwise, $b^2 + 1 = 0$.

Now, show that $c < a$. If $c \geqslant a$, then $a + 1 \leqslant c$. However, from Vieta's formulas, $a + 1 \leqslant c = \frac{b^2+1}{a} \leqslant a + \frac{1}{a}$, which is a contradiction. Thus, $c < a$. If $c = a$, then $a = b^2 + 1 > \frac{9}{4}b^2$, which is again a contradiction. Therefore, $c < a$. If $k > 3$, our assumption leads to a contradiction.

Hence, the only possible value for k is $k = 3$, as required. $\square$

Advance Topics in Divisibility Criteria

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Lemoine’s Conjecture states that every odd integer n greater than 5 can be expressed as the sum of a prime and twice a prime.
Lemoine’s Conjecture is closely related to Goldbach’s Conjecture, which states that every even integer greater than 2 can be expressed as the sum of two primes.

“

Lemoine and Goldbach are the «dynamic duo» of number theory — tackling even and odd numbers together!

Theory and Practice

Let's logically continue and deepen the topic started in the previous chapters.

Example 13.1. Consider all possible seven-digit natural numbers obtained by permuting the digits in the number 1234567. Prove that none of these numbers is divisible by another.

Solution: Recall that a number, when divided by 9, leaves the same remainder as the sum of its digits. Notice that no matter which numbers we form, they will have the same remainder when divided by 9, equal to 1. Let $a = kb$, where a and b are distinct seven-digit numbers formed by permuting the given digits. Then, $a - b = b(k - 1)$ is divisible by 9, implying that $k - 1$ is divisible by 9. However, this is impossible since clearly $k \leqslant \frac{7654321}{1234567} < 7$. □

Example 13.2. Does there exist a power of two from which another power of two can be obtained by permuting its digits?

Solution: Suppose such a power exists. Then, the remainder of dividing these powers of two by 9 will be the same since we are merely permuting the digits. However, the sequence of remainders when dividing 2^n by 9 (which is $2, 4, 8, 7, 5, 1, 2, \ldots$) has a period of 6. This implies that powers of two with the same remainder differ by at least $2^6 = 64$ times, leading to different digit amounts. Contradiction. □

Example 13.3. In how many ways can you fill in the blanks with digits (excluding nine) in the expression 2_0_2_0_0_2_ to form a number divisible by 45?

Solution: To make the number divisible by 45, it must be divisible by both 9 and 5. Since the number is a multiple of 5, the last digit must be 0 or 5.

1) If the number ends in 5, then the sum of the filled digits has a remainder of 2 when divided by 9. Thus, the sum of the remaining five digits must give a remainder of 7

when divided by 9. There are 9^4 ways to achieve this, as the first four digits uniquely determine the last digit (as we do not have 9 as a last digit).

2) If the number ends in 0, the problem is practically the same as in the previous case, resulting in another 9^4 possibilities.

Therefore, the total number of ways is $2 \cdot 9^4$. □

Example 13.4. Find all natural numbers n such that $(n^2 + 1)$ is divisible by $(n + 1)$.

Solution: $n^2 + 1 = n^2 - 1 + 2 = (n - 1)(n + 1) + 2$. Thus, 2 is divisible by $(n + 1)$, implying $n + 1 = 1$ or $n + 1 = 2$. The case $n + 1 = 1$ is not suitable since $n \geqslant 1$, so the only possible value for n is $n = 1$. □

Example 13.5. If $(3x + 2)$ is divisible by 7, prove that $(15x^2 - 11x - 14)$ is divisible by 7.

Proof. Notice that $15x^2 - 11x - 14 = (3x + 2)(5x - 7)$. Therefore, since $7s = 3x + 2$ for some integer s, and thus $15x^2 - 11x - 14 = 7s(5x - 7)$. Hence $15x^2 - 11x - 14$ is divisible by 7. □

We have already mentioned and utilized the fact that among any two consecutive integers, there is an even number; among any three, there is one divisible by 3, and so on. In fact, a more general theorem holds.

Theorem 3. The product of n consecutive integers is divisible by $n!$.

