Tatiana Babicheva is an inspiring teacher who believes that teaching is not just about imparting knowledge but also about being a lifelong learner. This passion for growth and discovery led her to earn two PhDs: one in applied mathematics and its applications in Russia, and another in computer science from Université Paris-Saclay in France. She has published 10 books in Russian and French in the field of popular science, including mathematics at the competition level.

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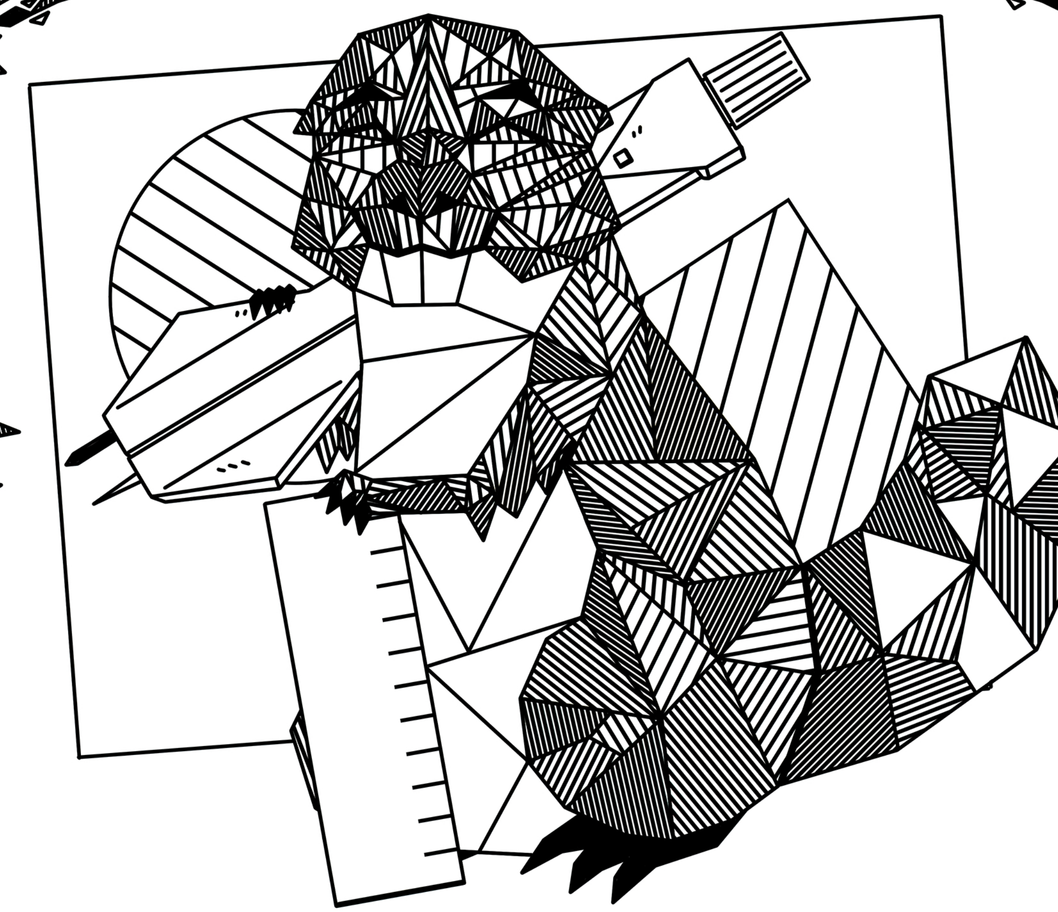
COMPETITIVE GEOMETRY

Mathematical Competitions.
Levels A1-A2

6.Competitive Geometry



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Mathematical Competitions. Levels A1-A2
Book 6. Competitive Geometry

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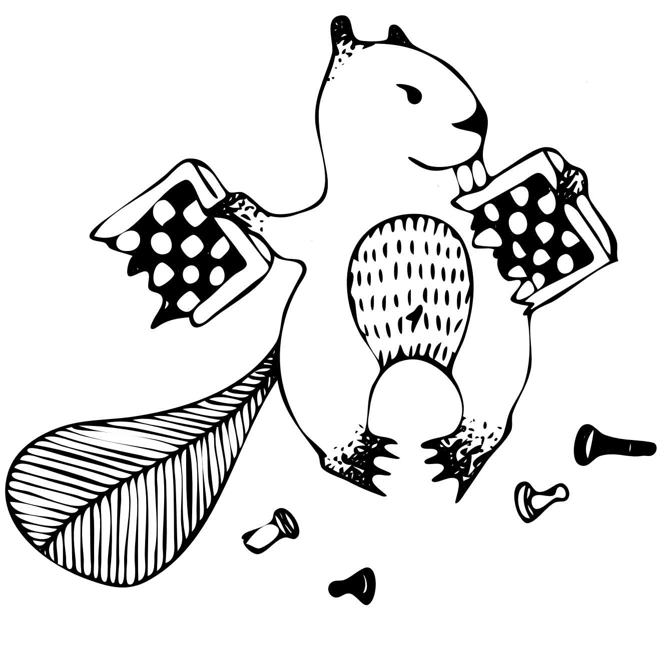
Dedication

To our esteemed colleagues, whose invaluable contributions made the realization of this book possible. A special acknowledgment goes to Michael Podaev for his exceptional support and insights.

Dedicated to our students, who discovered the joy of geometry through the pages of this book. Your enthusiasm and engagement have truly made the journey enjoyable.

Introduction



Introduction to the series

Begin your preparation for Competition Mathematics with our carefully crafted series. These books are designed to inspire a love for problem-solving and foster critical thinking. They are ideal for both budding mathematicians and passionate enthusiasts.

Inside, you will find a wide range of challenges, puzzles, and problems. Each one is selected to enhance your mathematical abilities. Experience the challenge of solving complex equations and gain confidence by deciphering complex geometric puzzles. Every book has engaging content to stimulate your mind and expand your skills.

If you're preparing for regional competitions, national tournaments, or simply want to deepen your mathematical knowledge, this series is an invaluable resource. The books provide clear explanations, strategic insights, and numerous practice problems. They aim to build your confidence and equip you with the skills needed to tackle any mathematical challenge.

While school mathematics forms a foundation, this series goes beyond it without requiring advanced knowledge to understand the material. Our course covers a wide range of topics, reflecting the diverse nature of Olympiad problems. Solving a geometry problem may require knowledge of combinatorics, while a number theory problem might involve understanding invariants and the pigeonhole principle.

Olympiad problems are generally not restricted to specific grade levels, making these books suitable for high school students. Some of the problems included have been featured in the final stages of national math Olympiads for higher grades. The goal is to demonstrate how to solve problems using straightforward and elegant methods, avoiding unnecessary complexity.

We have categorized competition mathematics into levels similar to the international standards used for foreign language proficiency. This approach is based on the concept of the «language» of competition mathematics. Traditional grade-based divisions are often outdated, as understanding a topic might only require elementary-level math. Moreover, the topics in these books are interconnected. Without a grasp of a topic at level A1, understanding its expanded form at level A2 can be challenging.

Here's what to expect at each level:

Let's use an analogy with foreign languages:

Level A1. You understand (generally) foreign speech and can talk about family, activities, hobbies, travels, weather, and buying things. In short, the standard tourist set. Can you conjugate basic verbs and be familiar with different tenses? The question «How are you?» doesn't stump you? Congratulations! You have a good A1 level! This is enough for survival.

Similarly, in olympiad math — you can «survive» at beginner-level olympiads, understand what is required in problems, and formulate solutions. You likely won't need math knowledge beyond seventh grade to understand topics at this level. (The problem might be from an 11th-grade olympiad, but the solving method remains the same)

At level A2, you can discuss preferences in art, cultural differences, and main social trends, etc. You form complex sentences («This is Peter, whose dad works at the bank. I've already told you about him»), can write to a friend on Facebook, describe a vacation, and understand the essence of any conversation in the language.

You can recognize and solve middle-level Olympiad problems. You will be able to avoid common mistakes and present your solutions effectively. Topics at this level typically require knowledge up to the eighth grade.

This series of books generally covers levels A1 and A2 of competition maths: you will understand any problem from most competitions, formulate your solution, and even change the solution of ChatGPT to match the real competition problem. However, you are still far from being a native speaker.

What is in these books?

This series uses a proof-based approach to problem-solving, which is usually reserved for advanced levels in countries like the USA and the UK. However, this method helps build a solid foundation in mathematics.

Each chapter is divided into four parts:

1. The first part covers the theoretical background and provides detailed solutions to typical problems.

2. The second part presents a problem set labeled by source. Olympiad problems are marked with notations like «Year.Grade/Round.Number.» For example, «ACM 2016.10A.5» is the fifth problem from the 10th-grade 10A variant of the ACM Olympiad 2016. Grade numbering may vary between countries, so adjust accordingly. Non-grade-specific Olympiads, like AIME, are marked by version (I or II) instead of grade.

You will encounter many problems from the Russian Olympiads (a country with a strong tradition in Olympiad mathematics) and various US mathematical competitions (such as AMC and AIME). We sincerely recommend not only finding the correct answer from the given AMC options but also approaching these problems from a proof-based perspective.

The problem number usually provides a sense of difficulty; generally, a higher number indicates a more challenging problem. However, this labeling doesn't always apply to some «independent» Olympiads, which can sometimes confuse genuine Olympiad participants.

3. The third part includes problems for independent solving, with some original problems introduced here.

4. Solutions are found in the fourth part.

The series consists of the following books.

1. Competitive Arithmetics
2. Ideas and Methods
3. Introduction to Discrete Mathematics
4. Introduction to Competitive Geometry
5. Competitive Number Theory
6. Competitive Geometry

This series is designed for both experienced Olympiad participants and newcomers to mathematical problem-solving. It offers a journey where theory and application meet, providing a rewarding experience. Welcome to a unique math adventure!

Introduction to this book

Within the pages of this book, we embark on a journey into the captivating realm of geometry. Allow me to make one thing clear – this is not an attempt to produce yet another run-of-the-mill geometry textbook. The existing school materials adequately cover the theoretical foundations essential for solving Olympiad-level geometry problems. Instead, our focus is on bridging the gap between possessing theoretical knowledge and mastering the art of conquering Olympiad geometry challenges.

In this book, we revisit essential geometric theory, providing both a reminder and a deeper understanding of the concepts crucial for Olympiad success. Additionally, we delve into various problem-solving methodologies tailored to the unique demands of Olympiad-style questions. Reading this book assumes you've already mastered school geometry, covering grades 7-9, and perhaps explored some aspects of AP Geometry (without trigonometry; it won't be necessary here).

In this book, in order to ensure the AMC problems pose a bit more challenge, we have selected those where reconstructing the diagram is possible purely from the textual conditions.

Embarking on this journey signifies your readiness to elevate your geometric prowess beyond the classroom. We won't rehash familiar theories but will delve into the intricacies, nuances, and strategic thinking required to crack the code of Olympiad geometry.

List of competitions used in this book

- «Математический праздник», in English mean «Mathematical festival». We note it in the book as «MF». The official site (in Russian) is https://olympiads.mccme.ru/matprazdnik/
- Городская устная математическая олимпиада для 6—7 классов, mean «City Oral Mathematical Olympiad for 6—7 grades». We note it in the book as «COM». The official site (in Russian) is https://olympiads.mccme.ru/ustn/
- Всероссийская олимпиада по геометрии им. И. Ф. Шарыгина (Москва), mean «Sharygin Geometry Olympiad». We note it in the book as «Sharygin». The official site (in Russian) is https://geometry.ru/
- Турнир городов, mean «Tournament of Towns». We note it in the book as «TOT». The official site is https://www.turgor.ru/en/
- Школьный этап Всероссийской олимпиады школьников, mean «first stage of All-Russian School Olympiad». We note it in the book as «1ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- Муниципальный этап Всероссийской олимпиады школьников, mean «second stage of All-Russian School Olympiad». We note it in the book as «2ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- Муниципальный этап Всероссийской олимпиады школьников (Москва), mean «second stage of All-Russian School Olympiad in Moscow». We note it in the book as «Mos2ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- Региональный этап Всероссийской олимпиады школьников, mean «third stage of All-Russian School Olympiad». We note it in the book as «3ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- Всероссийская олимпиада школьников, mean «All-Russian School Olympiad». We note it in the book as «ARSO». The official site (in Russian) is https://vserosolimp.edsoo.ru/
- American Mathematics Competitions. We note it in the book as «AMC». The official site is https://maa.org/math-competitions
- Московская математическая олимпиада, mean «Moscow Mathematical Olympiad». We note it in the book as «MMO». The official site (in Russian) is https://mmo.mccme.ru/
- Олимпиада им. Леонарда Эйлера, mean «Leonhard Euler Math Olympiad». We note it in the book as «LEO». The official site (in Russian) is http://matol.ru/
- Junior Mathematical Olympiad. We note it in the book as «JMO». The official site is https://ukmt.org.uk/

- Турнир им. Ломоносова, mean «Lomonosov Tournament». We note it in the book as «LT». The official site (in Russian) https://turlom.olimpiada.ru/
- Турнир Архимеда, mean «Archimedes Tournament». We note it in the book as «AT». The official site (in Russian) is http://www.arhimedes.org/
- Олимпиада «Ломоносов», mean «Lomonosov Competition», competition of Moscow State University. We note it in the book as «Lomonosov». The official site (in Russian) is https://olymp.msu.ru/
- Кружок МЦНМО, mean «Circle of Moscow Center for Continuous Mathematical Education». We note it in the book as «Mccme». The official site (in Russian) is https://mccme.ru/en/math-circles/circles-mccme/20232024/
- Московская математическая регата, mean «Moscow mathematical regatta». We note it in the book as «MMG». The official site (in Russian) is https://olympiads.mccme.ru/regata/
- «Покори Воробьёвы горы», mean «Conquer Vorobyovy Gory», competition of MSU. We note it in the book as «PVG». The official site (in Russian) is https://pvg.mk.ru/
- Nederlandse Wiskunde Olympiade. The official site is https://www.vwo.be/vwo/
- MATHCOUNTS competition. We note it in the book as «mathcounts». The official site is https://www.mathcounts.org/resources/mathcounts-minis
- Hamilton Mathematical Olympiad. We note it in the book as «HMO». The official site is https://ukmt.org.uk/
- Курчатов, mean «Kurchatov Competition». We note it in the book as «Kurchatov». The official site (in Russian) is https://olimpiadakurchatov.ru/
- Московская устная олимпиада по геометрии, mean « Moscow Geometry Oral Olympiad». We note it in the book as «OMGO». The official site (in Russian) https://olympiads.mccme.ru/ustn/
- Система задач Р. Гордина, mean «Gordin's system of problems». We note it in the book as «Gordin». The official site (in Russian) is https://zadachi.mccme.ru/
- МГУ Мехмат, mean «Moscow State University, Department of Mechanics and Mathematics». We note it in the book as «MechMat». The official site (in Russian) is https://math.msu.ru/
- МГУ, ДВИ, mean «Additional entrance test at Moscow State University». We note it in the book as «DVI». The official site (in Russian) is https://www.msu.ru/entrance/
- «РЕШУ ЕГЭ», математика, mean «I'll solve the Russian state exam in mathematics». We note it in the book as «EGE». The official site (in Russian) is https://ege.sdamgia.ru/

- ЦПМ, кружок по математике, mean «Mathematical Circle of Center for Teaching Excellence». We note it in the book as «ZPM». The official site (in Russian) is https://школа-цпм.рф/
- Каталог задач www.problems.ru, mean «Catalog of problems www.problems.ru». We note it in the book as «Problems.ru». The official site (in Russian) is problems.ru
- Прасолов В.В. Задачи по планиметрии, means the book «Prasolov V.V. Planimetry problems». We note it in the book as «Prasolov». The official site (in Russian) is https://math.ru/lib/files/pdf/planim5.pdf
- «Высшая проба», means the schoolchildren competition of HSE. We note it in the book as «HSE». The official site (in Russian) is https://olymp.hse.ru/mmo/

Contents

Geometric Inequalities

«Obvious» is the most dangerous word in mathematics.

—Eric Temple Bell

Theory and Practice

Although problems involving the triangle inequality in its pure form are quite rare in olympiads, it is often used in the solutions of «larger» problems.

Let's start by recalling the basic properties of triangles:

1. The larger angle of a triangle is opposite the larger side.
2. The larger side of a triangle is opposite the larger angle.
3. The sum of any two sides of a triangle is greater than the third side.

The last property is often referred to as the «triangle inequality,» and it is precisely this property that is most commonly used in problem-solving.

Example 1.1. In a triangle, the lengths of two sides are 1 and 2024 respectively. Find the length of the third side, given that it is expressed as an integer.

Solution: Denoting the side lengths as $a = 1$, $b = 2024$, and the unknown side as c, we have three inequalities:

$$\begin{aligned} 1 + 2024 &> c, \\ 1 + c &> 2024, \\ 2024 + c &> 1, \end{aligned}$$

which immediately implies $2023 < c < 2025$. Therefore, $c = 2024$. □

Example 1.2. Prove that in any triangle, the length of any side is less than half the perimeter of the triangle.

Solution: Let a, b, and c be the side lengths of the triangle.

Without loss of generality, let's prove that

$$a < \frac{a+b+c}{2}.$$

We know that $a < b+c$. Adding a to both sides of the inequality, we get $2a < a+b+c$. Dividing both sides of the inequality by the positive number 2 yields the desired result. □

Example 1.3. Prove that in any quadrilateral, the length of any diagonal is less than half the perimeter.

Solution: Let a, b, c, and d be the side lengths of the quadrilateral (see Figure 1.1).

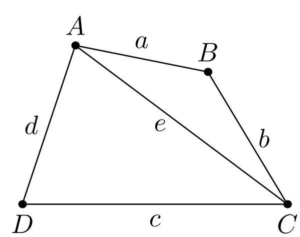


Figure 1.1: Side lengths of a quadrilateral.

Consider the diagonal AC and denote its length as e.

Then, by the triangle inequality, it is true that $a + b > e$ and $c + d > e$.

Adding these inequalities (since we can do this for inequalities with the same sign), we get $a + b + c + d > 2e$, and after dividing by the positive number 2, we obtain the desired statement.

Note that in formulating the inequalities, we did not use the convexity property of the quadrilateral, so the statement holds for arbitrary quadrilaterals. □

Example 1.4. Is it true that for any 10-line segments, there will be three that can form a triangle?

Solution: Here, it is sufficient to present 10 line segments from which it is impossible to choose any three to form a triangle.

Consider the set of the following 10 line segments: $1, 2, 4, 8, 16, 32, 64, 128, 256$, and 512. Let's assume a triangle with sides $a < b < c$ does exist. Given the values of the proposed lengths, $a < \frac{c}{2}$ and $b \leqslant \frac{c}{2}$. Adding these inequalities, we get $a + b < c$, which contradicts the triangle inequality. □

Inequalities in triangles are a powerful tool, especially for proving the impossibility of certain geometric constructions.

Problem Set

Problem 1.1. On the base AC of the isosceles triangle ABC, a point D is chosen, and on the extension of AC beyond vertex C, a point E is chosen such that $AD = CE$. Prove that $BD + BE > AB + BC$.

Problem 1.2. The diagonals AC and BD of the isosceles trapezoid $ABCD$ intersect at point O. It is also known that a circle can be inscribed in the trapezoid. Prove that $\angle BOC > 60°$.

Problem 1.3. (OMGO – 2019.7.5): A rectangular sheet of paper was folded along the diagonal. Can the perimeter of the resulting pentagon be equal to the perimeter of the original sheet?

Problem 1.4. (MMO — 1997.9): In a triangle, one side is three times smaller than the sum of the other two sides. Prove that the angle opposite to this side is the smallest angle in the triangle.

Problem 1.5. (MMG — 2012.9): On a plane, given a square and a point P, is it possible for the distances from point P to the vertices of the square to be 1, 1, 2, and 3?

Problem 1.6. (TOT — 2016.8.9): Prove that the sum of the lengths of any two medians of an arbitrary triangle
a) is not greater than $\frac{3P}{4}$, where P is the perimeter of the triangle;
b) is not less than $\frac{3p}{4}$, where p is the semiperimeter of the triangle.

Problem 1.7. (MMO — 2002.8.5): In triangle ABC, medians AD and BE intersect at point M. Prove that if angle AMB
a) is right;
b) is acute,
then $AC + BC > 3AB$.