Proof. Assume that the consecutive integers $m + 1, m + 2, \dots, m + n$ are positive. In this case, divisibility by $n!$ follows from the fact that binomial coefficients are integers: $C_{m+n}^{n} = \frac{(m+n)!}{n!m!} = \frac{(m+n)(m+n-1)\dots(m+1)}{n!}$. If one of the consecutive numbers is zero, their product is zero, and there is nothing to prove. If all n consecutive numbers are negative, we can multiply them by (-1), and in this case, we can apply the previous argument. $\square$

Problem Set

Problem 13.1. (MF — 2004.7.1): Leo thought of a prime three-digit number. All of the digits are distinct. What digit can this number end on if its last digit is equal to the sum of the first two digits?

Problem 13.2. (MF — 1995.7.5): From a natural number, its digit sum was subtracted, then from the resulting number, its digit sum was subtracted again, and so on. After eleven such subtractions, zero was obtained. What was the starting number?

Problem 13.3. (3ARSO — 2006.9.1): Find some nine-digit number N, consisting of distinct digits, such that among all numbers obtained by erasing seven digits from N, there is no more than one prime.

Problem 13.4. (MMO — 2007.9.1): This year is 2007. The number of the current competition is 70 (competition began in 1938). When the last two digits of 2007 are taken in reverse order, the number 70 is obtained, the same as the number of the current competition. How many more times will such a situation occur in this millennium?

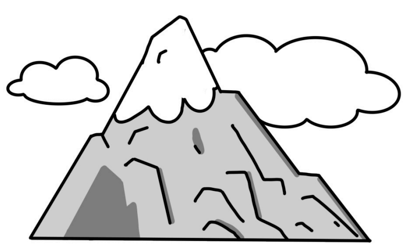
Problem 13.5. (3ARSO — 2007.10.5): In a natural number A, its digits were rearranged, resulting in the number B. It is known that $A - B = \underbrace{1\ldots1}_{n}$. Find the smallest possible value of n.

Problem 13.6. (MMO — 2015.8.3): Leo noticed that on the electronic display showing the exchange rate of the dollar to the ruble (4 digits, separated by a decimal point), the same four distinct digits are lit as a month ago but in a different order. At the same time, the rate increased by exactly 20%. Give two numbers that Leo could have seen.

Problem 13.7. (MMO — 2015.11.2): Last year, Max bought a phone that cost an integer, a four-digit amount of rubles. Going to the store this year, he noticed that the price of the phone increased by 20%. The new price consists of the same digits as the old price but in reverse order. How much did Max spend on the phone?

Problem 13.8. (3ARSO — 2005.9.2): 19 cards are given; any non-zero digit can be written on each card. The cards can be put alongside each other to form 19-digit numbers. Is it possible to have a combination of cards such that out of all 19-digit numbers that can be formed, only one is divisible by 11?

Problem 13.9. (3ARSO — 1993.9.2;10.2): Find the largest natural number where, if any number of digits are removed, a number divisible by 11 will never be obtained. Zero is divisible by 11.



Problem 13.10. (LEO — 2014.6): A natural number is called *mountainous* if there is one digit (not the first or the last) which is greater than all the others; all other digits are non-zero and increasing (each next digit is greater than or equal to the previous)

from the leftmost digit to the peak and then decreasing (each next digit is less than or equal to the previous one) from the peak to the rightmost digit. For example, the number 12243 is mountainous, while the numbers 3456 and 1312 are not. Prove that the sum of all hundred-digit mountainous numbers is a composite number.

Problem 13.11. (3ARSO — 1998.9.3): A ten-digit number is called *interesting* if it is divisible by 11111 and all its digits are distinct. How many interesting numbers are there?

Problem 13.12. (3ARSO — 2003.9.4): Two players play a game where they take turns writing arbitrary digits on the board from left to right. One player «loses» if after their move, two or more digits in a row form a number divisible by 11. Which player, knowing the right strategy, is guaranteed to win?

Problem 13.13. (ARSO — 1995.9.5): Natural numbers are called *similar* if they are written using the same set of digits (for example, for the set of digits 1, 1, 2, numbers 112, 121, 211 will be similar). Prove that three similar 1995-digit numbers (without zeroes in their representation) exist such that one of the numbers is equal to the sum of the other two.

Problem 13.14. (ARSO — 1996.9.5): Prove that in an arithmetic progression with the first term equal to 1 and a difference equal to 729, there are infinitely many terms that are powers of 10.

Problem 13.15. (Mock AIME — 2006-2007.4.10): Compute the remainder when

$$\binom{2007}{0} + \binom{2007}{3} + \ldots + \binom{2007}{2007}$$

is divided by 1000.

Problem 13.16. (AMC — 2017.12B.19): Let $N = 123456789101112\ldots4344$ be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

(A) 1 (B) 4 (C) 9 (D) 18 (E) 44

Problem 13.17. (UNCO Math Contest — 2016.II.4): How many positive integers less than 100 are divisible by exactly two of the numbers $2, 3, 4, 5, 6, 7, 8, 9$? For example, 75 is such a number: it is divisible by 3 and by 5, but is not divisible by any of the others on the list. (If you show the integers you find, then you may be assigned partial credit if you have accurately found most of them, even if you do not find all of them.)

Problem 13.18. (Pan African MO — 2004.5): Each of the digits 1, 3, 7, and 9 occurs at least once in the decimal representation of some positive integers. Prove that one can permute the digits of this integer such that the resulting integer is divisible by 7.

Skill Assessment Problems

Skill Assessment Problem 13.1. Prove that for any integers m and n the number $\frac{(2m)!(3n)!}{(m!)^2(n!)^3}$ is always an integer.

Skill Assessment Problem 13.2. Prove that for all integers n, the number $n^9 - 6n^7 + 9n^5 - 4n^3$ is divisible by 8640.

Skill Assessment Problem 13.3. Prove that for all natural numbers n, $(n!)!$ is divisible by $n!^{(n-1)!}$.

Solutions to Skill Assessment Problems

Solution to Problem 13.1: Let's use the theorem discussed in the chapter. Why is the number $\frac{(2m)!}{(m!)^2}$ always an integer? Because in the numerator, the first m numbers of the product will exactly constitute $m!$, and the product of the second consecutive m numbers will also be divisible by $m!$. Similar considerations apply to the components of $(3n)!$, but now there are now 3 groups instead of 2. □

Solution to Problem 13.2: Factorizing both expressions, $n^9 - 6n^7 + 9n^5 - 4n^3 = n^3(n-2)(n-1)^2(n+1)^2(n+2)$, $8640 = 2^6 \cdot 3^3 \cdot 5$.

The expression includes the product of 5 consecutive numbers; hence, it is divisible by 5. Now let's consider divisibility by 2^6.

- If n leaves a remainder of 0 when divided by 4, then the expression n^3 is already divisible by 2^6.
- If n leaves a remainder of 1 when divided by 4, then the expression $(n-1)^2(n+1)^2$ is divisible by 2^6.
- If n leaves a remainder of 3 when divided by 4, then the expression $(n-1)^2(n+1)^2$ is divisible by 2^6.
- If n leaves a remainder of 2 when divided by 4, then the expression $n^3(n-2)(n+2)$ is divisible by 2^6.

Now let's consider divisibility by 3^3.

- If n is divisible by 3, then the expression n^3 is already divisible by 3^3.
- If n leaves a remainder of 1 when divided by 3, then the expression $(n-1)^2(n+2)$ is divisible by 3^3.
- If n leaves a remainder of 2 when divided by 3, then the expression $(n+1)^2(n-2)$ is divisible by 3^3.

Thus, the expression is divisible by three mutually prime numbers, which implies divisibility by their product, concluding the proof. □

Solution to Problem 13.3: $(n!)!$ can be expressed as the product of $n!$ consecutive numbers, which is $(n-1)!$ groups of n consecutive numbers. The product of each of these groups is divisible by $n!$, which completes the proof. □

Numeral Systems: Fractions and Advanced Concepts

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Brocard’s Problem poses the question of whether there are any integer solutions to the equation $n! + 1 = m^2$, where $n!$ denotes the factorial of n.