Problem 1.8. (ARSO — 1993.9.2): The segments AB and CD of length 1 intersect at point O, and $\angle AOC = 60°$. Prove that $AC + BD \geqslant 1$.

Problem 1.9. (LEO — 2017.3): Diagonals of a convex quadrilateral $ABCD$ intersect at point E. It is known that $AB = BC = CD = DE = 1$. Prove that $AD < 2$.

Problem 1.10. (LEO — 2016.3): Given an equilateral triangle ABC, point D is chosen on the extension of side AB beyond point A, point E on the extension of BC beyond point C, and point F on the extension of AC beyond point C such that $CF = AD$ and $AC + EF = DE$. Find the angle $\angle BDE$.

Problem 1.11. The perpendicular bisector of side AB of triangle ABC intersects side AC at point K, and point K divides the broken line ACB into two parts of equal length. Prove that triangle ABC is isosceles.

Problem 1.12. (TOT — 2015.8-9): Prove that in any cyclic polygon, three sides exist that can form a triangle.

Problem 1.13. (TOT — 1985.7-8): A quadrilateral is inscribed in a rectangle (with one vertex on each side of the rectangle). Prove that the perimeter of the quadrilateral is not less than twice the diagonal of the rectangle.

Problem 1.14. (TOT — 1989.7-8): In triangle ABC, the median AM is drawn. Can the radius of the inscribed circle of triangle ABM be exactly twice the radius of the inscribed circle of triangle ACM?

Problem 1.15. (TOT — 2006.8-9): A segment of length 1 is divided into 11 segments, the length of each of which does not exceed a. For what values of a can we say that any three resulting segments can be used to form a triangle?

Problem 1.16. (TOT — 2014.8-9): Prove that among 100 red, 100 yellow, and 100 green sticks, where any three sticks of different colors can form a triangle, there must be a color for which any three sticks of that color can form a triangle.

Problem 1.17. (TOT — 2015.8-9): Given are N right-angled triangles. For each triangle, one leg was chosen, and the sum of their lengths was found. Then, the sum of the lengths of the remaining legs was found, and finally, the sum of the lengths of all hypotenuses was found. It turned out that these three found numbers are the side lengths of a certain right-angled triangle. Prove that for all original triangles, there is a consistent ratio between the longer leg and the shorter leg, if

a) $N = 2$;

b) N — is any number bigger than 1.

Problem 1.18. (TOT — 1991.8-9): The sides AB, BC, CD, and DA of quadrilateral $ABCD$ are respectively equal to the sides $A'B'$, $B'C'$, $C'D'$, and $D'A'$ of quadrilateral $A'B'C'D'$. Additionally, it is known that $AB \parallel CD$ and $B'C' \parallel D'A'$. Prove that both quadrilaterals are parallelograms.

Problem 1.19. (Sharygin — 2012.8.5): Does there exist a convex quadrilateral and a point P inside it such that the sum of the distances from P to the vertices is greater than the perimeter of the quadrilateral?

Problem 1.20. (MMO — 2011.11.3): In an isosceles triangle ABC, point D is chosen on the base BC, and points E and M are chosen on the side AB such that $AM = ME$, and the segment DM is parallel to side AC. Prove that $AD + DE > AB + BE$.

Problem 1.21. (mathcounts) How many non-congruent triangles are there with sides of integer length having at least one side of length five units and having no side longer than five units?

Problem 1.22. (AMC — 2003.12A.7): How many non-congruent triangles with perimeter 7 have integer side lengths?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 1.23. (UK Kangaroo): The lengths of two sides of a triangle are 5 cm and 2 cm. The length of the third side in cm is an odd integer. What is the length of the third side?

(A) 1cm (B) 3cm (C) 5cm (D) 7cm (E) 9cm

Problem 1.24. (UK Kangaroo): The lengths of the sides of a triangle are the integers $13, x, y$. It is given that $xy = 105$. What is the length of the perimeter of the triangle?

Problem 1.25. (HMMT): Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. Company XYZ wants to locate their base at the point P in the plane, minimizing the total distance to their workers, who are located at vertices A, B, and C. There are 1, 5, and 4 workers at A, B, and C, respectively. Find the minimum possible total distance Company XYZ's workers have to travel to get to P.

Problem 1.26. (Canadian math competition): The triangle with side lengths 6, 8, and 10 is right-angled, while the triangle with side lengths 6, 8, and 9 is an acute triangle, and the triangle with side lengths 6, 8, and 11 is an obtuse triangle. An obtuse triangle with a positive area has side lengths of 10, 17, and x. If x is an integer, what is the sum of all possible values of x?

Problem 1.27. (International Olympiad «Formula of Unity»): John has 12 sticks; the length of each stick is a positive integer not greater than 56. Prove that he has three sticks that could form a triangle.

Problem 1.28. (HMMT): What is the smallest possible perimeter of a triangle whose side lengths are all squares of distinct positive integers?

Skill Assessment Problems

Skill Assessment Problem 1.1. Prove that the sum of the diagonals of any convex quadrilateral is less than its perimeter but greater than half of its perimeter.

Skill Assessment Problem 1.2. The angle at vertex M of triangle ACM is $60°$. Prove that

$$AM + MC \leqslant 2 \cdot AC.$$

Solutions to Skill Assessment Problems

Solution to Problem 1.1: In the quadrilateral $ABCD$ (Figure 1.2), each diagonal forms 2 triangles, diagonal AC forms triangles ADC and ABC, and diagonal BD forms triangles BAD and BCD.

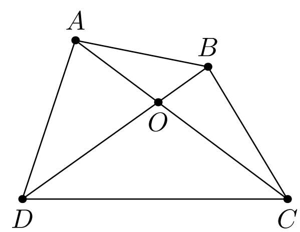


Figure 1.2: Sum of diagonals of a quadrilateral.

Writing down the inequalities for the larger triangles, we get

$$\begin{aligned} BD &< BA + AD, \\ BD &< BC + CD, \\ AC &< AD + DC, \\ AC &< AB + BC. \end{aligned}$$

Summing up these inequalities, we arrive at the inequality

$$2 \cdot BD + 2 \cdot AC < 2 \cdot AB + 2 \cdot BC + 2 \cdot CD + 2 \cdot DA,$$

from which the first estimate follows.

To solve the second part of the problem, let's consider smaller triangles involving point O. We obtain the following triangle inequalities:

$$\begin{aligned} AB &< BO + AO, \\ AD &< AO + DO, \\ DC &< DO + CO, \\ BC &< BO + CO. \end{aligned}$$

After summing up these inequalities, we get

$$AB + AD + DC + BC < 2 \cdot AO + 2 \cdot OC + 2 \cdot BO + 2 \cdot OD,$$

which, after dividing both sides, gives us the desired second estimate. □

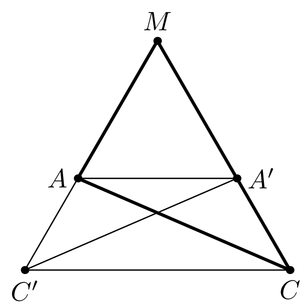


Figure 1.3: Triangle AMC.

Solution to Problem 1.2: If $MC = MA$, then the triangle is equilateral, and the equality $AM + MC = 2AC$ holds.

If the triangle is not equilateral, we extend the shorter side (let it be AM) by the length of side MC, forming an equilateral triangle MCC'. On side MC, we mark off segment $MA' = MA$.

Since $AC = A'C'$, then $2AC = AC + A'C' < CC' + AA' = MC + AM$. The statement of the problem is proven. □

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Angle Calculations

> It is worth repeating that geometry is the art of reasoning well from badly drawn figures.
>
> —Henri Poincaré

Theory and Practice

Often, when solving geometry problems, many forget to do the simplest thing: consider all possible angles arising in the problem. At the same time, one of the first facts of plane geometry that many encounter even before the beginning of the study of school geometry is the formula for the sum of angles in a triangle: «the sum of angles in a triangle is equal to 180 degrees.» Of course, the method of «calculating angles» is not limited to just using this fact, but often you can find problems involving combinations of triangles (isosceles, equilateral, right-angled), bisectors, and perpendiculars, where it is necessary to sequentially calculate angles, proving one fact or another. Calculation of angles is also popular in problems with circles, where formulas for an inscribed angle hint at the multiplicity of equal or easily expressible angles. However, in this context, we will focus, for warm-up, on straight lines and segments.

To find, or rather «calculate,» angles in many problems, it is enough to know about the sum of angles in a triangle, equality of vertical angles, the sum of adjacent angles, angles with parallel lines, and angles in a circle. Often, introducing one, two, and, in rare cases, three variables is sufficient.

Example 2.1. In a right-angled triangle ABC, altitude CK is drawn from the vertex of the right angle C, and in triangle ACK, angle bisector CE is drawn. Prove that $CB = BE$.

Figure 2.1: Illustrations for the solutions. a) First solution, b) second solution.

First Solution: In this solution, we will use Figure 2.1 a).

Let $\angle A = \alpha$. Then, we can determine the other angles: $\angle ACK = 90^\circ - \alpha$. Furthermore, $\angle ECK = \angle ACE = 45^\circ - \frac{\alpha}{2}$. Thus, $\angle KCB = \alpha$, and $\angle CEB = \angle A + \angle ACE = \alpha + 45^\circ - \frac{\alpha}{2} = 45^\circ + \frac{\alpha}{2}$. Similarly, $\angle BCE = \angle KCB + \angle ECK = \alpha + 45^\circ - \frac{\alpha}{2} = 45^\circ + \frac{\alpha}{2}$. The discovered angle equality proves that triangle CBE is isosceles and $CB = BE$. □

Second Solution: Let us consider a perpendicular EL to AC, as shown in Figure 2.1 b). Triangles ECK and ECL are congruent by the hypotenuse and acute angle, so $CK = CL$. But the segment CL is equal to the height of the triangle CBE dropped on side BC. Therefore, in triangle CBE, two altitudes are equal, that is, it is isosceles. □

Example 2.2. In an acute-angled scalene triangle, a height is drawn through one vertex, a median through another, and a bisector through the third. Prove that if the drawn lines intersect to form a triangle, it cannot be equilateral.

Solution: Let the height AH of triangle ABC intersect the median BM at point P, the bisector CL at point R, and the bisector CL and median BM intersect at point Q, as shown in Figure 2.2.

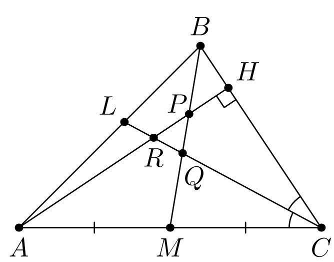


Figure 2.2: Height, median, and bisector from different angles of the triangle.

Suppose that triangle PQR is equilateral. Then $\angle RCH = 90^\circ - \angle CRH = 90^\circ - 60^\circ = 30^\circ, \angle C = 60^\circ$. Moreover, $\angle PBH = 90^\circ - \angle BPH = 90^\circ - \angle RPQ =$

$90^\circ - 60^\circ = 30^\circ, \angle BMC = 90^\circ$. Thus, BM is the height and median of triangle ABC. Therefore, triangle ABC is isosceles, and since one of its angles (C) is 60°, it is equilateral. Contradiction. □

Example 2.3. In a triangle with unequal sides AB and AC, a height AH and a bisector AD are drawn. Prove that the angle HAD is equal to half the difference between angles B and C.

Solution: Without the loss of generality, let $AC > AB$. Then point D lies between C and H (see Figure 2.3).

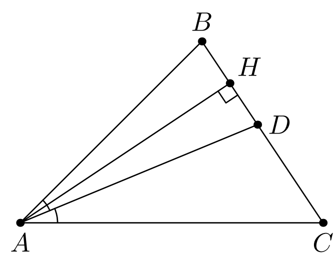


Figure 2.3: Figure for the problem.

Therefore, $\angle HAD = 90^\circ - \angle ADC = 90^\circ - (\angle DAC + \angle C) = 90^\circ - \angle ADC = 90^\circ - \frac{1}{2}\angle A - \angle C = \frac{1}{2}(\angle B - \angle C)$. □

Certainly, nobody prevents us from introducing variables when counting angles, and the number of variables does not necessarily have to be limited to one (after all, we participate in Olympiads; will we be scared of some algebra in geometry?!).

Example 2.4. The bisectors of angles A and B of triangle ABC are equally inclined to sides BC and AC. Find the relationship between angles A and B.

Solution: Let AK and BM be the bisectors of triangle ABC, and P be the point of their intersection (see Figure 2.4).

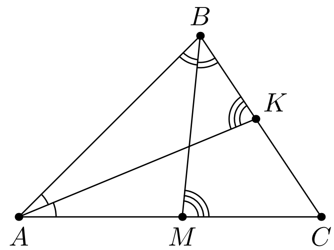


Figure 2.4: Two bisectors in a triangle.

Denote $\angle A = \alpha, \angle B = \beta$. From the condition, it follows that $\angle AKC = \angle BMC$ or $\angle AKC + \angle BMC = 180°$. In the first case, $\angle AKC = \angle KAB + \angle KBA = \frac{\alpha}{2} + \beta, \angle BMC = \angle MBA + \angle MAB = \alpha + \frac{\beta}{2}$. Then $\frac{\alpha}{2} + \beta = \alpha + \frac{\beta}{2}$. Consequently, $\alpha = \beta$. In the second case, similarly, we obtain that $\alpha + \beta = 120°$. $\square$

Problem Set

Problem 2.1. (Gordin — 1306): The bisector of the angle adjacent to angle C of triangle ABC intersects the extension of side AB beyond point B at point D, and the bisector of the angle adjacent to angle A intersects the extension of BC beyond point C at point E. It is known that $DC = CA = AE$. Find the angles of triangle ABC.

Problem 2.2. (Gordin — 2726): From point O on the plane, four rays emerge in clockwise order: OA, OB, OC, and OD. It is known that the sum of angles AOB and COD is $180°$. Prove that the bisectors of angles AOC and BOD are perpendicular.

Problem 2.3. (TOT — 1999.8.20): In triangle ABC, points A', B', and C' lie on sides BC, CA, and AB, respectively. It is known that $\angle AC'B' = \angle B'A'C$, $\angle CB'A' = \angle A'C'B$, $\angle BA'C' = \angle C'B'A$. Prove that points A', B', and C' are the midpoints of the sides of triangle ABC.

Problem 2.4. (MMG — 2011.7.3): Given square $ABCD$. On the side AD of the square, an equilateral triangle ADE is constructed inside the square. Diagonal AC intersects side ED of this triangle at point F. Prove that $CE = CF$.

Problem 2.5. (Gordin — 1103): The triangle ABC is given. On the extension of side AC beyond point A, a segment $AD = AB$ is marked, and beyond point C, a segment $CE = CB$ is marked. Find the angles of triangle DBE in terms of the angles of triangle ABC.

Problem 2.6. (Gordin — 1103): On the side BC of an isosceles triangle ABC ($AB = BC$), points N and M (N is closer to B than M) are taken such that $NM = AM$ and $\angle MAC = \angle BAN$. Find $\angle CAN$.

Problem 2.7. (1ARSO — 2014.7.2): It's half past nine. What is the angle between the hour and minute hands?

Problem 2.8. (1ARSO — 2018.7.3): At some point in time, Esther measured the angle between the hour and minute hands of her clock. Exactly one hour later, she measured the angle again, and it turned out to be the same. What could this angle be? (Consider all possible cases.)

Problem 2.9. (2ARSO — 2018.7.1): Draw four rays OA, OB, OC, and OD with a common starting point in such a way that the following angles can be found on this diagram: $100°$, $110°$, $120°$, $130°$, and $140°$. Write down which angles have the specified measures.

Problem 2.10. (Lomonosov — 2016.7.8.3): In the rectangle, $ABDF$ with sides $BD = 2$ and $DF = 3$, points C and E are chosen on the sides BD and DF, respectively, in such a way that triangle AFE is congruent to triangle EDC. Then triangles ABC, CDE, and AFE are cut off from the rectangle $ABDF$. Find the angles of the remaining triangle.

Problem 2.11. (OMGO — 2013.7.3): A triangle was folded from a square sheet of paper (see Figure 2.5). Find the angle measure of the marked corner.

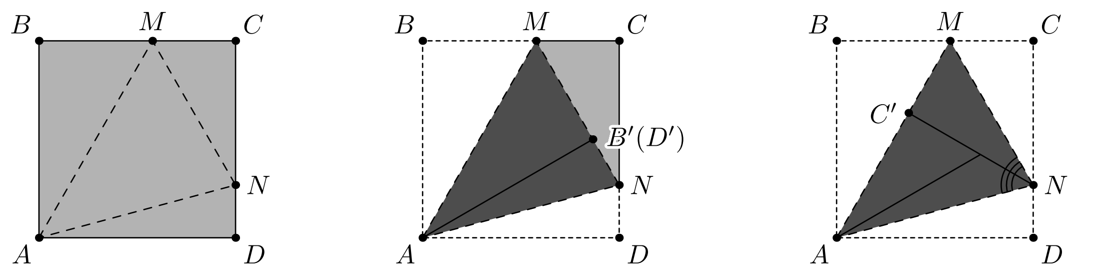


Figure 2.5: Square folded into a triangle

Problem 2.12. (MF — 2019.7.3): Two congruent triangles are located inside a square, as shown in Figure 2.6. Find their angles.

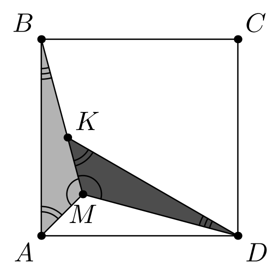


Figure 2.6: Triangles in a square

Problem 2.13. (Kurchatov — 2015.7.4): In a right-angled triangle ABC, the bisector AL was drawn, and a point K was marked on the hypotenuse AB such that $AB = 3BK$. It turned out that the angle ALK is right. Prove that $AL = BL$.

Problem 2.14. (HSE — 2015.7.6): In a triangle ABC where $AB = BC$ and $\angle ABC = 90^\circ$, the altitude BH is drawn. On side CA, a point P is chosen such that $AP = AB$, and on side CB, a point Q is chosen such that $BQ = BH$. Prove that the lines PQ and AB are parallel.

Problem 2.15. (OMGO — 2013.7.8): The bisectors of triangle ABC intersect at point I, where $\angle ABC = 120^\circ$. On the extensions of sides AB and CB beyond point B, points P and Q are marked, respectively, such that $AP = CQ = AC$. Prove that angle PIQ is a right angle.

Problem 2.16. (MF — 2014.7.2): Two identical right-angled triangles made of paper were placed one on top of the other, as shown in Figure 2.7 (with one triangle's right angle vertex on the side of the other). Prove that the shaded triangle is equilateral.

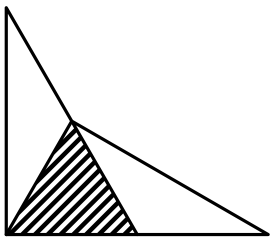


Figure 2.7: Overlapping triangles.

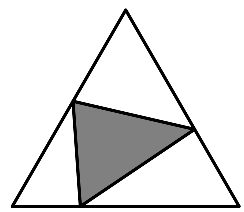


Figure 2.8: Folded triangle.

Problem 2.17. (MF — 2015.7.5): A paper equilateral triangle was folded along a straight line so that one of the vertices landed on the opposite side (see Figure 2.8). Prove that the angles of the two white triangles at the bottom are respectively equal.

Problem 2.18. (OMGO — 2012.7.5): In triangle ABC, the bisector of angle C intersects side AB at point M, and the bisector of angle A intersects segment CM at point T. It turned out that segments CM and AT divided triangle ABC into three isosceles triangles. Find the angles of triangle ABC.

Problem 2.19. (Lomonosov — 2014.8): In an isosceles triangle ABC, each angle contains a non-integer number of degrees. It is known that through one of the vertices of triangle ABC, one can draw a straight cut that divides the given triangle into two isosceles triangles. Find the angles of triangle ABC.

Problem 2.20. (MMG — 2003.7.1.3): Does there exist a triangle in which the degree measure of each angle is expressed by a prime number?

Problem 2.21. (Gordin — 1104): The altitudes of triangle ABC drawn from vertices A and C intersect at point M. Find $\angle AMC$ if $\angle A = 70°$, $\angle C = 80°$.

Problem 2.22. (PVG 2014.8-9): Leo wanted to draw an equilateral triangle ABC. However, due to his imprecise drawing, he obtained a triangle with angles $\angle A = 59°$ and $\angle B = 63°$. Later, Leo drew altitudes CE and BD. However, since the right angle in the ruler was slightly distorted, he got angles $\angle ADB = \angle AEC = 92°$. Find the degree measure of angle AED.