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The solutions $n = 4$ and $m = 5$ are the «starter pack» of Brocard’s Problem — showing us that solutions do exist!

Theory and Practice

In this chapter, we will focus on different numeral systems and their applicability to solving olympiad problems.

As a reminder, the formula for converting numbers to the decimal system from a numeral system with base s is: $\overline{a_n \ldots a_1 a_0}_s = a_n s^n + \ldots + a_1 s^1 + a_0 s^0$. In such a numeral system, the digits range from 0 to $s-1$. If the numeral system has a base greater than 10, Latin letters $A, B, \ldots$ are usually used as digits, representing values beyond 9.

In previous chapters, we proved divisibility criteria for various numbers. Now, let's explore how divisibility criteria change when transitioning to another numeral system.

Example 14.1. Consider the numeral system with base n. When will a number in this system be divisible by $n-1$?

Solution: Let's first understand what we need to prove. In the decimal system, the divisibility criterion for 9 was based on the sum of the digits of a number. Maybe here it will be the same?

It is the same! A $k+1$-digit number in the base-n numeral system looks like:

$$\overline{a_k \ldots a_1 a_0}_n = a_k n^k + \ldots + a_1 n^1 + a_0 n^0.$$

We can notice that $n = 1 \cdot (n-1) + 1$, $n^2 = (n-1) \cdot (n-1) + 1$, $n^3 = (n^2 - n + 1) \cdot (n-1) + 1$, or, in terms of congruence modulo, $n \equiv 1 (\bmod (n-1))$, which implies $\forall i \geqslant 0 \quad n^i \equiv 1 (\bmod (n-1))$.

Then, $a_k n^k + \ldots + a_1 n^1 + a_0 n^0 \equiv a_k + \ldots + a_1 + a_0 (\bmod (n-1))$. Thus, we have even proven a stronger statement: in the base-n numeral system, a number will have the same remainder when divided by $n-1$ as the sum of its digits. □

Example 14.2. Consider the numeral system with base 16 (which is used by computer scientists). Prove the following divisibility criteria:

a) A number is divisible by 15 if and only if the sum of its digits is divisible by 15.

b) A number is divisible by 5 if and only if the sum of its digits is divisible by 5.

c) A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

Solution: Part a) obviously follows from the previous problem. What about the remaining two parts? But 5 and 3 are just factors of 15. Therefore, since the remainders, when dividing by 15 for the number and the sum of its digits, are equal, the remainders, when dividing by 5 and 3, will also be equal. □

Example 14.3. Does the number

$$123456789ABCDEF101112\ldots FDFEFF_{16}$$

divide by 15?

Solution: Let's find the sum of the digits of this number. It will have the same remainder when divided by 15 as the sum $1 + 2 + 3 + \ldots FF_{16} = 100_{16} \cdot 128$, as $FF_{16} = 15 \cdot 16 + 15 = 15 \cdot 17 = 255$.

128 is not divisible by 15, and $100_{16} = 256$ is also not divisible by 15. Therefore, the sum of the digits of the original number is not divisible by 15, which implies that the number itself is not divisible by 15. □

Example 14.4. Find all natural numbers $k \geq 1$ such that the following statement is true: A natural number is divisible by 2 if and only if the sum of its digits in the base-k numeral system is divisible by 2.

Solution: The congruence

$$a_n q^n + \ldots + a_1 q + a_0 \equiv a_n + \ldots + a_1 + a_0 \pmod{m}$$

is equivalent to the congruence

$$a_n(q^n - 1) + \ldots + a_1(q - 1) \equiv 0 \pmod{m},$$

which holds independently of each digit of the number, a_i, if and only if $q - 1 \equiv 0 \pmod{m}$. In particular, for $m = 2$, any odd base-k numeral system is suitable. □

Example 14.5. Find the smallest natural number n such that in the base-n numeral system, the following divisibility criteria are simultaneously true:

1) a number is divisible by 5 if and only if the sum of its digits is divisible by 5;

2) a number is divisible by 7 if and only if the number formed by its last two digits is divisible by 7.