Problem 2.23. (HSE — 2018.7-8.4): Let there be a quadrilateral $ACDE$ such that vertices D and E are on the same side of the line AC. Take point B on side AC such that triangle BCD is isosceles with base BC (i.e., $BD = CD$). Suppose that angles $\angle BDC$, $\angle ABE$, and $\angle ADE$ are all equal to $80°$. Find the angle EAD.

Problem 2.24. (MechMat — 2003.05.3): In triangle ABC with $\angle B = 50°$ and side $BC = 3$, a point D is taken on the altitude BH such that $\angle ADC = 130°$ and $AD = \sqrt{3}$. Find the angle between the lines AD and BC, and also find $\angle CBH$.

Problem 2.25. (MechMat — 1999.03.4): The diagonals of the convex quadrilateral $ABCD$ intersect at point E. It is given that $AB = AD$, CA is the angle bisector of $\angle C$, $\angle BAD = 140°$, and $\angle BEA = 110°$. Find the angle CDB.

Problem 2.26. (Mos2ARSO — 2014.10): Point F is the midpoint of side BC of square $ABCD$. A perpendicular AE is drawn to segment DF. Find the angle CEF.

Problem 2.27. (LEO — 2018.8.4): Inside the parallelogram $ABCD$, a point E is chosen such that $AE = DE$, and $\angle ABE = 90°$. Point M is the midpoint of segment BC. Find the angle DME.

Problem 2.28. (OMGO — 2012.8.9): In a trapezoid $ABCD$, where sides AD and BC are parallel, and $AB = BC = BD$, the height BK intersects diagonal AC at point M. Find $\angle CDM$.

Problem 2.29. (OMGO – 2011.8.9): In a trapezoid $ABCD$ where $AB = BC = CD$, and CH is the height, prove that the perpendicular from H to AC passes through the midpoint of BD.

Problem 2.30. (AMC – 2016.10B.17): The ratio of the measures of two acute angles is $5 : 4$, and the complement of one of these two angles is twice as large as the complement of the other. What is the sum of the degree measures of the two angles?
(A) 75 (B) 90 (C) 135 (D) 150 (E) 270

Problem 2.31. (AMC – 2012.10A.10): Mary divides a circle into 12 sectors. The central angles of these sectors, measured in degrees, are all integers, and they form an arithmetic sequence. What is the degree measure of the smallest possible sector angle?
(A) 5 (B) 6 (C) 8 (D) 10 (E) 12

Problem 2.32. (AMC – 2011.10B.7): The sum of two angles of a triangle is $6/5$ of a right angle, and one of these two angles is 30° larger than the other. What is the degree measure of the largest angle in the triangle?
(A) 69 (B) 72 (C) 90 (D) 102 (E) 108

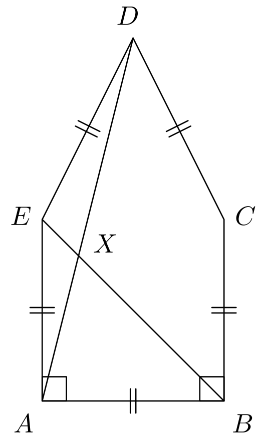
Problem 2.33. (AMC – 2010.10B.6): A circle is centered at O, $\overline{AB}$ is a diameter, and C is a point on the circle with $\angle COB = 50^\circ$. What is the degree measure of $\angle CAB$?
(A) 20 (B) 25 (C) 45 (D) 50 (E) 65

Problem 2.34. (International Olympiad «Formula of Unity»): Let BK be a bisector of triangle ABC. Given that $AB = AC$ and $BC = AK + BK$, find the angles of the triangle.

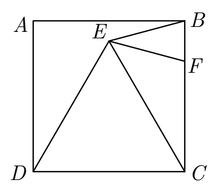
Problem 2.35. (HMMT): Let P be a point inside regular pentagon $ABCDE$ such that $\angle PAB = 48^\circ$ and $\angle PDC = 42^\circ$. Find $\angle BPC$, in degrees.

Problem 2.36. (HMO): The diagram shows a pentagon $ABCDE$ in which all sides are equal in length, and two adjacent interior angles are 90°. The point X is the point of intersection of AD and BE.

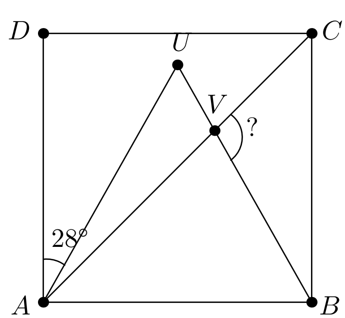
Prove that $DX = BX$.



Problem 2.37. (JMO): In the diagram shown, $ABCD$ is a square, and point F lies on BC. Triangle DEC is equilateral, and $EB = EF$. What is the size of $\angle CEF$?



Problem 2.38. (Nederlandse Wiskunde Olympiade): In the square $ABCD$ lies a point U such that BU and AB have the same length. Point V is the intersection of BU and the diagonal AC. The size of angle DAU is 28 degrees. What is the size of the angle at V in triangle BVC?



Skill Assessment Problems

Skill Assessment Problem 2.1. A five-pointed star (self-intersecting) is given. Find the sum of the angles at its five vertices.

Skill Assessment Problem 2.2. Find the degree measures of the angles indicated in the figure by x.

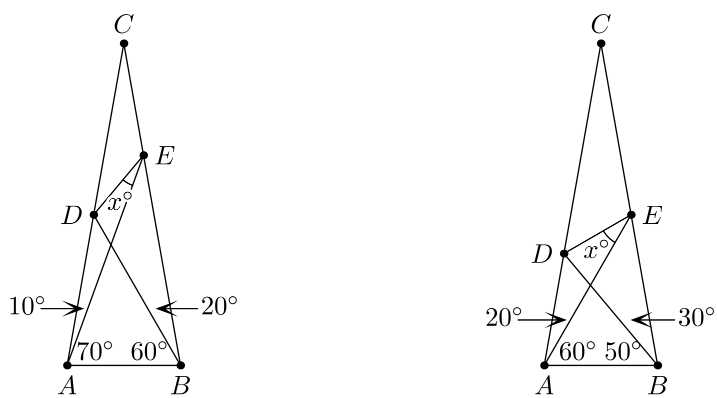


Figure 2.9: Angles to find.

Skill Assessment Problem 2.3. Point M lies inside triangle ABC. Prove that $\angle BMC > \angle BAC$.

Solutions to Skill Assessment Problems

Solution to Problem 2.1: Denote the vertices of the star sequentially: A_1, A_2, A_3, A_4, A_5 (Figure 2.10).

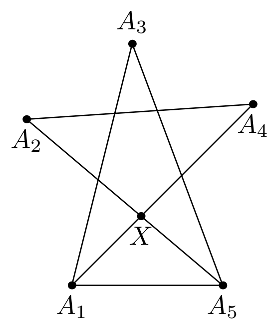


Figure 2.10: Star with additional construction.

Let X be the intersection point of the segments A_1A_4 and A_2A_5. Express the sums of angles in triangles A_2A_4X and A_1A_5X : $\angle A_2A_4X + \angle A_2XA_4 + \angle XA_2A_4 = 180°$, $\angle A_1A_5X + \angle A_1XA_5 + \angle XA_1A_5 = 180°$. Then, the sum of angles A_2 and A_4 is equal to the sum of angles XA_1A_5 and XA_5A_1. However, the sum of angles in triangle $A_1A_3A_5$ is 180 degrees, and as we have already shown, this sum is equal to the sum of all angles of the star. Therefore, the sum of angles at the vertex of the star is 180 degrees.

□

Solution to Problem 2.2: Let's solve the problem step by step for each of the proposed figures to demonstrate one of the possible reasoning processes.

First, let's find the angle x for the first figure.

Step 1. Find some angles:
$\angle ACB = 180 - (10 + 70) - (60 + 20) = 20°$,

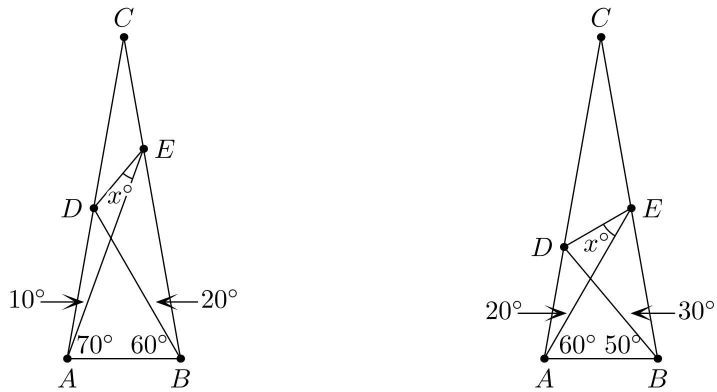


Figure 2.11: Angles to find.

$\angle AEB = 180 - 70 - (60 + 20) = 30°$.

Step 2. Draw a line through point D parallel to AB that intersects BC at point F (Figure 2.12). Then,
$\angle CFD = \angle CBA = 60 + 20 = 80°$,
$\angle DFB = 180 - 80 = 100°$,
$\angle CDF = \angle CAB = 70 + 10 = 80°$,
$\angle ADF = 180 - 80 = 100°$,
$\angle BDF = 180 - 100 - 20 = 60°$.

Step 3. Let FA intersect DB at point G, then we can calculate the following angles:
$\angle AFD = \angle BDF = 60°$,
$\angle DGF = 180 - 60 - 60 = 60° = \angle AGB$,
$\angle GAB = 180 - 60 - 60 = 60°$.
Thus, from the triangles with all angles equal to 60 degrees, we can identify the equilateral triangles DFG and AGB.

Step 4. CFA has two angles of 20° each, so $FC = FA$.

Step 5. Draw CG, which bisects the angle ACB:
$FC - CE = FA - AG = FE = FG$.
On the other hand, $FG = FD$, so $FE = FD$.

Step 6. With two equal sides, triangle DFE is isosceles, hence:

$\angle DEF = 30 + x = (180 - 80)/2 = 50°$.
Thus, we obtain $x = 20°$.

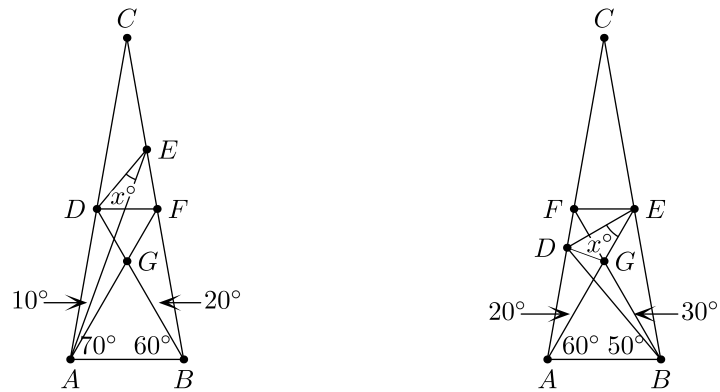


Figure 2.12: Additional constructions for the problem.

Now, let's find the angle x for the second figure.

Step 1. Calculate some angles:
$\angle ACB = 180 - (10 + 70) - (60 + 20) = 20°$,
$\angle AEB = 180 - 60 - (50 + 30) = 40°$.

Step 2. Draw a line through E parallel to AB, intersecting AC at point F (Figure 2.12):
$\angle CEF = \angle CBA = 50 + 30 = 80°$,
$\angle FEB = 180 - 80 = 100°$,
$\angle AEF = 100 - 40 = 60°$,
$\angle CFE = \angle CAB = 60 + 20 = 80°$,
$\angle EFA = 180 - 80 = 100°$.

Step 3. Let FB intersect AE at point G:
$\angle AFB = \angle BEA = 40°$,
$\angle BFE = \angle AEF = 60°$,
$\angle FGE = 180 - 60 - 60 = 60° = \angle AGB$.
Then,
$\angle ABG = 180 - 60 - 60 = 60°$.

Step 4. Draw DG. Since $AD = AB$ and $AG = AB$, we have $AD = AG$.
Thus, triangle DAG is isosceles.
Therefore, $\angle ADG = \angle AGD = (180 - 20)/2 = 80°$.

Step 5. $\angle DGF = 180 - 80 - 60 = 40°$, so triangle FDG is isosceles, having two angles of $40°$, and therefore $DF = DG$.

Step 6. $EF = EG$, so triangles DEF and DEG are congruent.
In this case, $\angle DEF = \angle DEG = x$.
Moreover, $\angle FEG = 60 = x + x$, which leads to the answer $x = 30°$.

There is another approach to solving the problem — you can notice that the triangle itself is enclosed within the sides and diagonals of a regular 18-gon, and the segments drawn inside it lie on the diagonals of this regular polygon (Figure 2.13):

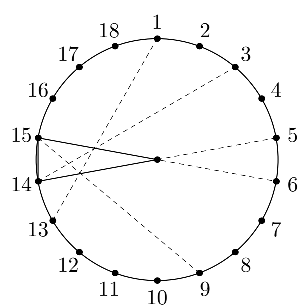


Figure 2.13: Application of a regular polygon.

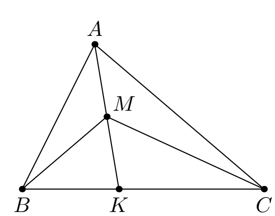


Figure 2.14: Additional constructions for the problem.

Of course, you and I agree that this is very easy to notice, but such people exist. □

Solution to Problem 2.3: Extend AM to intersect side BC at point K (Figure 2.14). Then, $\angle BMK > \angle BAM$, and $\angle CMK > \angle CAM$. Therefore, $\angle BMC = \angle BMK + \angle CMK > \angle BAM + \angle CAM = \angle BAC$. □

Double the Median in Triangles

> "There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.
>
> —Nikolai Ivanovich Lobachevsky

Theory and Practice

Let us recall the basic theory that will be useful to you in this chapter.

Definition 1. A parallelogram is a quadrilateral whose both pairs of opposite sides are parallel.

Of course, like any geometric figure, a parallelogram has its properties. The following ones are usually studied in school:

- Opposite angles of a parallelogram are equal.
- Opposite sides of a parallelogram are equal.
- The diagonals of a parallelogram bisect each other.

These are the properties of parallelograms, meaning they are criteria that can be used to determine whether a quadrilateral is a parallelogram. Perhaps the most useful among them is the criterion that if the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

Example 3.1. The median AM of triangle ABC is extended beyond point M by a distance equal to AM, and a point D is marked at the end of the segment (see Figure 3.1). Prove that the quadrilateral $ABDC$ is a parallelogram.

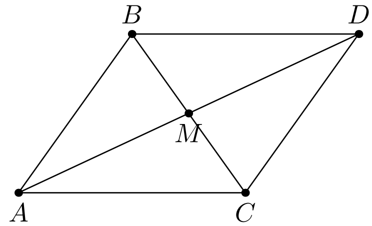


Figure 3.1: Doubling the median

Proof. The diagonals of the quadrilateral $ABDC$ intersect at point M and divide each other in half, so the quadrilateral $ABDC$ is a parallelogram. □

The fact we proved is probably one of the most well-known properties that aid in additional construction. It is extremely useful for a wide range of problems, and we'll explore some of them in this chapter.

Example 3.2. (Mccme — 2017.8-9.2): The median BM of triangle ABC forms angles $40°$ and $70°$, with its sides emanating from B. Prove that this median is equal to half of one of the sides of the triangle (Figure 3.2a).

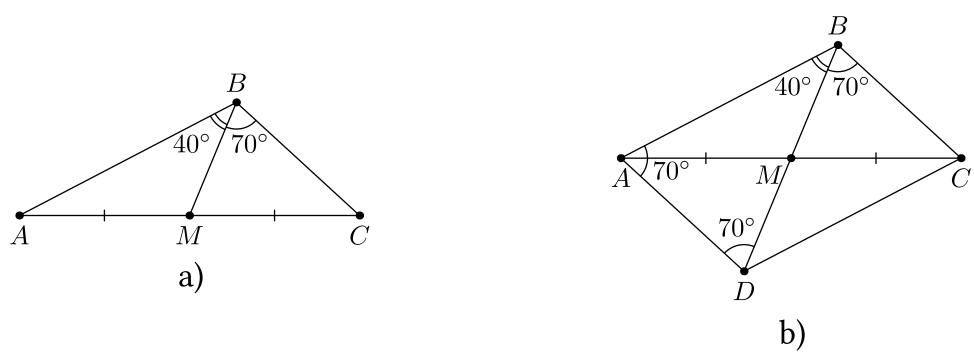


Figure 3.2: Problem of doubling the median.

Proof. Extend BM by its own length to form the parallelogram $ABCD$ (Figure 3.2b). The angle of $40°$ is the angle on the longer side; let it be ABD. Angles DBC and BDA are both $70°$ as alternate interior angles, and the angle $BAD = 180° - 110° = 70° = ADB$. Hence, $AB = BD$, and therefore, the median $BM = \frac{1}{2}BD = \frac{1}{2}AB$. □

Example 3.3. (Mccme — 2017.8-9.3): On the sides AB and BC of triangle ABC squares $ABDE$ and $BCKF$ are constructed outside it (Figure 3.3). Prove that the segment DF is twice the median BM of the triangle.

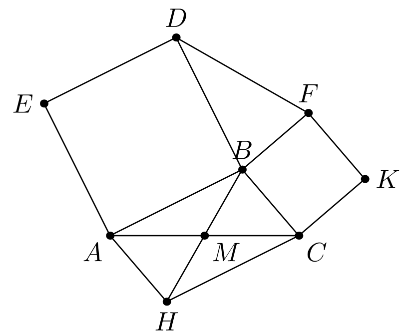


Figure 3.3: Outer squares.

Proof. Let us double the median BM, obtaining the segment $BH = 2BM$. Consider the triangles DBF and BAH. There, $DB = BA$ (as the sides of a square). Furthermore, $AH = BC$ (according to the properties of a parallelogram) $= BF$ (as the sides of a square). Also, $\angle DBF + \angle ABC = 180°$ (since the other two angles at vertex B are right and add up to $180°$), $\Rightarrow \angle DBF = 180° - \angle ABC = \angle BAC + \angle BCA = \angle BAC + \angle CAH = \angle BAH$. Therefore, triangles DBF and BAH are equal by two sides and the angle between them, and $BM = 1/2BH = 1/2DF$. □

Example 3.4. (Mccme — 2017.8-9.4): In a convex pentagon $ABCDE$, $AE = AD$, $AC = AB$ and $\angle DAC = \angle AEB + \angle ABE$. Prove that side CD is twice the median AK of triangle ABE.

Proof. On the extension of the median AK beyond point K, let us plot the segment KF equal to AK (Figure 3.4).

From the equality of triangles BKF and EKA it follows that $BF = AE = AD$ and $\angle KBF = \angle KEA$. So $\angle ABF = \angle ABK + \angle KBF = \angle DAC$. Also, $AB = AC$. Therefore, triangles ABF and CAD are congruent. Hence, $CD = AF = 2AK$. □

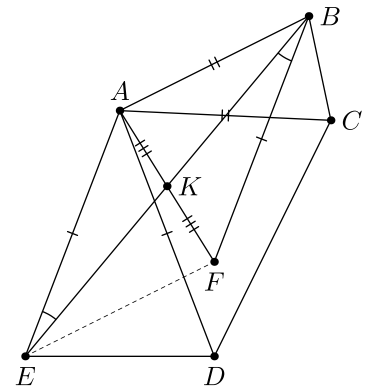


Figure 3.4: Convex pentagon.

Problem Set

Problem 3.1. (Gordin — 1209): Construct a triangle, given two sides, and draw the median to the third side using a compass and a straightedge.

Problem 3.2. (TOT — 2008.8.3): Sergey drew triangle ABC and constructed median AD. He then told Ilya the length of median AD and the length of side AC in this triangle. Based on this information, Ilya proved that angle CAB is obtuse and angle DAB is acute. Find the ratio $AD : AC$ (and prove that for any triangle with this ratio, Ilya's statement holds true).

Problem 3.3. (MMO — 2012.8.3): In triangle ABC, the median drawn from vertex A to side BC is four times smaller than side AB and forms a $60°$ angle with it. Find angle A.

Problem 3.4. (Gordin — 5835): The median AD and the altitude CE of an isosceles triangle ABC $(AB = BC)$ intersect at point P. Find the area of triangle ABC if $CP = 5$ and $PE = 2$.

Problem 3.5. (Gordin — 4011): Prove that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of all of its sides.

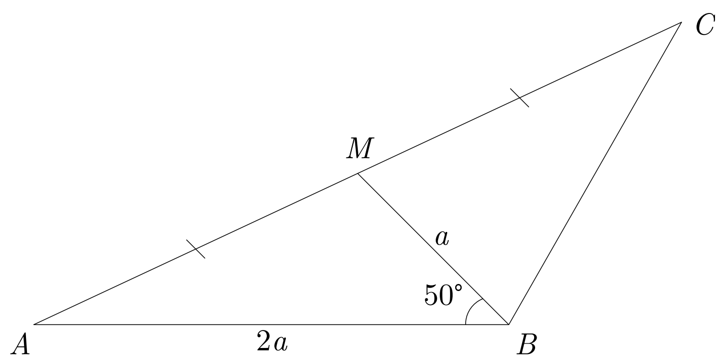
Problem 3.6. (Gordin — 2651): The sides of a triangle are $11, 13$, and 12. Find the length of the median drawn to the longest side.

Problem 3.7. (Gordin — 2652): In a triangle, two sides are 11 and 23, and the median drawn to the third side is 10. Find the third side.

Problem 3.8. (Gordin — 4047): Prove that the ratio of the sum of the squares of the medians of a triangle to the sum of the squares of its sides is $\frac{3}{4}$.

Problem 3.9. Prove that if two medians in a triangle are equal in length, then the triangle is isosceles.

Problem 3.10. (Mid-Michigan Mathematical Olympiad): In triangle ABC, the median BM is drawn.



The length $|BM| = |AB|/2$. The angle $\angle ABM = 50°$. Find the angle $\angle ABC$.

Skill Assessment Problems

Skill Assessment Problem 3.1. BM is the median of triangle ABC such that the sum of angles A and C of this triangle is equal to angle ABM. Find the ratio $BC : BM$.

Skill Assessment Problem 3.2. BM is the median of triangle ABC, where $AB = 2BM$, and $\angle ABM = 40°$. Find angle ABC.

Solutions to Skill Assessment Problems

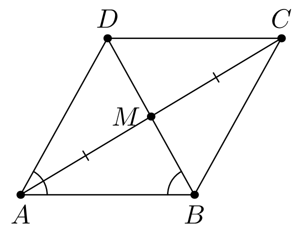


Figure 3.5: Doubling the median in the problem.

Solution to Problem 3.1: Extend BM by its length to form the parallelogram $ABCD$ (Figure 3.5). From the equality of angles, we have $\angle ACB = \angle DAC, \angle CAB + \angle BCA = \angle ABM \Rightarrow \angle DAB = \angle DBA$, which implies $DA = DB = CB$. Therefore, the required ratio is $CB : BM = 2 : 1$. □

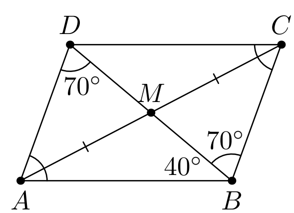


Figure 3.6: Doubling the median in the problem.

Solution to Problem 3.2: Extend BM by its length to form parallelogram $ABCD$ (Figure 3.6). $AB = BD$, which implies $\angle BAD = \angle BDA = 70° = \angle DBC \Rightarrow \angle ABC = 110°$. □

Thales's Theorem

> Mathematicians are a kind of Frenchmen. Whenever you say anything or talk to them, they translate it into their own language, and right away, it is something completely different.
>
> —Johann Wolfgang von Goethe

Theory and Practice

The intercept theorem, also known as Thales's theorem, states the following.

Theorem 1 (Thales)**.** If parallel lines intersect the sides of an angle, forming equal segments on one side of the angle, then they also form equal segments on the other side of the angle.

Proof. Let the parallel lines A_1B_1, A_2B_2, and A_3B_3 intersect the sides of angle AOB, such that $A_1A_2 = A_2A_3$ (see Figure 4.1). It is required to prove that $B_1B_2 = B_2B_3$.

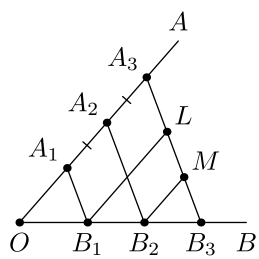


Figure 4.1: Thales's Theorem

Draw B_1L and B_2M parallel to OA (B_1L intersects A_2B_2 at point K).

The quadrilaterals $A_1A_2KB_1$ and $A_2A_3MB_2$ are parallelograms. Therefore, $B_1K = A_1A_2$ and $B_2M = A_2A_3$. Thus, $B_1K = B_2M$.

Triangles B_1KB_2 and B_2MB_3 are congruent by side and two adjacent angles. Hence, $B_1B_2 = B_2B_3$. Thales's Theorem is proven. □

An important consequence of Thales's theorem is the theorem on proportional segments.

Theorem 2 (On Proportional Segments)**.** Parallel lines intersecting the sides of an angle cut these sides into proportional segments.

Proof. Let AC and BD be parallel lines. The theorem states that

$$\frac{OA}{OB} = \frac{OC}{OD}.$$

Suppose the equality is not satisfied. Let, for example,

$$\frac{OA}{OB} < \frac{OC}{OD},$$

then

$$OC > \frac{OA \cdot OD}{OB}.$$

Construct on the ray OD a segment

$$OE = \frac{OA \cdot OD}{OB}.$$

Point E lies between O and C because $OE < OC$.

Take a natural number n and divide the segment OD into n equal segments (Figure 4.2). Let the length of one segment be y; then $OD = ny$. Draw lines through the ends of these equal segments, parallel to BD. By Thales's theorem, they divide the segment OB into n equal segments. Let x be the length of each of these segments; then $OB = nx$.

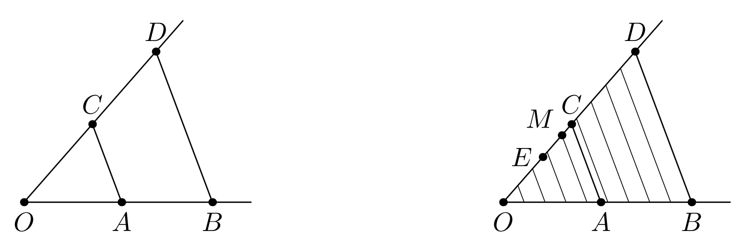


Figure 4.2: Proof of the theorem on proportional segments

For a sufficiently large n, points of division of the segment OD can be found inside the segment EC. Let M be such a point, and $OM = my$. The corresponding line intersects OB at point K, and then $OK = mx$. We have:

$$\frac{OM}{OD} = \frac{my}{ny} = \frac{m}{n} = \frac{mx}{nx} = \frac{OK}{OB}.$$

Now, since $OE < OM$ and $OK < OA$, we obtain:

$$\frac{OE}{OD} < \frac{OM}{OD} = \frac{OK}{OB} < \frac{OA}{OB},$$

which implies

$$OE < \frac{OA \cdot OD}{OB}.$$

The obtained contradiction, therefore, proves the theorem. □

Thales' theorem forms the basis for many other important theorems and problem-solving principles. Let's consider one of them.

Example 4.1. Using a compass and a ruler, divide a given segment into 2024 equal parts.

Solution: Let AB be the given segment. Take an arbitrary point C outside the line AB. Along the ray AC, mark 2024 consecutive equal segments. Connect the endpoint D of the last one with point B. Draw lines through the endpoints of the marked segments parallel to DB. According to Thales's theorem, these lines will also divide the segment AB into 2024 equal parts. □

Problem Set

Problem 4.1. (Gordin – 6275): Points D and E are chosen on sides AB and BC of triangle ABC respectively, such that $\frac{AD}{DB} = \frac{BE}{EC} = 2$ and $\angle C = 2\angle DEB$. Prove that triangle ABC is isosceles.

Problem 4.2. (Gordin — 3397): On the median AM of triangle ABC, point K is taken such that $AK : KM = 1 : 3$. Find the ratio in which the line passing through point K, parallel to side AC, divides side BC.

Problem 4.3. (COM — 2004.10.1): Points E and F are the midpoints of sides DC and AD of convex quadrilateral $ABCD$. Prove that segment EF divides diagonals AC and BD in equal ratio.

Problem 4.4. (Gordin — 1321): Point M is taken on side CB of triangle ABC, and point P is taken on side CA. It is known that $CP : CA = 2CM : CB$. A line through point M is drawn parallel to line CA, and a line through point P is drawn parallel to line AB. Prove that these constructed lines intersect on median CN.

Problem 4.5. (Gordin — 1379): A line is drawn through a point on the side of a quadrilateral, parallel to one diagonal, until it intersects the adjacent side. Through the obtained point, a line parallel to the other diagonal is drawn, and so on. Prove that the fifth point obtained in this way coincides with the original point.

Problem 4.6. (Gordin — 1380): Through a point on the side of a triangle, a line parallel to another side is drawn until it intersects the third side of a triangle. Through the obtained point, a line parallel to the first side of the triangle is drawn, and so on.

a) If the original point coincides with the midpoint of a side of the triangle, prove that the fourth point obtained in this way coincides with the original point.

b) If the original point is different from the midpoint of a side of the triangle, prove that the seventh point obtained in this way coincides with the original point.

Problem 4.7. (Gordin — 3110): In triangle ABC with an area of 6, point K is taken on side AB such that $AK : BK = 2 : 3$, and point L is taken on side AC such that $AL : LC = 5 : 3$. The distance between point Q, the intersection of lines CK and BL, and the AB is equal to 1.5. Find the length of side AB.

Problem 4.8. (Gordin — 2608): Given segments a, b, and c. Using a compass and straightedge, construct a segment x such that $x : a = b : c$.

Problem 4.9. (Gordin — 2814): On side AB of triangle ABC, point K is taken, and on side BC, points M and N are taken such that $AB = 4AK$, $CM = BN$, $MN = 2BN$. Find the ratios $AO : ON$ and $KO : OM$, where O is the intersection of lines AN and KM.

Problem 4.10. (2ARSO — 2011.8.3): On side AB of triangle ABC, point K is chosen. Segment CK intersects the median AM of the triangle at point P. It turns out that $AK = AP$. Find the ratio $BK : PM$.

Problem 4.11. (MMG — 2014.8.3.2): In triangle ABC, points D, E, and F are chosen on sides AB, AC, and BC, respectively, such that $BF = 2CF$, $CE = 2AE$, and angle DEF is right. Prove that DE is the angle bisector of angle ADF.

Problem 4.12. (Gordin — 6621): In triangle ABC, point M is the midpoint of side AC, and point P lies on side BC. Segment AP intersects BM at point O. It turns out that $BO = BP$. Find the ratio $OM : PC$.

Skill Assessment Problems

Skill Assessment Problem 4.1. Point T lies on side AB of triangle ABC. Q is the intersection point of CT and the median AM of triangle ABC. Find the ratio $BT : QM$ if $AT = AQ$.

Skill Assessment Problem 4.2. In triangle ABC, point P on side AB and point M on side AC are positioned such that $AP : PB = 3 : 2$, and $AM : MC = 4 : 5$. Find the ratio in which the line passing through point P and parallel to side BC divides segment BM.

Solutions to Skill Assessment Problems

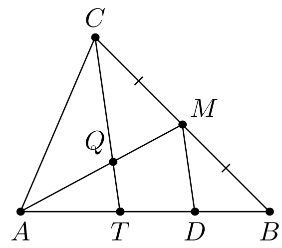


Figure 4.3: Additional construction.

Solution to Problem 4.1: Extend a line through point M parallel to CT that intersects AB at point D (Figure 4.3).

By Thales's theorem, $BD = TD$. By the theorem of proportional segments, $QM = TD = BT/2$. □

Solution to Problem 4.2: Let F be the intersection point of the mentioned line with side AC, and Q be the intersection point of BM and PF. Draw a line through vertex B parallel to AC. Let T be the intersection point of this line with line PF (Figure 4.4). By the theorem of proportional segments, $FC = 2/5AC$. Since $TBCF$ is a

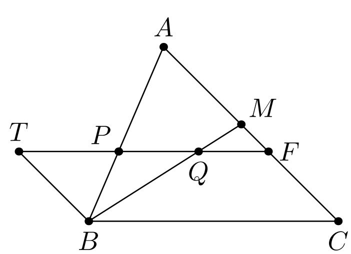


Figure 4.4: Additional construction.

parallelogram, $TB = FC = 2/5AC$. From the similarity of triangles TQB and

FQM, it follows that

$$\frac{BQ}{QM} = \frac{TB}{MF} = \frac{TB}{MC - FC} = \frac{\frac{2}{5}AC}{\frac{7}{45}AC} = \frac{18}{7}.$$

□

Additional Constructions

> The art of solving geometric problems is somewhat reminiscent of the tricks of illusionists — sometimes, even knowing the solution, it is difficult to understand how one could have come to it.
>
> —I. Novikov

Theory and Practice

The main idea of additional or complementary constructions is that the drawing for a problem, where the relation between given data and needed quantities is not easily noticeable, is complemented with new (additional) elements. After that, these relationships become more perceptible or even evident. There are problems where a complementary construction defines the only way of solving; in such cases, the solution usually starts with needed construction. In other problems, a mixed approach is used, where the complementary construction is only implemented in a part of the solution. In the third category of problems, it is applied as one of several possible methods alongside others, even though it might not be the best. In many cases, employing a complementary construction allows for solving the problem more intuitively.

Often, the one solving the problem intuitively uses a complementary construction but may not recognize it as a method and might not see the potential of applying it to other, more complex, or analogous problems.

Arguably, the most popular method of complementary construction, the method of doubling a median, was discussed in the previous chapter. What are some other classical methods of complementary constructions?

Method 1. If a certain secant is given in a triangle, a ray parallel to one of the sides can be drawn through its base into the interior of the triangle until it intersects the other side (Figure 5.1).

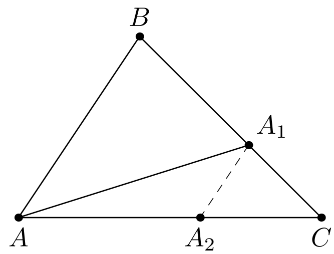


Figure 5.1: Method 1: Illustration.

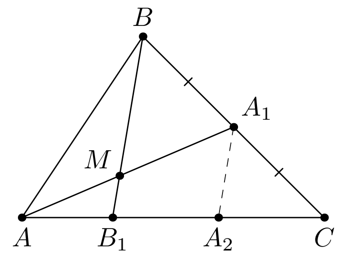


Figure 5.2: Method 2: Illustration.

As a result, parallel lines will appear on the diagram: $A_1A_2 \parallel AB$, which, thanks to Thales's theorem, determine proportional segments on the sides:

$$\frac{CA_2}{CA_1} = \frac{AA_2}{BA_1}.$$

This can be very useful, especially in problems where certain ratios need to be found.

Method 2. If a median and some arbitrary secant (including height, bisector, or another median) are given in a triangle, drawn from different vertices, a ray parallel to the given secant can be drawn through the base of the median into the interior of the triangle until it intersects a side of the triangle (Figure 5.2).

As a result, two pairs of parallel lines will appear on the diagram:

1) $A_1A_2||BB_1$;

2) $MB_1||A_1A_2$, where $M = AA_1 \cap BB_1$.

From the first pair, it indicates that A_2 is the midpoint of B_1C, and from the second pair, the following relationship holds:

$$\frac{AM}{AB_1} = \frac{MA_1}{B_1A_2}.$$

Method 3. If two arbitrary secants are given in a triangle, drawn from different vertices, a ray parallel to one of them can be drawn through the base of one secant into the interior of the triangle until it intersects a side of the triangle (Figure 5.3).

One can notice that this construction differs from the previous one only in that here the first pair of parallel lines gives proportional segments, not equal ones.

Method 4. If two secants are given in a triangle, drawn from different vertices, a line parallel to one of them can be drawn through the origin of one of them (the vertex of the triangle) into the interior of the triangle until it intersects the extension

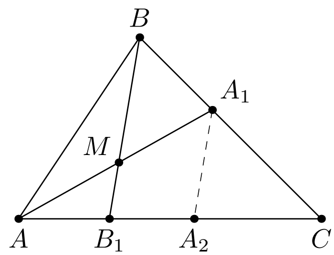


Figure 5.3: Method 3: Illustration.

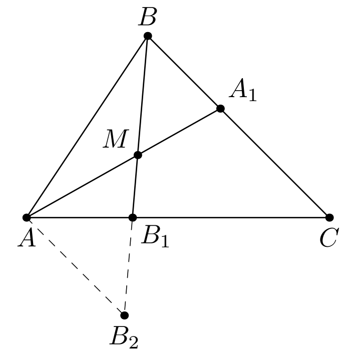


Figure 5.4: Method 4: Illustration.

of the other secant (Figure 5.4). As a result of the construction, two pairs of similar triangles are formed.

Thus, in Figure 5.4, $\Delta AMB_2 \sim \Delta A_1MB$; $\Delta AB_1B_2 \sim \Delta CB_1B$.

Method 5. If a secant and a segment with endpoints on two sides of a triangle are given, intersecting the secant and not parallel to the third side, you can either:

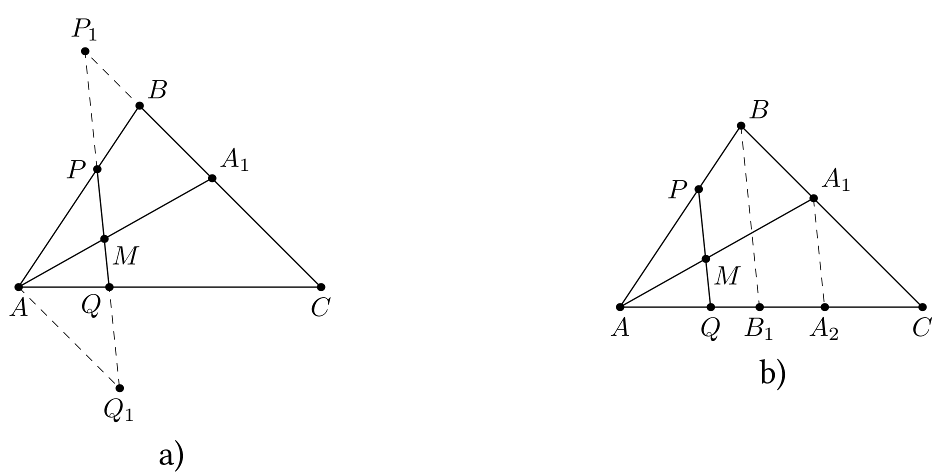


Figure 5.5: Method 5: Illustrations.

1) Extend the given segment in both directions until it intersects with the extension

of the third side and with the line parallel to this side passing through the vertex from which the secant originates (Figure 5.5 a);

2) Draw rays through the base of the secant and one of the vertices not coinciding with its starting point, parallel to the given segment, into the interior of the triangle until they intersect a side of the triangle (Figure 5.5 b).

In the first case, three pairs of similar triangles appear on the diagram. In the illustration, we have:

$$\Delta APQ1 \sim \Delta BPP_1, \Delta AQQ1 \sim \Delta CQP_1, \Delta AMQ1 \sim \Delta A_1MP_1.$$

In the second case, several pairs of parallel lines appear, providing an opportunity to use Thales's theorem.

Method 6. If a segment with endpoints on the sides of a triangle is given and if the extension of this segment intersects the line containing the third side of the triangle, it can be useful to either:

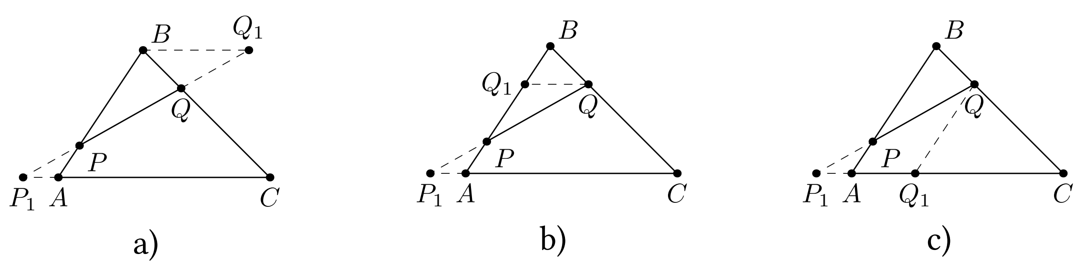


Figure 5.6: Method 6: Illustrations

1) Extend this segment until it intersects with the line drawn through the vertex of the triangle parallel to the third side (Figure 5.6 a);

2) Draw a ray through the other end of this segment into the interior of the triangle, parallel to one of the sides, until it intersects the other side (Figure 5.6 b, c).