Solution: From the solution to the previous problem, we get that $n - 1 \equiv 0 (\bmod 5)$.

How do we consider the second condition? The congruence $a_x q^x + \ldots + a_1 q + a_0 \equiv a_1 q + a_0 (\bmod m)$ is equivalent to the congruence $a_x q^x + \ldots + a_2 q^2 \equiv 0 (\bmod m)$. It holds independently of a_i if and only if $q^2 \equiv 0 (\bmod m)$. In this case, as we consider the base n, we can say that $q = n$. Thus, $n^2 \equiv 0 (\bmod 7)$, that is, $n \vdots 7$.

The combination of these conditions gives the minimum base of the numeral system, which is 21. □

Fractions, of course, can also exist in different numeral systems.

Example 14.6. Express the number 13/16 in the base-6 numeral system.

Solution: Let

$$\frac{13}{16} = \frac{a_1}{6} + \frac{a_2}{6^2} + \ldots$$

Multiplying both sides by 6, we get:

$$\frac{39}{8} = \frac{a_1}{+}\frac{a_2}{6} + \dots.$$

$\frac{39}{8} = 4 + \frac{7}{8}$, thus on one side there will be 4 plus a number less than 1, and on the other side, there will be a_1 plus a number less than 1.

Therefore, $a_1 = 4$. Substituting this into the original expression, we get

$$\frac{7}{48} = \frac{a_2}{6^2} + \frac{a_3}{6^3} + \dots,$$

which, after multiplying by 6^2, is reduced to $a_2 = 5$.

Continuing similar reasoning, we obtain that $13/16 = 0.4513_6$ □

Example 14.7. The increasing sequence $1, 3, 4, 9, 10, 12, 13, \dots$ consists of integers that are either powers of three or the sum of different powers of three. Find the 100th term of this sequence.

Solution: If we write the terms of this sequence in the base-3 number system, we get a sequence of integers that do not contain the digit 2. Then, these numbers represent $1, 10, 11, 100, 101, 110, 111, \dots$. Imagining that these numbers were in the binary numeral system, we get $1, 2, 3, \dots$. To get the 100th term of the sequence, it is necessary to write the number 100 in the binary numeral system $100 = 1100100_2$, and then, imagining that this number was still written in the base-3 numeral system, convert it to the decimal system: $1100100_3 = 3^6 + 3^5 + 3^2 = 981$. □

Problem Set

Problem 14.1. (AU): There are scales with two pans and a load of 61 grams. There are also weights: 1 gram, 3 grams, 9 grams, 27 grams and 81 grams, however, there are only one of each of these weights. How can the scales be balanced when this load is placed on one of the sides?

Problem 14.2. (AU): A bag of sugar, a set of scales, and a 1 g weight are given. Is it possible to measure 1 kg of sugar in 10 weighings?

Problem 14.3. (AU): There are two-pan scales and 4 weights. The two-pan scales do not have an arrow; hence, it is possible to judge which side is heavier, but not how many grams heavier one side is than the other. How many different weights can you exactly weigh with these weights if
a) the weights can only be placed on one pan of the scales;
b) the weights can be placed on both pans of the scales?

Problem 14.4. (AU): You are allowed to choose 4 weights of any value. What should these weights be so that all loads from 1 to 40 kg can be exactly balanced on a two-pan scale?

Problem 14.5. (AU): a) There are two ropes. If either of them is ignited from one end, it will burn in an hour. The ropes burn unevenly. For example, it cannot be guaranteed that half of the rope will burn in 30 minutes. How, having two such ropes, can you measure a time interval of 15 minutes?
b) How many time intervals (including zero) can be measured with three such ropes?

Problem 14.6. (AU): a) Jacob's basement is illuminated by three light bulbs. The switches for these light bulbs are outside the basement. Jacob never remembers which switch lights up which bulb. After turning on any of the switches, Jacob has to go down to the basement to see exactly which bulb lit up. One day, he comes up with a way to determine for each switch to turn on the bulb, going down to the basement exactly once. How does Jacob do this?

b) What is the largest number of light bulb-switch combinations that can be identified if it is allowed to go down to the basement twice?