In the illustrations in Figure 5.6 b, c, we have $\Delta PP_1A \sim \Delta QP_1Q_1$; $\Delta ACB \sim \Delta Q_1CQ$.

Method 7. If the bisector of one of the internal angles is given in a triangle, parallelogram, or trapezoid, a rhombus can be introduced into the drawing, two sides of which lie directly along the sides of the given figure, and this bisector is one of the diagonals.

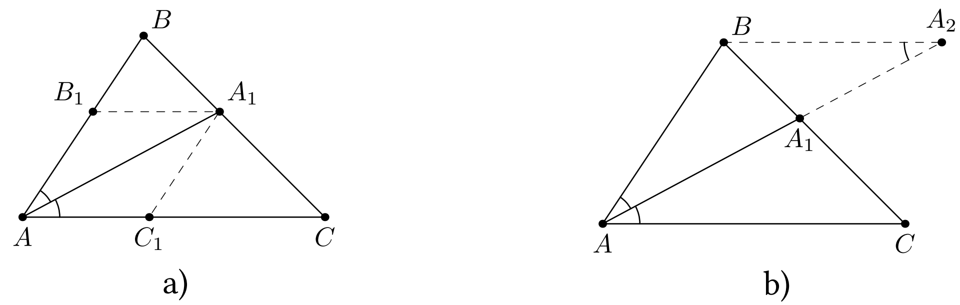


Figure 5.7: Method 7: Illustrations.

As you can see in the illustrations in Figure 5.7 a) and b), the rhombus can be either inside or outside the triangle. In Figure 5.7 b), we constructed not a rhombus but an isosceles triangle, which can also be quite useful.

In this chapter, we won't show you the application of specific techniques in problems (after all, we hope you have already considered the relevance of these methods), but you can practice on problems from textbooks or try solving problems independently.

By the way, a little secret is that sometimes the key to the most effective complementary construction is already hidden in the problem statement.

Consider the problem below:

Example 5.1. Point E is the midpoint of the lateral side CD of trapezoid $ABCD$. Point K is taken on side AB such that lines CK and AE are parallel. Segments CK and BE intersect at point O.

a) Prove that $CO = KO$.
b) Find the ratio of the bases BC and AD in the trapezoid if the area of triangle BCK is $\frac{9}{100}$ of the area of trapezoid $ABCD$.

Solution: First construction. Extend BC and AE, resulting in congruent triangles AED and ZEC (Figure 5.8). Since $AE = EZ$, BE is the median of triangle ABZ. KC and AZ are parallel. The median is helpful because it bisects not only the side of the triangle but also any segment parallel to that side. Therefore, $CO = KO$.

The area of triangle ABZ is the same as the area of trapezoid $ABCD$ due to the congruency of triangles AED and ZEC. Triangle KBC is similar to triangle ABZ. It is known that the ratio of the areas of two similar triangles is equal to the square of the similarity ratio. The segment BC divided by the segment BZ is $3 : 10$. Seven parts correspond to the segment CZ. The answer to part «b» is $3 : 7$.

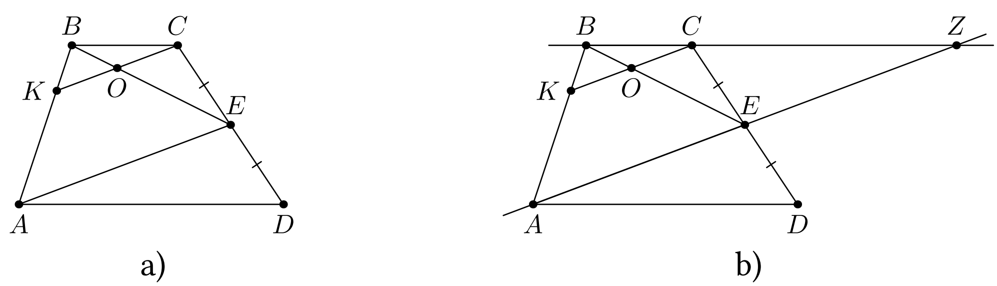


Figure 5.8: Complementary constructions.

Second construction. Extend the sides AB and CD (Figure 5.9). Using the converse of Thales's theorem, prove that the ratio of $KB : BY$ is equal to ratio $YE : ED$, and that segment KD is parallel to BE. In triangle KCD, segment OE is parallel to KD and passes through the midpoint — hence, it is the median and passes through the midpoint of KC. Part «a» of the problem is solved. Dealing with part «b» is more complicated with this construction. The ratio KB to BY being the same as ED to YE leads to the answer $3 : 7$.

$\square$

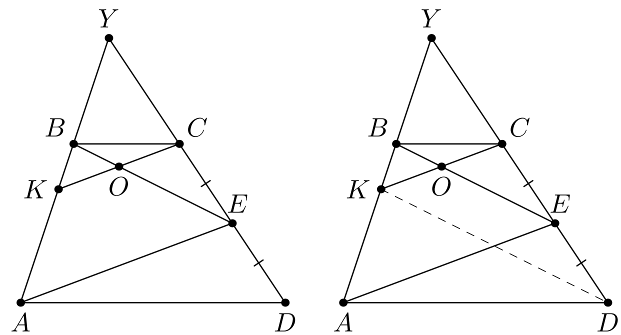


Figure 5.9: Complementary constructions: second solution.

Problem Set

Problem 5.1. (LT — 1996.9.2): The length of the altitude AB of the right trapezoid $ABCD$ is equal to the sum of the lengths of the bases AD and BC. In what ratio does the angle bisector of angle B divide the side CD?

Problem 5.2. (TOT — 2003.9.3): On the lateral sides AB and BC of an isosceles triangle ABC, points K and L are taken, respectively, so that $AK + LC = KL$. From the midpoint M of segment KL, a line parallel to BC is drawn, and this line intersects side AC at point N. Find the measure of angle KNL.

Problem 5.3. (TOT — 2014.9.3): In triangle ABC, angle C is a right angle. On the leg CB, a semicircle is constructed outward with the diameter, and point N is the midpoint of this semicircle. Prove that the line AN bisects the angle C bisector.

Problem 5.4. (Gordin — 2047): The bases of a trapezoid are a and b, and the angles at the larger base are $30°$ and $45°$. Find the area of the trapezoid.

Problem 5.5. (ZPM): On the legs AC and BC of an isosceles right-angled triangle, points M and L are marked respectively, such that $MC = BL$. Point K is the midpoint of the hypotenuse AB. Prove that triangle MKL is also a right-angled isosceles triangle.

Problem 5.6. (ZPM): In triangle ABC, the angle bisector AE is congruent to segment EC. Moreover, $2AB = AC$. Find the angles of triangle ABC.

Problem 5.7. (ZPM): In the isosceles right-angled triangle ABC, points M and N are taken on the hypotenuse AB (N between M and B) such that $\angle MCN = 45°$. Prove that segments MN, AM, NB can form a right-angled triangle.

Problem 5.8. (ZPM): In the isosceles triangle ABC ($AB = BC$), a perpendicular is dropped from A to the lateral side BC. Point L is the base of the perpendicular from H to side AB. It turns out that $AL = \frac{AB}{4}$. Find the angles of triangle ABC.

Problem 5.9. (ZPM): On the lateral sides AB and AC of an isosceles triangle ABC, points K and L are marked, respectively, such that $AK = CL$ and $\angle ALK + \angle LKB = 60°$. Prove that $KL = BC$.

Problem 5.10. (ZPM): On the hypotenuse AC of the right-angled triangle ABC, point D is selected such that $BC = CD$. On leg BC, point E is chosen such that $DE = CE$. Prove that $AD + BE = DE$.

Problem 5.11. (ZPM): In the square $ABCD$, points K and M belong to the sides BC and CD, respectively, such that AM is the bisector of the angle $\angle KAD$. Prove that $AK = DM + BK$.

Problem 5.12. (ZPM): In the isosceles triangle ABC with the base AC, the bisector AD is drawn. It is known that $AD + BD = AC$. Find the angles of the triangle.

Problem 5.13. (ZPM): On the median BM of triangle ABC, a perpendicular AL is dropped, and a perpendicular DK is dropped from some point D on side AB (L and K are distinct points lying inside BM). It turned out that $BK = LM$. Prove that $CD = BD + BA$.

Problem 5.14. (MMG — 2004.10.9.2.2.): Do there exist four segments with lengths a, b, c, and d such that two trapezoids can be formed: one with bases a and b and diagonals c and d, and the other with bases c and d and diagonals a and b?

Problem 5.15. On the median BM of triangle ABC, a point P is taken such that $AP = BC$. The line AP intersects segment BC at point D. Prove that $BD = PD$.

Problem 5.16. On the plane, a triangle ABC is given, and at the same plate, there are points D and E such that $\angle ADB = \angle BEC = 90°$. Prove that the length of segment DE does not exceed half the perimeter of triangle ABC.

Problem 5.17. (Gordin — 1423): Two equilateral triangles ABC and CDE are located on the same side of line AE and have a unique common point C. Let M, N, and K be the midpoints of segments BD, AC, and CE, respectively. Prove that triangle MNK is equilateral.

Problem 5.18. In the convex quadrilateral $ABCD$: $AD = BC$; $\angle ABD + \angle CDB = 180°$. Prove that $\angle BAD = \angle BCD$.

Problem 5.19. (LEO — 2009.1.6): In the convex quadrilateral $ABCD$, $AB = BD$ and $\angle ABD = \angle DBC$. On diagonal BD, a point K is found such that $BK = BC$. Prove that $\angle KAD = \angle KCD$.

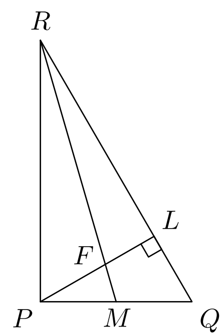
Problem 5.20. In the right-angled triangle ABC, points M and N are marked on the legs AB and BC, respectively, such that $AM = CB$ and $MB = CN$. Prove that the angle between the segments AN and CM is $45°$.

Problem 5.21. On the sides BC and CD of the square $ABCD$, points M and N are taken, respectively, such that $\angle MAN = 45°$. AH is the altitude of the triangle AMN. Prove that $AH = AB$.

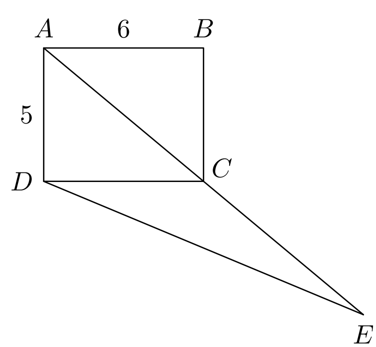
Problem 5.22. On the sides of a quadrilateral, four circles are constructed with the sides as diameters. Prove that the common chord of the circles constructed on two adjacent sides is parallel to the common chord of the other two circles, or these chords lie on the same line.

Problem 5.23. In parallelogram $ABCD$, a point Q is taken such that $\angle ABQ = \angle ADQ$. Prove that $\angle DAQ = \angle DCQ$.

Problem 5.24. (Pascal Contest, Canada – 2008.9.25): In the diagram, $\triangle PQR$ is right-angled at P and has $PQ = 2$ and $PR = 2\sqrt{3}$. Altitude PL intersects median RM at F. What is the length of PF?

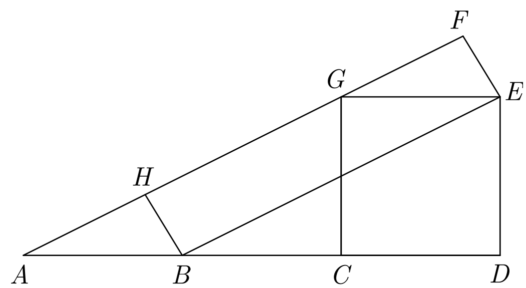


Problem 5.25. (mathcounts): Rectangle $ABCD$ is shown with $AB = 6$ units and $AD = 5$ units. If AC is extended to point E such that AC is congruent to CE, what is the length of DE?



Problem 5.26. (mathcounts): In the figure below, quadrilateral $CDEG$ is a square with $CD = 3$, and quadrilateral $BEFH$ is a rectangle. If $BE = 5$, how many units

is BH? Express your answer as a mixed number.



Skill Assessment Problems

Skill Assessment Problem 5.1. Point M divides side BC of triangle ABC into two equal parts. CL is perpendicular to AM (L is between A and M). On AM, there is a point K such that $AK = 2LM$. Prove that $\angle BKM = \angle CAM$.

Skill Assessment Problem 5.2. The lengths of the bases of a trapezoid are 202 and 2020. If it is known that its diagonals are mutually perpendicular, find the square root of the sum of their squares.

Solutions to Skill Assessment Problems

Solution to Problem 5.1: Extend segment LM to point N such that $NM = LM$ (see Figure 5.10). Then triangles CLM and BNM are congruent (by two sides and the included angle). Therefore, $\angle BNM = \angle CLM = 90°$ and $BN = CL$.

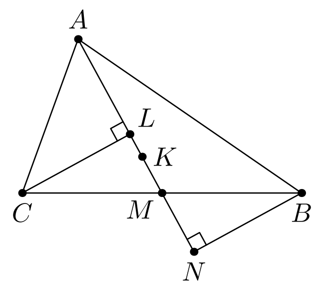


Figure 5.10: Illustration for the problem.

Since $KN = KL + 2LM = KL + AK = AL$, right triangles BNK and ALC are congruent (by two legs). Thus, $\angle BKM = \angle CAM$. □

Solution to Problem 5.2: Let's denote the angles of our trapezoid as A, B, C, D, where $AC \perp BC$, and $AD = 2020$, $BC = 202$ (Figure 5.11).

As an additional construction, draw $CE \parallel BD$ (Figure 5.12).

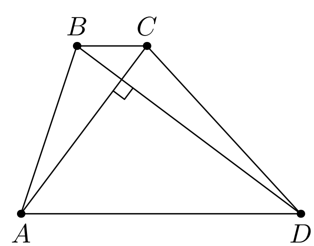


Figure 5.11: Trapezoid with perpendicular diagonals.

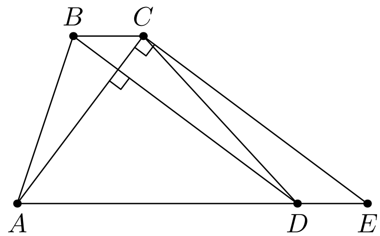


Figure 5.12: Additional construction.

Then $CEBD$ is a parallelogram, so $\angle ACE = 90^\circ$ and $BD = CE$. Thus, $AC^2 + DB^2 = AC^2 + CE^2$. Using the Pythagorean theorem, $AC^2 + CE^2 = AE^2 = (202 + 2020)^2$. Therefore, the square root of the sum of the squares of the diagonals is 2222. □

Ceva's and Menelaus's Theorems

> No one expects you to derive all of the analytical geometry on the exam. It has been done by many smart people before you. Some things just need to be known.

—Dmitry Vladimirovich Beklemishev, university professor of one of authors

Theory and Practice

In geometry, a lot of facts have «if and only if» condition. The «necessary and sufficient» statement means that the former statement is true if and only if the latter is true. In general, a necessary condition is one (possibly one of multiple conditions) that must be present in order for another condition to occur, while a sufficient condition is one that produces the said condition. We will divide the proofs of «if and only if» theorems to proofs of necessity and sufficiency.

The Ceva's Theorem is one of the classical theorems in elementary geometry, devised in 1678 by the Italian engineer Giovanni Ceva.

Theorem 3 (of Ceva)**.** On the sides AB, BC, CA of triangle ABC, take points C_1, A_1, and B_1 respectively (Figure 6.1). Then, AA_1, BB_1, and CC_1 intersect at a single point if and only if the following equality holds true:

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1.$$

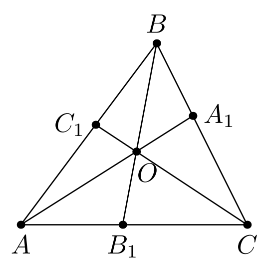


Figure 6.1: Ceva's Theorem.

Proof of Necessity. We will prove that if the segments AA_1, BB_1, and CC_1 intersect at a single point, then the required equality is satisfied. To do this, draw a line through point B parallel to line AC and denote by D and C the points of intersection of lines CC_1 and AA_1 with this line, respectively (Figure 6.2).

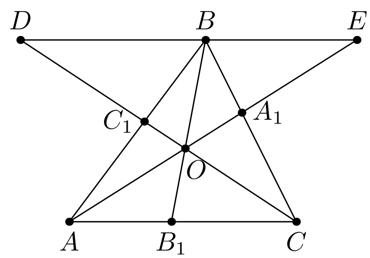


Figure 6.2: Ceva's Theorem: proof of necessity.

From the similarities of triangles, we have:

$$\Delta AC_1C \sim \Delta DC_1B \Rightarrow \frac{AC_1}{C_1B} = \frac{AC}{DB},$$

$$\Delta AA_1C \sim \Delta BA_1E \Rightarrow \frac{BA_1}{A_1C} = \frac{BE}{AC},$$

$$\Delta CB_1O \sim \Delta DBO \Rightarrow \frac{CB_1}{OB_1} = \frac{DB}{OB},$$

$$\Delta AOB_1 \sim \Delta BOE \Rightarrow \frac{OB_1}{B_1A} = \frac{OB}{BE}.$$

Multiplying the obtained equalities, we get:

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{OB_1} \cdot \frac{OB_1}{B_1A} = \frac{AC}{DB} \cdot \frac{BE}{AC} \cdot \frac{DB}{OB} \cdot \frac{OB}{BE} \Rightarrow$$

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1.$$

□

Proof of Sufficiency. Let's prove that if the condition that the product is equal to 1 is satisfied, then the segments AA_1, BB_1, and CC_1 intersect at a single point. Assume the opposite. Let AA_1 and CC_1 intersect at point O, and suppose that segment BB_1 does not pass through point O (Figure 6.3 a).

Let some segment BB' pass through point O (Figure 6.3 b).

Figure 6.3: Ceva's Theorem: proof of sufficiency.

Since the segments AA_1, BB_1, and CC_1 intersect at one point, the equality is satisfied:

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB'}{B'A} = 1.$$

Dividing this expression by the given one, we obtain:

$$\frac{CB'}{B'A} \cdot \frac{B_1A}{CB_1} = 1,$$

which implies:

$$\frac{CB'}{B'A} = \frac{CB_1}{B_1A},$$

and hence B' and B_1 coincide. □

Definition 2. Cevian is a segment in a triangle that connects a vertex of the triangle to a point on the opposite side.

Three such segments that intersect at one point are sometimes collectively called cevians.

The stated theorem extends to the case where the cevians intersect inside the triangle. Let's consider another case.

Theorem 4 (Generalized Ceva's Theorem)**.** Extend the sides of the triangle ABC — AB and CB beyond point B. Mark points C_1, A_1 on the extensions of these sides, and on side CA, take point B_1 (Figure 6.4 a).

The lines AA_1, BB_1, and CC_1 intersect at one point, or they are parallel if and only if the following condition is satisfied:

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1.$$

Figure 6.4: Generalized Ceva's Theorem.

Proof of necessity. Let's prove the statement: if the lines AA_1, BB_1, CC_1 intersect at one point, then the required equality holds true.

Draw a line through B parallel to the base of the triangle, and intersecting AA_1 and CC_1 at points D and E respectively (Figure 6.4 b).

The following equalities follow from the similarity of triangles.

$$\Delta AC_1C \sim \Delta BC_1E \Rightarrow \frac{AC_1}{C_1B} = \frac{AC}{BE},$$

$$\Delta AA_1C \sim \Delta DA_1B \Rightarrow \frac{BA_1}{A_1C} = \frac{BD}{AC},$$

$$\Delta ODB \sim \Delta OAB_1 \Rightarrow \frac{DB}{AB_1} = \frac{OB}{OB_1},$$

$$\Delta BOE \sim \Delta B_1OC \Rightarrow \frac{CB_1}{BE} = \frac{OB_1}{OB}.$$

Multiplying these equalities, we get

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{DB}{AB_1} \cdot \frac{CB_1}{BE} = \frac{AC}{BE} \cdot \frac{BD}{AC} \cdot \frac{OB}{OB_1} \cdot \frac{OB_1}{OB} \Rightarrow$$

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1$$

□

Proof of sufficiency. This can be carried out similarly to the previous theorem. □

Using Ceva's theorem, one can quickly prove that the medians AA_m, BB_m, CC_m of triangle ABC intersect at one point:

$$\frac{AC_m}{C_mB} \cdot \frac{BA_m}{A_mC} \cdot \frac{CB_m}{B_mA} = 1.$$

Example 6.1. Prove, using Ceva's theorem, that the altitudes of an acute-angled triangle intersect at one point.

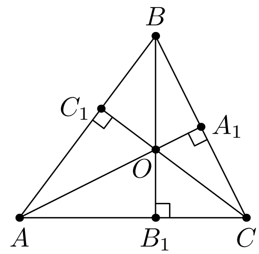


Figure 6.5: Property of triangle altitudes.

Proof. Right-angled triangles AA_1C and BB_1C (Figure 6.5) are similar due to two equal pair of angles, so

$$\frac{CA_1}{B_1C} = \frac{CA}{BC}.$$

In addition, from the similarity of right-angled triangle pairs AA_1B and CC_1B, BB_1A and CC_1A, it follows that the following equalities hold:

$$\frac{BC_1}{A_1B} = \frac{BC}{AB}, \frac{AB_1}{C_1A} = \frac{AB}{CA}.$$

Multiplying these equalities, we get

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1.$$

□

Example 6.2. *Gergonne's point.* In triangle ABC, there is an inscribed circle tangent to the lines AB, BC, CA at points C_1, A_1, B_1 respectively. Prove that the segments AA_1, BB_1, CC_1 intersect at a single point (Gergonne's point).

Proof. Using the property of tangents drawn from a single point, we can assert that $AC_1 = AB_1, BC_1 = BA_1, CB_1 = CA_1$. Therefore,

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = \frac{AC_1}{B_1A} \cdot \frac{BA_1}{C_1B} \cdot \frac{CB_1}{A_1C} = 1.$$

□

Another important theorem in planar geometry is Menelaus's theorem. It is named after Menelaus of Alexandria, the author of the book «Sphaerica» (published in around 100 AD), who proved the theorem. In this book, Menelaus first proves the theorem for the planar case and then extends it to the sphere using central projection. According to some sources, the planar case of the theorem, considered in classical plane geometry, might have been discussed earlier in the lost works of Euclid known as «Porisms.» This theorem was very popular throughout the history of classical plane geometry, being one of its essential «building blocks.» In fact, Ceva's theorem discussed earlier in this chapter also, in a sense, relied on this «building block.»

Theorem 5 (of Menelaus)**.** Let ABC be a triangle, and points B_1 and A_1 are marked on its sides AC and CB, respectively. Additionally, point C_1 is marked on the extension of side AB (see Figure 6.6).

a) (Direct Menelaus's Theorem) If points A_1, B_1, C_1 lie on the same line, then

$$\frac{AB_1}{B_1C} \cdot \frac{CA_1}{A_1B} \cdot \frac{BC_1}{C_1A} = 1.$$

b) (Converse Menelaus's Theorem) If the equality from a) holds, then points A_1, B_1, C_1 lie on the same line.

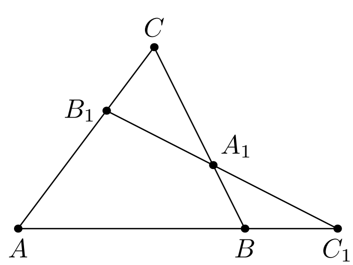


Figure 6.6: Menelaus's Theorem.

Proof. a) Let's draw a line through C parallel to B_1C_1. It intersects AB at N (see Figure 6.7).

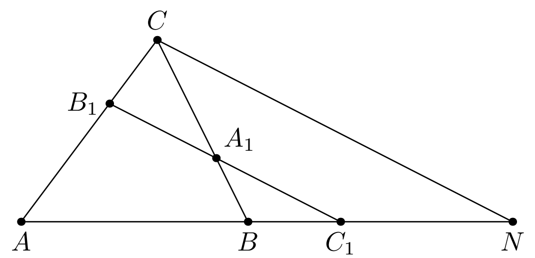


Figure 6.7: Menelaus's Theorem: proof.

According to the theorem on proportional segments, we have:

$$\frac{AB_1}{B_1C} = \frac{C_1A}{C_1M}, \frac{CA_1}{A_1B} = \frac{C_1M}{BC_1}.$$

Then,

$$\frac{AB_1}{B_1C} \cdot \frac{CA_1}{A_1B} \cdot \frac{BC_1}{C_1A} = \frac{C_1A \cdot C_1M \cdot BC_1}{C_1M \cdot BC_1C_1A} = 1.$$

b) The proof in the reverse direction resembles a similar proof, as in Ceva's theorem.

Firstly, note that $\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \neq 1$, since, by assumption, this expression is equal to $\frac{B_1A}{CB_1} \neq 1$. Therefore, the lines A_1C_1 and AC are not parallel.

Let's draw a line through points C_1 and A_1. Let it intersect line AC at some point B_2. Then, for points A_1, C_1, and B_2, the equality from the direct Menelaus theorem holds, i.e., $\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_2}{B_2A} = 1$.

Thus, $\frac{CB_2}{B_2A} = \frac{BC_1}{C_1A} \cdot \frac{CA_1}{A_1B} = \frac{CB_1}{B_1A}$.

From this equality, it follows that both points B_1 and B_2 lie on the extension of segment AC beyond a common point. Since to the right of C, this ratio is less than 1, and to the left of A, it is strictly greater than 1, both points must coincide. Let $CB_1 = x$, $CB_2 = y$, and $AC = b$. Then, considering that $B_1A = x + b$ and $B_2A = y + b$, we rewrite the obtained equality as $\frac{x}{x+b} = \frac{y}{y+b} \Leftrightarrow xy + xb = xy + yb \Leftrightarrow x = y$.

From the equality $CB_1 = CB_2$, it follows that $B_1 = B_2$, and consequently, point B_1, coinciding with B_2, lies on the line A_1C_1. □

Problem Set

Problem 6.1. (3ARSO — 2009.10.3): Three lines pass through the point O and form pairwise equal angles. Points A_1 and A_2 are taken on one of them, and B_1 and B_2 on another, such that the point C_1 of the intersection of the lines A_1B_1 and A_2B_2 lies on the third line. Let C_2 be the point of intersection of A_1B_2 and A_2B_1. Prove that the angle C_1OC_2 is right.

Problem 6.2. (3ARSO — 2000.11.6): A circle with center O inscribed in triangle ABC is tangent to side AC at point K. The second circle, also with center O, intersects all sides of triangle ABC. Let E and F be its points of intersection with sides AB and BC, respectively, closer to vertex B; B_1 and B_2 are the points of intersection with side AC, with B_1 closer to A. Prove that points B, K, and point P, an intersection of segments B_2E and B_1F, are collinear.

Problem 6.3. (Gordin — 1624): Through the point P lying on the median CC_1 of triangle ABC, lines AA_1 and BB_1 are drawn (points A_1 and B_1 lie on sides BC and CA respectively). Prove that $A_1B_1||AB$.

Problem 6.4. (Sharygin — 2015.10.1): Let K be a point on side BC of triangle ABC, and KN be the bisector of triangle AKC. Lines BN and AK intersect at point F, and lines CF and AB intersect at point D. Prove that KD is the bisector of triangle AKB.

Problem 6.5. (Sharygin — 2013.9.5): Points E and F are taken on sides AB and AC of triangle ABC, respectively. Lines EF and BC intersect at point S. Points M and N are the midpoints of segments BC and EF, respectively. The line passing through vertex A and parallel to MN intersects BC at point K. Prove that $BK : CK = FS : ES$.

Problem 6.6. (OMGO — 2010.10.4): From vertex A of parallelogram $ABCD$, we drop altitudes AM onto BC and AN onto CD. P is the point of intersection of BN and DM. Prove that lines AP and MN are perpendicular.

Problem 6.7. (OMGO – 2004.10.1): Points E and F are the midpoints of sides BC and AD of the convex quadrilateral $ABCD$. Prove that the segment EF divides diagonals AC and BD in the same ratio.

Problem 6.8. (HMMT): The three points A, B, C form a triangle. $AB = 4$, $BC = 5$, $AC = 6$. Let the angle bisector of $\angle A$ intersect side BC at D. Let the foot of the perpendicular from B to the angle bisector of $\angle A$ be E. Let the line through E parallel to AC meet BC at F. Compute DF.

Problem 6.9. (ARML): Let $ABCD$ be a parallelogram with $\angle ABC$ obtuse. Let BE be the altitude to side AD of $\triangle ABD$. Let X be the point of intersection of AC and BE, and let F be the point of intersection of lines AB and DX. If $BC = 30$, $CD = 13$, and $BE = 12$, compute the ratio AC/AF.

Skill Assessment Problems

Skill Assessment Problem 6.1. Circles S and S_1 are tangent at point A_1, and circles S and S_2 are tangent at point A_2. Prove that at least one of the points of intersection of the common external or common internal tangents to the circles S_1 and S_2 lies on A_1A_2.

Skill Assessment Problem 6.2. Prove, using Ceva's theorem, that the angle bisectors of a triangle intersect at one point.

Solutions to Skill Assessment Problems

Solution to Problem 6.1: Let O, O_1, and O_2 be the centers of circles S, S_1, and S_2 respectively. Let X be the intersection point of the lines O_1O_2 and A_1A_2 (Figure 6.8).

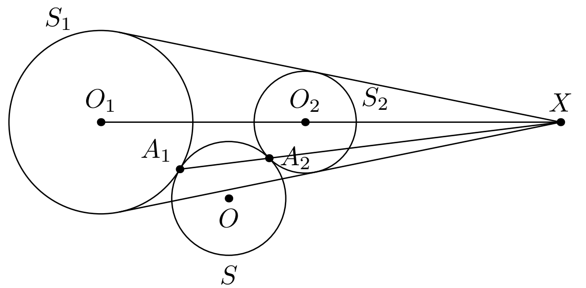


Figure 6.8: Three circles.

Applying Menelaus's theorem to triangle OO_1O_2 and points A_1, A_2, and X, we get

$$\frac{O_1X}{O_2X} \cdot \frac{O_2A_2}{OA_2} \cdot \frac{OA_1}{O_1A_1} = 1,$$

which implies $\frac{O_1X}{O_2X} = \frac{R_1}{R_2}$, where R_1 and R_2 are the radii of circles S_1 and S_2. Therefore, X is the intersection point of the common external or common internal tangents to circles S_1 and S_2 (this follows, for example, from the similarity of triangles). $\square$

Solution to Problem 6.2: Let three angle bisectors AA_1, BB_1, and CC_1 be drawn in triangle ABC (Figure 6.9).

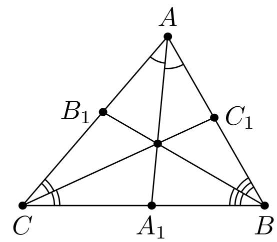


Figure 6.9: Proving the intersection of triangle bisectors at one point using Ceva's theorem.

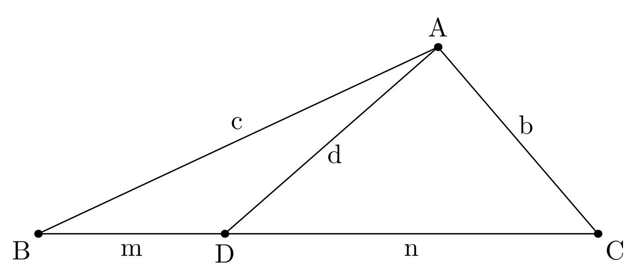
From the properties of angle bisectors and Ceva's theorem:

$$\frac{AB_1}{B_1C} \cdot \frac{CA_1}{A_1B} \cdot \frac{BC_1}{C_1A} = \frac{AB}{BC} \cdot \frac{CA}{AB} \cdot \frac{BC}{CA} = 1,$$

so the angle bisectors intersect at one point. □

Ratio Calculations

Given a triangle $\triangle ABC$ with sides of length a, b, c and opposite vertices A, B, C, respectively. If cevian AD is drawn so that $BD = m$, $DC = n$ and $AD = d$, we have that $b^2m + c^2n = amn + d^2a$. (This is also often written $man + dad = bmb + cnc$, a phrase which invites mnemonic memorization, i.e., «A man and his dad put a bomb in the sink.») That is Stewart's Theorem. I know, it's easy to memorize.



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Theory and Practice

Although geometry problems encountered in Olympiads usually have beautiful geometric solutions, sometimes combinatorial methods help prove certain statements or even find the solution to the entire problem. The most cumbersome method is the coordinate method, which is usually not applicable to more complex problems: the formulas become too complicated, making it very difficult to work with them without the help of a computer, and one mistake can render the entire solution incorrect.

Here, we will discuss some kind of «crutches»: theorems that help understand how segments and angles are related. In solving a problem, it is not necessary to write down all the combinatorial theorems; there may be too many of them to work with reasonably. Let's present a few more useful theorems in addition to those covered in the two previous sections.

A crucial consequence of Menelaus's theorem is the following theorem.

Theorem 6 (Proportional Segments in a Triangle). Consider a triangle ABC and points A_1 and B_1 on its sides BC and AC, respectively. Let AA_1 and BB_1 intersect at point O.

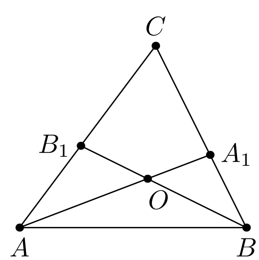


Figure 7.1: Theorem on proportional segments in a triangle.

Then (see Figure 7.1)

$$\frac{AO}{OA_1} = \frac{AB_1}{B_1C} \cdot \left(1 + \frac{CA_1}{A_1B}\right).$$

Proof. Consider the triangle ACA_1 and the secant B_1B. According to Menelaus's theorem, $\frac{AB_1}{B_1C} \cdot \frac{CB}{BA_1} \cdot \frac{A_1O}{OA} = 1$. The proposed theorem follows directly from this. □

This theorem often simplifies problem-solving. It might even be applicable at the national level in most countries. For example, in 2009, participants of the Republican Olympiad in Mathematics (Pridnestrovian Moldavian Republic) were given the following problem:

Example 7.1. On sides AC and BC of triangle ABC, points M and N are marked, respectively, such that $AM : MC = 4 : 5$ and $BN : BC = 0.25$. Segments BM and AN intersect at point P. Find the length of AP if $PN = 10$.

Solution: Let's construct the diagram shown in Figure 7.2.

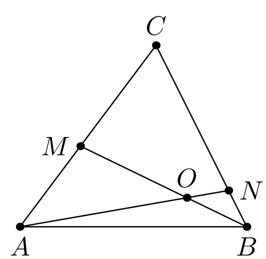


Figure 7.2: Proportional segments in a triangle problem.

By the theorem on proportional segments in a triangle,

$$\frac{AP}{PN} = \frac{AM}{MC} \cdot \left(1 + \frac{CN}{NB}\right).$$

Now substitute everything given in the problem.

$$\frac{AP}{10} = \frac{4}{5} \cdot (1 + 3),$$

from which $AP = 32$. □

As you can see, with the help of this theorem, the problem turned into solving a single equation.

Example 7.2. In triangle ABC with area 1, point D is located on segment BC and divides it in the ratio $BD : DC = 1 : 2$, and point E lies on the midpoint of segment AB. Segments CE and AD intersect at point G. Find the area of triangle AGE.

Solution: Extend ray BG, which intersects segment AC at point F (see Figure 7.3).

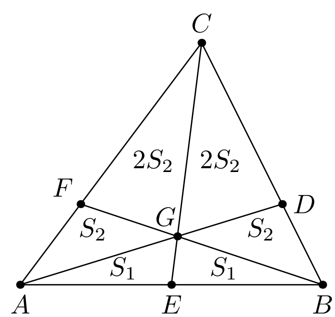


Figure 7.3: Extending the ray.

By Ceva's theorem,

$$\frac{AF}{FC} \cdot \frac{CD}{DB} \cdot \frac{BE}{EA} = 1,$$

which implies $AF : FC = 1 : 2$. Let's denote the area of triangle AGE as S_1 and the area of triangle AGF as S_2. Then, the area of triangle $BEG = S_1$ because these triangles have equal bases and a common height. $S_{CFG} = 2S_2$. Since both $S_{ADB} = \frac{1}{3}$ and $S_{ABF} = \frac{1}{3}$, we have $S_{BDG} = S_2$, and $S_{CDG} = 2S_2$. Equality of areas S_{ACE} and S_{CEB} implies $S_{CFG} = 2S_2$. Now, we have triangle ADC with an area of $\frac{2}{3}$, containing only areas equal to S_2. Therefore, $S_2 = \frac{2}{3} : 5 = \frac{2}{15}$. $S_{ABF} = S_2 + S_1 + S_1 = \frac{1}{3}$, so $S_1 = \frac{1}{10}$. □

Often, when solving geometry problems, many forget to do the simplest thing — consider all possible angles that arise in the problem. To calculate angles in many problems, it is sufficient to have knowledge about the sum of the angles in a triangle, the equality of vertical angles, the sum of adjacent angles, angles formed by parallel lines, and angles in a circle. Often, introducing one, two, or, in rare cases, three variables is sufficient.

Let's recall three signs of similar triangles: Triangle ABC is similar to triangle $A_1B_1C_1$ if:

1. $AB : A_1B_1 = BC : B_1C_1 = AC : A_1C_1$ (SSS rule),

2. $\angle ABC = \angle A_1B_1C_1$ and $AB : A_1B_1 = BC : B_1C_1$ (SAS rule),

3. $\angle ABC = \angle A_1B_1C_1$ and $\angle BAC = \angle B_1A_1C_1$ (AA rule).

By calculating all the angles present in a given problem, it becomes much easier to discover similar triangles. Typically, there are many triangles in a diagram, and finding similar ones among them requires some skill. Counting angles can also help identify isosceles triangles.

For example, this method can help prove the following useful theorem.

Theorem 7 (Altitude of a Right-Angled Triangle)**.** Consider a right-angled triangle ABC with a right angle at C, from which the altitude CH is drawn. Then, $CH = \sqrt{AH \cdot BH}$.

Proof. After calculating all possible angles, it can be observed that $\triangle ABC \sim \triangle ACH \sim \triangle BCH$.

From the similarity of triangles ACH and BCH, it follows that $AH : CH = CH : BH \Rightarrow CH = \sqrt{AH \cdot BH}$ (see Figure 7.4), meaning the altitude drawn from the right angle is equal to the geometric mean of the segments into which it divides the hypotenuse. □

This chapter, in which we presented several useful counting methods, will allow you

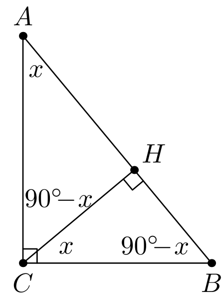


Figure 7.4: Property of the altitude of a right-angled triangle.

to find shorter solutions to the classical Olympiad problems.

Problem Set

Problem 7.1. In triangle ABC, the side AB is c, BC is a, and AC is b. In what ratio, in terms of a,b, and c, do the angle bisectors AK, BL, CM divide their common point F?

Problem 7.2. Point X divides side AB of triangle ABC in the ratio $2 : 1$. Point Y lies on side AC, and BY is divided by XC in the ratio $5 : 2$. In what ratio does point Y divide side AC?

Problem 7.3. A line divides the base AB of an isosceles triangle ABC with a side length of 3 in the ratio $2 : 1$ and intersects ray CA at point M such that $CM = 5$. In what ratio does it divide the other side?

Problem 7.4. A line bisects the base AB of an isosceles triangle ABC with a side length of 3 and intersects rays CA and CB at points M and N, respectively. Find the length of CM if the length of CN is 2.

Problem 7.5. Points C_1, B_1, A_1 divide the sides AB, AC, BC of triangle ABC in the ratios $AC_1 : C_1B = 1 : 4$, $CB_1 : B_1A = 2 : 1$, $BA_1 : A_1C = 1 : 1$. Let point P

be the intersection point of segments AA_1 and CC_1, and Q be the intersection point of segments AA_1 and BB_1. In what ratio do points P and Q divide the segment AA_1?

Problem 7.6. In triangle ABC, point N is taken on side BC such that $NC = 3BN$. On the extension of side AC beyond point A, point M is taken such that $MA = AC$. The line MN intersects side AB at point F. Find the ratio $BF : FA$.