Problem 14.7. (AU): Two operations are allowed: «double it» and «increase it by 1». The starting number is 0. What is the least number of operations that can be done to the starting number for which you can get a) the number 100; b) the number n?

Problem 14.8. (AU): *Binary exponentiation method.* Suppose you need to raise the number x to the power of n. For example, if $n = 16$, this can be done by performing 15 multiplications $x^{16} = x \cdot x \cdot \ldots \cdot x$, or it can be done with only four:

$$x_1 = x \cdot x = x^2, x_2 = x_1 \cdot x_1 = x^4,$$
$$x_3 = x_2 \cdot x_2 = x^8, x_4 = x_3 \cdot x_3 = x^{16}.$$

Let $n = 2^{e_1} + 2^{e_2} + \ldots + 2^{e_r}$ $(e_1 > e_2 > \ldots > e_r \geqslant 0)$.

Devise an algorithm that allows you to calculate x^n using $b(n) = e_1 + v(n) - 1$ multiplications, where $v(n) = r$ — the number of ones in the binary representation of the number n.

Problem 14.9. (AU): Let $l(n)$ be the smallest number of multiplications required to find x^n. Using the examples of numbers $n = 15$ and $n = 63$, show that the binary exponentiation method is not always optimal, i.e., for some n, the inequality $l(n) < b(n)$ holds.

Problem 14.10. (AU): Leo thought of a number from 1 to 31 inclusive. In front of Leo lay the 5 given cards:

1	3	5	7
9	11	13	15
17	19	21	23
25	27	29	31

2	3	6	7
10	11	14	15
18	19	22	23
26	27	30	31

4	5	6	7
12	13	14	15
20	21	22	23
28	29	30	31

8	9	10	11
12	13	14	15
24	25	26	27
28	29	30	31

16	17	18	19
20	21	22	23
24	25	26	27
28	29	30	31

Leo picked out the cards on which his number was present. How, knowing the cards he chose and the cards he did not choose, can the intended number always be determined? How many numbers must be on each card, and how must they be written such that the intended number between 1 and 63 can always be determined?

Problem 14.11. (AU): *Card trick.* a) Take a deck of 27 cards (of different suites). Your friend chooses one of the cards (without telling you). Keep all the cards face-up as you spread them into three equal piles, putting one card each time (in the first pile, then in the second, then in the third, then again in the first, etc.). Your friend points to the pile in which his card lies. Then, you put all three piles together, inserting the indicated pile between the other two piles. This procedure is repeated two more times. At what place in the deck will the chosen card be after you put the three piles together for the third time?

b) at what place in the deck will the guessed card be if there were initially $3n$ (where $n < 9$) cards?

Problem 14.12. (AU): Max thought of a number: 1, 2, or 3. You can only ask him one question to which he can only answer «yes», «no», or «I don't know». Can you guess the number by only asking him one question?

Problem 14.13. (AU): Jean thought of a number from 1 to 200. How many questions do you need to ask him to guess the number if he answers each question with:
a) «yes» or «no»;
b) «yes», «no», or «I don't know»?
c) «yes» or «no», however he is allowed to lie exactly once.

Problem 14.14. (Mock AIME — 2005.2.9): Let

$$(1+x^3)\left(1+2x^{3^2}\right)\dots\left(1+kx^{3^k}\right)\dots\left(1+1997x^{3^{1997}}\right)=$$

$$= 1 + a_1 x^{k_1} + a_2 x^{k_2} + \ldots + a_m x^{k_m}$$

where $a_i \neq 0$ and $k_1 < k_2 < \ldots < k_m$. Determine the remainder obtained when a_{1997} is divided by 1000.

Problem 14.15. (AIME — 2023.II.15): For each positive integer n let a_n be the least positive integer multiple of 23 such that $a_n \equiv 1 \pmod{2^n}$. Find the number of positive integers n less than or equal to 1000 that satisfy $a_n = a_{n+1}$.