Problem 7.7. Triangle ABC, circumscribed around a circle, has $AB = 13$, $DC = 12$, $AC = 9$. A_1 and C_1 are the points of tangency on sides BC and AB, respectively. Q is the intersection point of segments AA_1 and BB_1. Q lies on side BB_1. Find the ratio $BQ : QB_1$.

Problem 7.8. The sides of a triangle are 5, 6, and 7. Find the ratio into which the bisector of the larger angle of this triangle divides the opposite side.

Problem 7.9. Determine the sides of a triangle if the median and altitude drawn from a vertex of one angle divide this angle into three equal parts, and the median itself is equal to 10.

Problem 7.10. (Gordin – 2916): On the sides AB and BC of triangle ABC, points M and N are located such that $\frac{AM}{MB} = \frac{3}{5}$ and $\frac{BN}{NC} = \frac{1}{4}$. Lines CM and AN intersect at point O. Find the ratios: $\frac{OA}{ON}$ and $\frac{OM}{OC}$.

Problem 7.11. (AMC – 2020.10A.20): Quadrilateral $ABCD$ satisfies $\angle ABC = \angle ACD = 90°$, $AC = 20$, and $CD = 30$. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at point E, and $AE = 5$. What is the area of quadrilateral $ABCD$?

(A) 330 (B) 340 (C) 350 (D) 360 (E) 370

Problem 7.12. (MechMat – 1999.07.4): In trapezoid $ABCD$ with lateral sides $AB = 9$ and $CD = 5$, the angle bisector of angle D intersects the angle bisectors of angles

A and C at points M and N, respectively. The angle bisector of angle B intersects the same angle bisectors at points L and K, where point K lies on the base AD.

a) In what ratio does the line LN divide the side AB, and the line MK divide the side BC?

b) Find the ratio $MN : KL$ if $LM : KN = 3 : 7$.

Problem 7.13. (AMC – 2016.10A.19): In rectangle $ABCD$, $AB = 6$ and $BC = 3$. Point E between B and C, and point F between E and C are such that $BE = EF = FC$. Segments $\overline{AE}$ and $\overline{AF}$ intersect $\overline{BD}$ at P and Q, respectively. The ratio $BP : PQ : QD$ can be written as $r : s : t$ where the greatest common factor of r,s and t is 1. What is $r + s + t$?

(A) 7 (B) 9 (C) 12 (D) 15 (E) 20

Problem 7.14. (AMC – 2016.10B.19): Rectangle $ABCD$ has $AB = 5$ and $BC = 4$. Point E lies on $\overline{AB}$ so that $EB = 1$, point G lies on $\overline{BC}$ so that $CG = 1$. and point F lies on $\overline{CD}$ so that $DF = 2$. Segments $\overline{AG}$ and $\overline{AC}$ intersect $\overline{EF}$ at Q and P, respectively. What is the value of $\frac{PQ}{EF}$?

(A) $\frac{\sqrt{13}}{16}$ (B) $\frac{\sqrt{2}}{13}$ (C) $\frac{9}{82}$ (D) $\frac{10}{91}$ (E) $\frac{1}{9}$

Problem 7.15. (AMC – 2014.10B.21): Trapezoid $ABCD$ has parallel sides $\overline{AB}$ of length 33 and $\overline{CD}$ of length 21. The other two sides are of lengths 10 and 14. The angles at A and B are acute. What is the length of the shorter diagonal of $ABCD$?

(A) $10\sqrt{6}$ (B) 25 (C) $8\sqrt{10}$ (D) $18\sqrt{2}$ (E) 26

Problem 7.16. (AMC – 2013.10A.12): Given $\triangle ABC$, $AB = AC = 28$ and $BC = 20$. Points D,E, and F are on sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, respectively, such that $\overline{DE}$ and $\overline{EF}$ are parallel to $\overline{AC}$ and $\overline{AB}$, respectively. What is the perimeter of the parallelogram $ADEF$?

(A) 48 (B) 52 (C) 56 (D) 60 (E) 72

Problem 7.17. (AMC — 2010.10A.14): Triangle ABC has $AB = 2 \cdot AC$. Let D and E be on $\overline{AB}$ and $\overline{BC}$, respectively, such that $\angle BAE = \angle ACD$. Let F be the intersection of segments AE and CD, and suppose that $\triangle CFE$ is equilateral. What is $\angle ACB$?

(A) 60° (B) 75° (C) 90° (D) 105° (E) 120°

Problem 7.18. (AMC — 2010.10A.16): Nondegenerate $\triangle ABC$ has integer side lengths, $\overline{BD}$ is an angle bisector, $AD = 3$, and $DC = 8$. What is the smallest possible value of the perimeter?

(A) 30 (B) 33 (C) 35 (D) 36 (E) 37

Skill Assessment Problems

Skill Assessment Problem 7.1. In an acute-angled triangle ABC, altitudes AA_1 and BB_1 are drawn. Prove that $A_1C \cdot BC = B_1C \cdot AC$.

Skill Assessment Problem 7.2. Point M lies on side BC of triangle ABC, and point K lies on the extension of side AB beyond vertex B. Additionally, $\frac{BM}{MC} = \frac{4}{5}$ and $\frac{BK}{AB} = \frac{1}{5}$. N is the point of intersection of KM and AC. Find the ratio $\frac{CN}{AN}$.

Solutions to Skill Assessment Problems

Solution to Problem 7.1: Let's construct the Figure 7.5.

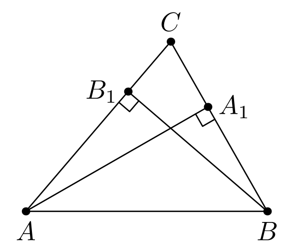


Figure 7.5: Two altitudes in a triangle

Notice that triangles AA_1C and BB_1C are similar by two angles: they share angle ACB, and $\angle BB_1C = \angle AA_1C = 90°$. Therefore, we can write the ratios for the sides of the triangles:

$$\frac{BB_1}{AA_1} = \frac{B_1C}{A_1C} = \frac{BC}{AC},$$

and from the second equality, the desired statement follows: $A_1C \cdot BC = B_1C \cdot AC$. □

Solution to Problem 7.2: Draw a line through point C parallel to AB. Let KM intersect it at point T. Suppose $BK = a$ and $AB = 5a$ (Figure 7.6).

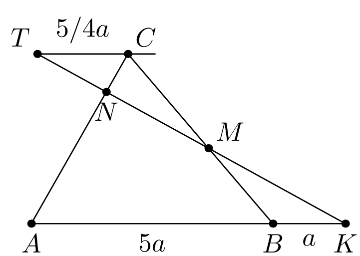


Figure 7.6: Additional construction.

From the similarity of triangles CMT and BMK with $k = 5/4$, we can find that

$$CT = \frac{5}{4} BK = \frac{5}{4} a.$$

Then, from the similarity of triangles CNT and ANK,

$$\frac{CN}{NA} = \frac{CT}{AK} = \frac{\frac{5}{4} a}{5a + a} = \frac{5}{24}.$$

□

Triangle Bisector Properties

> Geometry … is the science that it hath pleased God hitherto to bestow on mankind.
>
> —Thomas Hobbes

Theory and Practice

When you see the word «bisector,» you often think only about the equal angles it creates. However, there is another important fact. The bisector is the set of points equidistant from the sides of the given angle.

Example 8.1. Prove that the bisectors of a triangle intersect at one point.

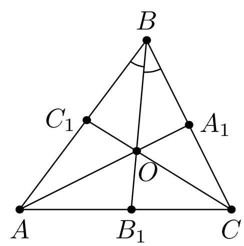


Figure 8.1: Bisectors of a triangle.

Proof. Let's draw two bisectors, AA_1 and BB_1, intersecting at point O (Figure 8.1). Obviously, O is equidistant from AB, AC, and from BA, BC. Furthermore, it is also equidistant from CA, CB – meaning it lies on the bisector of the third angle. Therefore, bisectors intersect at one point. □

Example 8.2. Construct the bisector of a given angle using a compass and a straightedge.

Solution: On a given figure, let's construct a circle with a center at the given point A and an arbitrary radius R. Let it intersect the sides of given angle A at points B and C (see Figure 8.2).

Let's construct a circle with a center at point B and radius R (see Figure 8.3).

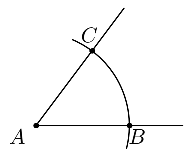


Figure 8.2: Constuction of the bisector: Step 1.

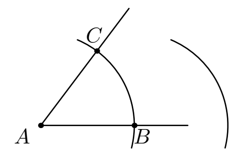


Figure 8.3: Constuction of the bisector: Step 2.

Let's construct a circle with a center at point C and radius R. Let these two circles intersect at point D (see Figure 8.4).

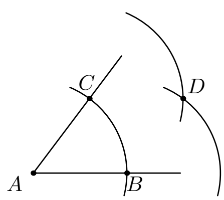


Figure 8.4: Constuction of the bisector: Step 3.

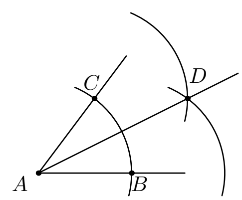


Figure 8.5: Constuction of the bisector: Step 4.

Let's draw ray AD — this is the bisector of the angle (see Figure 8.5).

To prove it, it is enough to note that triangles ABD and ACD are congruent by SSS (three sides are equal by construction), and angles BAD and DAC are congruent as corresponding elements of congruent triangles. $\square$

Example 8.3. Solve the previous problem in the case when the vertex of the angle is lying outside the drawing.

Solution: Consider arbitrary points A and B on different sides of the angle (Figure

8.6). Let's construct the bisectors of angles OAB and OBA (O is the inaccessible vertex of the given angle).

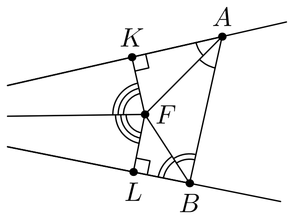


Figure 8.6: Bisector of an inaccessible angle.

Let F be their point of intersection. Drop perpendiculars FK and FL onto the sides of the angle. Then, the bisector of angle KFL will be the desired bisector since segments FK and FL, and therefore, the lines OA and OB, are symmetric with respect to the bisector of angle KFL. □

Example 8.4. a) In triangle ABC, a bisector BD of an internal or external angle is drawn. Prove that $AD : DC = AB : BC$.

b) Prove that the center O of the inscribed circle of triangle ABC divides the bisector AA_1 in the ratio $AO : OA_1 = (b + c) : a$, where a, b, c are the lengths of the sides of the triangle.

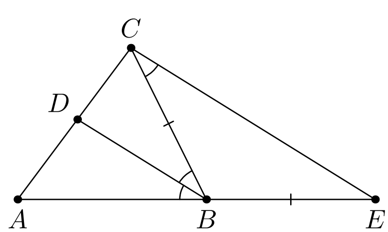


Figure 8.7: Proportions in triangle bisectors.

Solution: a) First method: Let's prove the statement for the internal angle bisector.

Draw a line through point C parallel to BD until it intersects line AB at point E. Since $\angle BEC = \angle CBD = \angle BCE$, triangle CBE is isosceles ($BC = BE$). By Thales' theorem, $AD : DC = AB : BE = AB : BC$.

For the external angle bisector, the proof is similar.

Second method: Point D is equidistant from lines AB and BC. Therefore, $AD : DC = S_{ABD} : S_{BCD} = AB : BC$.

b) From a), it follows that $BA_1 = ac/(b+c)$. Since BO is the bisector of triangle ABA_1, then $AO : OA_1 = AB : BA_1 = (b+c) : a$. □

Example 8.5. The external angle bisector of angle A in triangle ABC intersects the extension of side BC at point M. Prove that $BM : MC = AB : AC$.

Proof. Draw a line through vertex B parallel to AM. Let this line intersect side AC at point P. Mark point Q on the extension of side AC beyond point A. Then, $\angle APB = \angle QAM = \angle MAB = \angle ABP$, which means triangle APB is isosceles, and $PA = AB$. By the theorem of proportional segments, $BM : MC = PA : AC = AB : AC$. □

Example 8.6. Prove that side BC of triangle ABC is seen from the incenter O at an angle of $90° + \frac{\angle A}{2}$, and from the excenter O_1 opposite to vertex A at an angle of $90° - \frac{\angle A}{2}$.

Proof. Since O is the point of intersection of the angle bisectors of triangle ABC, $\angle BOC = 180° - \angle OBC - \angle OCB = 180° - \frac{\angle B}{2} - \frac{\angle C}{2} = 180° - \frac{\angle B + \angle C}{2} = 90° + \frac{\angle A}{2}$.

As BO_1 and CO_1 are the bisectors of the external angles of triangle ABC, $\angle OBO_1 = \angle OCO_1 = 90°$. Therefore, $\angle BO_1C = 180° - \angle BOC = 180° - (90° + \frac{\angle A}{2}) = 90° - \frac{\angle A}{2}$. □

Problem Set

Problem 8.1. (TOT – 2000.10.1): In triangle ABC, the intersection point of the angle bisectors is connected with the vertices, dividing the triangle into three smaller triangles. One of the smaller triangles is similar to the original one. Find its angles.

Problem 8.2. (MMG – 2014.8.4): On the equal sides AB and BC of triangle ABC, points M and N are chosen such that $AC = CM$ and $MN = NB$. The altitude from vertex B intersects the segment CM at point H. Prove that NH is the bisector of angle MNC.

Problem 8.3. (Sharygin – 2015.9.4): In parallelogram $ABCD$, trisectors (@@@lignes that divide the angle to 3 equal parts@@@) of angles A and B were drawn. The trisectors closer to side AB intersect at point O. Let A_1 be the intersection of the trisector AO with the second trisector of angle B, and B_1 be the intersection of the trisector BO with the second trisector of angle A. If M is the midpoint of A_1B_1, and line MO intersects side AB at N, prove that triangle A_1B_1N is equilateral.

Problem 8.4. (ARSO – 2007.8.6): In triangle ABC, the bisectors intersect at point I. A line through I intersects sides AB and BC at points M and N, respectively. Triangle BMN is acute. Points K and L are chosen on side AC such that $\angle ILA = \angle IMB$ and $\angle IKC = \angle INB$. Prove that $AM + KL + CN = AC$.

Problem 8.5. (MMG – 2017.8.3): In triangle ABC, with $AC = 8$ and $BC = 5$, a line parallel to the external bisector of angle C passes through the midpoint of side AB and point E on side AC. Find AE.

Problem 8.6. (MMG – 2013.9.3): In convex quadrilateral $ABCD$, the bisectors of angles CAD and CBD intersect on side CD. Prove that the bisectors of angles ACB and ADB intersect on side AB.

Problem 8.7. (ARSO — 2015.10.1): Let K be a point on side BC of triangle ABC, and KN be a bisector of triangle AKC. Lines BN and AK intersect at point F, and lines CF and AB intersect at point D. Prove that KD is the bisector of triangle AKB.

Problem 8.8. In triangle ABC, bisectors BB_1 and CC_1 are drawn. Prove that if the circumcircles of triangles ABB_1 and ACC_1 intersect at a point on side BC, then $\angle A = 60°$.

Problem 8.9. (Mccme — 05.031): In triangle ABC with $\angle A = 120°$, the bisectors AA_1, BB_1, and CC_1 intersect at point O. Prove that $\angle A_1C_1O = 30°$.

Problem 8.10. (MMG — 2012.8.4.2): On the sides, AB, BC, and AC of an equilateral triangle ABC, points K, M, and N are chosen, respectively, such that the angle MKB is equal to the angle MNC, and the angle KMB is equal to the angle KNA. Prove that NB is the bisector of angle MNK.

Problem 8.11. (COM — 2013.11.7.8): The bisectors of triangle ABC intersect at point I, and $\angle ABC = 120°$. On the extensions of sides AB and CB beyond point B, points P and Q are marked, respectively, such that $AP = CQ = AC$. Prove that the angle PIQ is a right angle.

Problem 8.12. (Mccme — 05.035): In triangle ABC, bisectors BB_1 and CC_1 are drawn. Prove that if $\angle CC_1B_1 = 30°$, then either $\angle A = 60°$ or $\angle B = 120°$.

Problem 8.13. (Gordin — 55): In triangle ABC where $\angle B = 60°$, bisectors AD and CE intersect at point O. Prove that $OD = OE$.

Problem 8.14. (AIME — 2009.I.5): Triangle ABC has $AC = 450$ and $BC = 300$. Points K and L are located on AC and AB, respectively, so that $AK = CK$, and CL is the angle bisector of angle C. Let P be the point of intersection of BK and CL, and let M be the point on line BK for which K is the midpoint of PM. If $AM = 180$, find LP.

Problem 8.15. (ARML): In triangle ABC, $AB = BC$. A trisector of $\angle B$ intersects AC at D. If AB, AC, and BD are integers and $AB - BD = 7$, compute AC.

Problem 8.16. (HMMT): Consider triangle ABC with $\angle A = 2\angle B$. The angle bisectors from A and C intersect at D, and the angle bisector from C intersects AB at E. If $\frac{DE}{DC} = \frac{1}{3}$, compute $\frac{AB}{AC}$.

Skill Assessment Problems

Skill Assessment Problem 8.1. O is the intersection point of the bisectors BD and CE in triangle ABC. Prove that if $OD = OE$, then either the triangle is isosceles or its angle at vertex A is $60°$.

Skill Assessment Problem 8.2. BB_1 and CC_1 are the bisectors of triangle ABC, and I is the point of intersection of the bisectors. It is known that AI is perpendicular to B_1C_1. Prove that triangle ABC is isosceles.

Solutions to Skill Assessment Problems

Solution to Problem 8.1: Triangles ADO and AEO are congruent by two sides and an angle. Therefore, they are either congruent or $\angle ADO + \angle AEO = 180°$. In the first case, the triangle is isosceles due to symmetry with respect to the line AO. In the second case, $\angle A + \angle DOE = 180°$. The angle DOE between the bisectors BD and CE is $90° + \frac{1}{2}\angle A$. From here, we can find the angle A. □

Solution to Problem 8.2: Let AI be the angle bisector of angle A. Segments AI and B_1C_1 intersect at point K. In triangle AB_1C_1, the bisector AK is also an altitude, so triangle B_1AC_1 is isosceles. Thus, AI is the median and perpendicular bisector of segment B_1C_1. Lines AB and AC are symmetric with respect to AI, and lines B_1I and C_1I are also symmetric with respect to AI. Therefore, the point B of intersection of AB and B_1I is symmetric to the point C of intersection of AC and C_1I. Hence, $AB = AC$. □

Remarkable Property of the Trapezoid

> Trapezium, in British and other forms of English, a trapezoid, a quadrilateral that has exactly one pair of parallel sides
>
> Trapezium, in North American English, an irregular quadrilateral with no sides parallel.
>
> — wiki dictionary

Theory and Practice

Example 9.1. Prove that the intersection point of the extensions of the lateral sides of a trapezoid, the midpoints of its bases, and the intersection point of its diagonals lie on the same straight line.

Proof. Let's break the problem into two parts:

1. Prove that the midpoints of the bases and the intersection point of the extensions of the lateral sides lie on the same straight line.

2. Prove that the midpoints of the bases and the intersection point of the diagonals lie on the same straight line.

Let $ABCD$ be the trapezoid. Let M be the midpoint of the shorter base BC.

1) Let X be the intersection point of the extensions of the lateral sides. Draw a line through points X and M. Let it intersect side AD at point N. Prove that N is the midpoint of the base AB (Figure9.1a). Triangles XMB and XAN are similar by three angles, so the ratio $XM/XN = BM/AN$ holds. Similarly, from the similarity of triangles XMC and XND, we get $XM/XN = MC/ND$. Thus, $BM/AN = MC/ND$, but since $BM = MC$, we also have $AN = ND$.

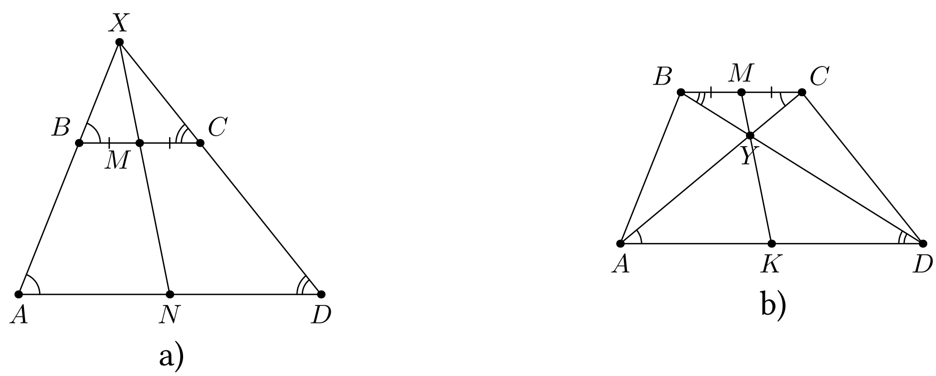


Figure 9.1: Remarkable property of the trapezoid.