Problem 14.16. (CIME — 2020.II.4): The probability a randomly chosen positive integer $N < 1000$ has more digits when written in base 7 than when written in base 8 can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 14.17. (AIME — 2020.II.5): For each positive integer n, let $f(n)$ be the sum of the digits in the base-four representation of n and let $g(n)$ be the sum of the digits in the base-eight representation of $f(n)$. For example, $f(2020) = f(133210_4) = 10 = 12_8$, and $g(2020) =$ the digit sum of $12_8 = 3$. Let N be the least value of n such that the base-sixteen representation of $g(n)$ cannot be expressed using only the digits 0 through 9. Find the remainder when N is divided by 1000.

Problem 14.18. (AIME — 2010.I.10): Let N be the number of ways to write 2010 in the form $2010 = a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0$, where the a_i's are integers, and $0 \leqslant a_i \leqslant 99$. An example of such a representation is $1 \cdot 10^3 + 3 \cdot 10^2 + 67 \cdot 10^1 + 40 \cdot 10^0$. Find N.

Problem 14.19. (iTest — 2008.92): Find [the decimal form of] the largest prime divisor of 100111011_6.

Problem 14.20. (AIME — 2001.I.8): Call a positive integer N a 7-10 double if the digits of the base-7 representation of N form a base-10 number that is twice N. For example, 51 is a 7-10 double because its base-7 representation is 102. What is the largest 7-10 double?

Problem 14.21. (AIME — 2023.II.2): Recall that a palindrome is a number that reads the same forward and backward. Find the greatest integer less than 1000 that is a palindrome both when written in base ten and when written in base eight, such as $292 = 444_{\text{eight}}$.

Problem 14.22. (AMC — 2012.12B.11): In the equation below, A and B are consecutive positive integers, and A, B, and $A + B$ represent number bases:

$$132_A + 43_B = 69_{A+B}.$$

What is $A + B$?

(A) 9 (B) 11 (C) 13 (D) 15 (E) 17

Problem 14.23. (AMC — 2002.10B.9): Using the letters A, M, O, S, and U, we can form five-letter «words». If these «words» are arranged in alphabetical order, then the «word» $USAMO$ occupies position

(A) 112 (B) 113 (C) 114 (D) 115 (E) 116

Problem 14.24. (AJHSME — 1990.12): There are twenty-four 4-digit numbers that use each of the four digits 2, 4, 5, and 7 exactly once. Listed in numerical order from smallest to largest, the number in the 17th position in the list is

(A) 4527 (B) 5724 (C) 5742 (D) 7245 (E) 7524

Problem 14.25. (AHSME — 1986.10): The 120 permutations of $AHSME$ are arranged in dictionary order as if each were an ordinary five-letter word. The last letter of the 86th word in this list is:

(A) A (B) H (C) S (D) M (E) E

Skill Assessment Problems

Skill Assessment Problem 14.1. Prove that the number 4.41 is a perfect square in any numeral system.

Skill Assessment Problem 14.2. Let C be an increasing sequence of numbers that, when expressed in base 3, do not contain the digit 2. Prove that there are no three consecutive numbers a_1, a_2, a_3 in the sequence C such that $a_3 - a_2 = a_2 - a_1$.

Solutions to Skill Assessment Problems

Solution to Problem 14.1: We are considering 4.41 to be the square of a non-integer rational number. If we consider a numeral system with base r, then $4.41 = 4 + \frac{4}{r} + \frac{1}{r^2} = \left(2 + \frac{1}{r}\right)^2$, which is a perfect square of a decimal number. □

Solution to Problem 14.2: These numbers are the ones we already considered in this chapter; that is, the sequence consists of integers that are either powers of three or sums of different powers of three. It follows that there will be no carryovers when adding such numbers. The problem setting can be rewritten as $a_3 + a_1 = 2 \cdot a_2$. The number on the right side consists only of zeros and twos. However, If the numbers on the left side were different, having a one in some digit of the first number and a zero in the corresponding digit of the second number would result in a sum of one. Hence, equality cannot be achieved. The obtained contradiction completes the solution to the problem. □

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