2) Let Y be the intersection point of the diagonals. Draw a line through points M and Y. Let it intersect side AD at point K. Prove that K is the midpoint of the base AB (Figure9.1b). Triangles YMB and YKD are similar by three angles, so the ratio $MY/YK = BM/KD$ holds. Similarly, from the similarity of triangles YMC and YKA, we get $MY/YK = MC/AK$. Thus, $BM/KD = MC/AK$, but since $BM = MC$, we also have $AK = KD$.

Points N and K coincide – it is the midpoint of side AD. On the one hand, we have proved that M, N, and X lie on the same straight line, and on the other hand, that points M, N, and Y lie on the same straight line. This means that all these 4 points lie on the same straight line.

Note that this problem has another, more elegant solution, using homothety, similarity transformation of a plane. However, we leave this advanced method for the future. ☐

Example 9.2. In trapezoid $ABCD$ $(BC \parallel AD)$, points P, M, Q, N are the midpoints of sides AB, BC, CD, and DA, respectively. Prove that the segments AQ, PD, and MN intersect at a single point.

Solution. Since quadrilateral $PMQN$ is a parallelogram, the point F, the intersection of segments PQ and MN, is the midpoint of segment PQ (Figure 9.2).

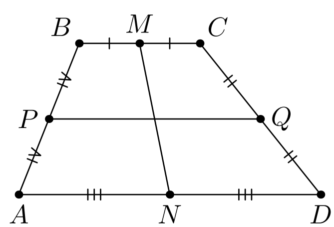


Figure 9.2: Trapezoid with a secant parallel to the bases.

In trapezoid $APQD$, the point of intersection of diagonals AQ and DP lies on the line passing through the midpoints F and N of the bases PQ and AD. Thus, all three proposed lines intersect at a single point, which completes the proof. ☐

As you can see, the remarkable property of the trapezoid allows this problem to be solved much faster than using proportional segments or additional constructions.

Problem Set

Problem 9.1. (Sharygin — 2016.9.6): Extensions of the lateral sides of trapezoid $ABCD$ intersect at point P, and its diagonals intersect at point Q. Point M on the smaller base BC is such that $AM = MD$. Prove that $\angle PMB = \angle QMB$.

Problem 9.2. (Gordin — 5725): Different parallelograms $ABCD$ and $AKLD$ are arranged such that their sides BC and KL lie on the same line. Additionally, lines AC and KD are not parallel. Prove that the point of intersection of lines AK and DC, the point of intersection of lines AB and DL, and the point of intersection of lines AC and KD lie on the same line.

Problem 9.3. (MMG — 2017.10.2): Diagonals of the quadrilateral $ABCD$ intersect at point O. M and N are the midpoints of sides BC and AD, respectively. Segment MN divides the area of the quadrilateral in half. Find the ratio $OM : ON$ if $AD = 2BC$.

Problem 9.4. (Gordin — 1646): Through point D taken on side AB of triangle ABC, a line parallel to AC is drawn, intersecting side BC at point E. Prove that lines AE, CD, and the median drawn from vertex B intersect at a single point.

Problem 9.5. (Gordin — 1546): In a trapezoid, the point of intersection of the diagonals is equidistant from the lines on which the lateral sides lie (@@@the not the bases lines@@@). Prove that the trapezoid is isosceles.

Problem 9.6. (Gordin — 1649): On the board, a trapezoid was drawn, and its midline EF and perpendicular OK from the point O of the intersection of the diagonals to the longer base were drawn. Then, the trapezoid was erased. How can the drawing be restored based on the preserved segments EF and OK?

Problem 9.7. (Gordin – 3040): The area of trapezoid $ABCD$ is 6. Let E be the point of intersection of the extensions of its lateral sides. Through point E and the point of intersection of the diagonals of the trapezoid, a line is drawn, intersecting the shorter base BC at point P and the longer base AD at point Q. Point F lies on the segment EC, and $EF : FC = EP : EQ = 1 : 3$. Find the area of triangle EPF.

Problem 9.8. (Gordin – 3041): The area of triangle MNP is 7. Through point Q on side MN, a line is drawn parallel to side MP, intersecting side NP at point R. Points A and B are taken on the segment QR such that $QR : MP = QA : QB = 1 : 5$. The line NB passes through the intersection point of lines MR and QP. Find the area of triangle NAR.

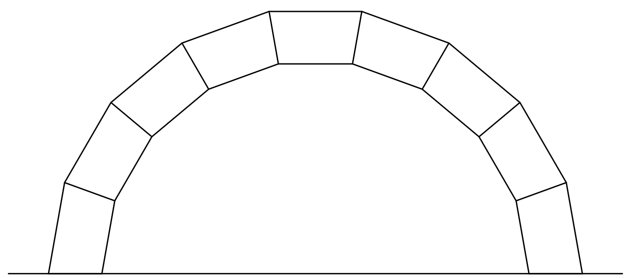
Problem 9.9. (Gordin – 1625): The line connecting point P of intersection of the diagonals of the quadrilateral $ABCD$ with point Q of intersection of the lines AB and CD divides the side AD in half. Prove that it also bisects side BC.

Problem 9.10. (Gordin – 3042): In triangle ABC, a line DE is drawn parallel to AC (points D and E are the intersections of this line with sides AB and BC respectively). The line passing through vertex B and the point of intersection of the diagonals of trapezoid $ADEC$ intersects side AC at point P. Point Q is taken on segment BD. Find the area of triangle QBP if it is known that the area of triangle DBE is 8 and $\frac{QB}{AQ} = \frac{DE}{AC} = \frac{1}{7}$.

Problem 9.11. (Gordin – 1539): Given two parallel lines l and l_1. Use only a straightedge to bisect the given segment AB lying on l.

Problem 9.12. (HMMT): Let $ABCD$ be a trapezoid with $AB \parallel CD$. The bisectors of $\angle CDA$ and $\angle DAB$ meet at E, the bisectors of $\angle ABC$ and $\angle BCD$ meet at F, the bisectors of $\angle BCD$ and $\angle CDA$ meet at G, and the bisectors of $\angle DAB$ and $\angle ABC$ meet at H. Quadrilaterals $EABF$ and $EDCF$ have areas 24 and 36, respectively, and triangle ABH has area 25. Find the area of triangle CDG.

Problem 9.13. (AMC — 2009.10B.24): The keystone arch is an ancient architectural feature. It is composed of congruent isosceles trapezoids fitted together along the non-parallel sides, as shown. The bottom sides of the two end trapezoids are horizontal. In an arch made with 9 trapezoids, let x be the angle measured in degrees of the larger interior angle of the trapezoid. What is x?



Skill Assessment Problems

Skill Assessment Problem 9.1. A segment on a line parallel to the bases of a trapezoid, contained within the trapezoid, divides its diagonals into three parts. Prove that the segments adjacent to the lateral sides are equal.

Skill Assessment Problem 9.2. Two parallelograms $ABCD$ and $AKLD$ are positioned such that sides BC and KL lie on the same line, and lines AC and KD are not parallel. Prove that the points of intersection of the lines AK and DC, AB and DL, and AC and KD lie on the same line.

Solutions to Skill Assessment Problems

Solution to Problem 9.1: Let's denote the trapezoid as $ABCD$. Suppose the segment given in the problem intersects the sides and diagonals at points K, L, M, N, as shown in Figure 9.3.

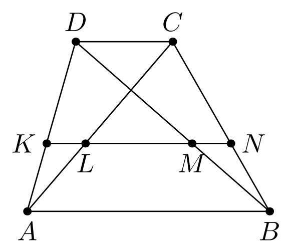


Figure 9.3: Trapezoid with a secant parallel to the bases.

The goal is to prove that $KL = MN$. From the similarity of triangles AKL and ABC, we have $\frac{KL}{BC} = \frac{AK}{AB}$. From the similarity of triangles MND and BCD, we have $\frac{MN}{BC} = \frac{DN}{DC}$. Using Thales' theorem for lines AB and CD and the fact that BC, KN, and AD are parallel, we conclude that $\frac{AK}{AB} = \frac{DN}{DC}$. Therefore, $\frac{KL}{BC} = \frac{MN}{BC} \Rightarrow KL = MN$, which completes the proof. □

Solution to Problem 9.2: Let P be the point of intersection of AK and CD, Q be the point of intersection of AB and DL, and T be the point of intersection of AC and KD. According to a remarkable property of the trapezoid ($ACKD$), points P, T, E (the midpoint of CK), and F (the midpoint of AD) lie on the same line (Figure 9.4).

The line EF, drawn through the midpoints E and F of the bases BL and AD of the trapezoid $ABLD$, passes through point Q, the intersection of the extensions of the lateral sides AB and DL of this trapezoid. Therefore, points P, Q, and T lie on the line EF.

Similarly, other cases can be analyzed. □

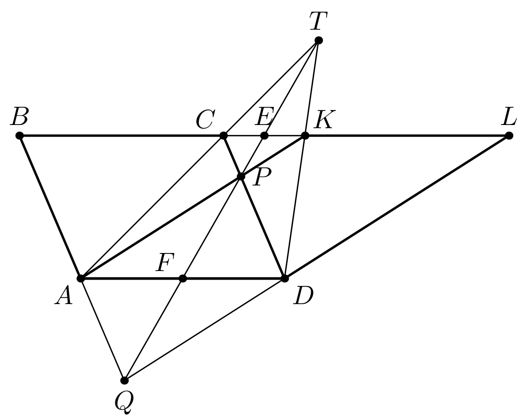


Figure 9.4: Illustration for the solution.

Cyclic Quadrilaterals

> The circle is the soul of geometry. Understand the circle, and you will not only comprehend geometry but also elevate your soul...
>
> —Igor Sharygin

Theory and Practice

It is well known that, a circle can be circumscribed around any triangle. However, for quadrilaterals, this is generally not the case, and we can use one of the criteria to check (See Figure 10.1):

1. The quadrilateral $ABCD$ is inscribed $\Leftrightarrow$ the sum of its opposite angles is equal to 180°.

2. The quadrilateral $ABCD$ is inscribed $\Leftrightarrow$ the angles formed by one side are equal (note that there are a total of 4 pairs of such angles, and equalities of angles from one pair are sufficient).

3. The quadrilateral $ABCD$ is inscribed $\Leftrightarrow$ $AT \cdot CT = BT \cdot DT$, where T is the intersection point of the diagonals.

4. The quadrilateral $ABCD$ is inscribed $\Leftrightarrow$ $AB \cdot CD + BC \cdot AD = AC \cdot BD$ (Ptolemy's theorem).

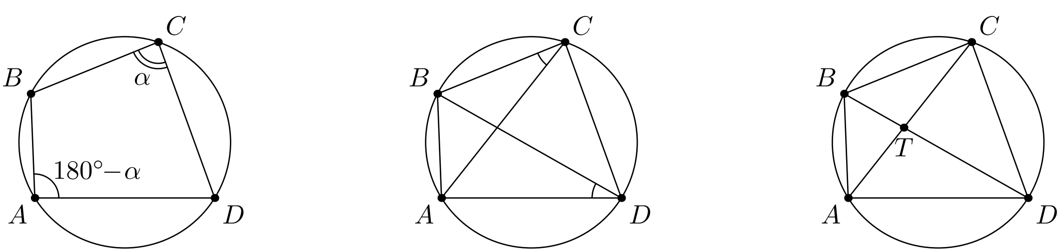


Figure 10.1: Criteria for an inscribed quadrilateral

Example 10.1. Two circles intersect at points A and B. Through point A, a line is drawn that intersects the circles at points C and D, respectively, and through point B, a line is drawn that intersects the circles at points E and F, respectively. Prove that $CE \parallel DF$ (see Figure 10.2).

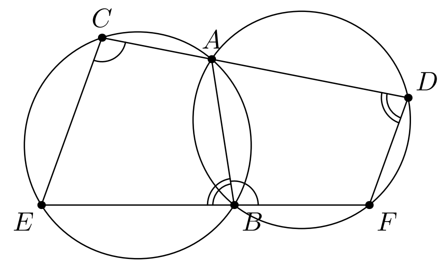


Figure 10.2: Diagram for the problem with intersecting circles.

Solution: In the diagram, two inscribed quadrilaterals can be identified: $ABEC$ and $ADFB$. Let's use the first criterion for $ABEC$: let $\angle ECA = \alpha$, then $\angle EBA = 180° - \alpha$. Angle ABF is equal to α as it is adjacent to angle EBA, and according to the first criterion for $ABDF$, we have $\angle ADF = 180° - \alpha$. The sum of angles ECD and CDF is $180°$, and therefore, lines CE and DF are parallel. □

Example 10.2. (Napoleon's Problem). On the sides of an acute triangle ABC, equilateral triangles ABC_1, BCA_1, and ACB_1 are constructed externally. Prove that the segments AA_1, BB_1, and CC_1 intersect at one point and are equal in length.

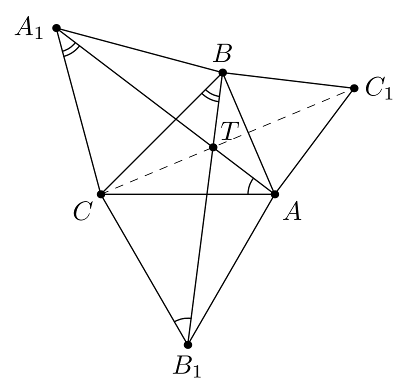


Figure 10.3: Diagram for the problem with external equilateral triangles.

Solution: Let's draw the diagram (Figure 10.3). Suppose the segments AA_1 and BB_1 intersect at point T. We will prove that points C, T, and C_1 lie on the same line. This will mean that all three segments AA_1, BB_1, and CC_1 intersect at one point.

$\triangle B_1CB = \triangle ACA_1$, as $B_1C = AC$, $BC = CA_1$, and $\angle B_1CB = \angle ACA_1 = 60^\circ + \angle ACB$. Hence, $\angle AA_1C = \angle BB_1C$ and $\angle CAA_1 = \angle CB_1B$.

Quadrilateral $CTBA_1$ is inscribed because two angles in it are based on one side (second criterion): $\angle CA_1T = \angle CBT$. Thus, the sum of opposite angles is 180°. Since $\angle CA_1B = 60^\circ$, then $\angle CTB = 120^\circ$. Similarly, quadrilateral $ATCB_1$ is inscribed, and $\angle CTA = 120^\circ$.

Let's find the angle ATB, which is $360^\circ - \angle ATC - \angle BTC = 360^\circ - 120^\circ - 120^\circ = 120^\circ$. Therefore, quadrilateral $ATBC_1$ is also inscribed because the sum of its opposite angles is 180°. From the inscribed property of this quadrilateral, $\angle ATC_1 = \angle ABC_1 = 60^\circ$ (second criterion). Thus, $\angle CTC_1 = \angle CTA + \angle ATC_1 = 120^\circ + 60^\circ = 180^\circ$, which means that points C, T, and C_1 lie on the same line.

The equality of the segments AA_1 and BB_1 follows from the congruency of triangles B_1CB and ACA_1. Similarly, the equality of the segments AA_1 and CC_1 follows from the congruency of triangles AA_1B and C_1BC. □

Example 10.3. (Gordin – 1714): Points A, B, C, and D are arranged sequentially on a circle, and the center O of the circle is located inside the quadrilateral $ABCD$. Points K, L, M, and N are the midpoints of segments AB, BC, CD, and AD, respectively. Prove that $\angle KON + \angle MOL = 180^\circ$.

Solution: Consider the isosceles triangles AOB, BOC, COD, and DOA. Their medians OK, OL, OM, and ON are also the angle bisectors of $\angle AOB$, $\angle BOC$, $\angle COD$, and $\angle DOA$, respectively. Therefore, $\angle KON + \angle MOL = (\frac{1}{2}\angle AOD + \frac{1}{2}\angle AOB) + (\frac{1}{2}\angle BOC + \frac{1}{2}\angle COD) = 180^\circ$. □

Example 10.4. (Gordin – 4780): Prove that the bisectors of angles of a convex quadrilateral form an inscribed quadrilateral.

Solution: Denote the angles of the given quadrilateral $ABCD$ by 2α, 2β, 2γ, 2δ, respectively. Let the bisectors of angles A and B intersect at point M, those of angles B and C at point N, those of angles C and D at point K, and those of angles A and

D at point L. Consider the case when the quadrilateral formed by these bisectors is $KNML$. Then $\angle AMB + \angle CKD = 180° - (\alpha + \beta) + 180° - (\gamma + \delta) = 360° - (\alpha + \beta + \gamma + \delta) = 360° - \frac{1}{2}360° = 180°$. Therefore, a circle can be circumscribed around quadrilateral $KNML$. □

Problem Set

Problem 10.1. (Gordin — 486): In the convex quadrilateral $ABCD$, it is known that $\angle CBD = 58°$, $\angle ABD = 44°$, and $\angle ADC = 78°$. Find $\angle CAD$.

Problem 10.2. (Gordin — 305): In the convex quadrilateral $ABCD$, it is given that $\angle ABC = 116°$, $\angle ADC = 64°$, $\angle CAB = 35°$, and $\angle CAD = 52°$. Find the angle between the diagonals of this quadrilateral.

Problem 10.3. (LT — 1999.9.5): Triangle ABC is inscribed in a circle. Point D is the midpoint of the arc AC, and points K and L are chosen on sides AB and CB, respectively, such that KL is parallel to AC. Let K' and L' be the points of intersection of the lines DK and DL with the circle, respectively. Prove that a circle can be circumscribed around the quadrilateral $KLL'K'$.

Problem 10.4. (Gordin — 47): In the inscribed quadrilateral $ABCD$, the angles are given as follows: $\angle DAB = \alpha$, $\angle ABC = \beta$, $\angle BKC = \gamma$, where K is the point of intersection of the diagonals. Find the angle ACD.

Problem 10.5. (Gordin — 68): The extension of the median AM of triangle ABC intersects its circumscribed circle at point D. Find BC if $AC = DC = 1$.

Problem 10.6. (Gordin — 66): In the convex quadrilateral $MNPQ$, diagonal NQ is the bisector of angle PNM and intersects diagonal PM at point S. Find NS if it is known that a circle can be circumscribed around quadrilateral $MNPQ$, and $PQ = 12$, $SQ = 9$.

Problem 10.7. (OMGO — 2013.10.1): The diagonals of the inscribed quadrilateral $ABCD$ intersect at point O. The circumcircles of triangles AOB and COD intersect at point M on side AD. Prove that point O is the center of the inscribed circle of triangle BMC.

Problem 10.8. (Gordin — 1307): Find the angles of quadrilateral $ABCD$, where the vertices are located on a circle, if $\angle ABD = 74°$, $\angle DBC = 38°$, and $\angle BDC = 65°$.

Problem 10.9. (Problems.ru — 35179): A billiard table has the shape of a convex quadrilateral $ABCD$. A billiard ball was released from point K on side AB, reflected at points L, M, N on sides BC, CD, DA, returned to point K, and again entered the trajectory $KLMN$. Prove that quadrilateral $ABCD$ can be inscribed in a circle.

Problem 10.10. (Gordin — 303): Can a circle be circumscribed around a quadrilateral whose angles have the following ratios : a) $2:4:5:3$; b) $5:7:8:9$?

Problem 10.11. (Gordin — 440): In triangle ABC, bisectors AD and BE are drawn, intersecting at point O. It is known that $OE = 1$, and points C, D, E, and O lie on the same circle. Find the sides and angles of triangle EDO.

Problem 10.12. (MMG — 2013.9.2.2): In triangle ABC, angle C is $135°$. A square with center O is constructed on side AB outside the triangle. Find OC if $AB = 6$.

Problem 10.13. (ARSO — 2015.4.10.3): On side AB of convex quadrilateral $ABCD$, points K and L are taken (point K lies between A and L), and on side CD, points M and N are taken (point M lies between C and N). It is known that $AK = KN = DN$ and $BL = BC = CM$. Prove that if $BCNK$ is an inscribed quadrilateral, then $ADML$ is also inscribed.

Problem 10.14. (MMO — 1992.55.11.2): Find the angles of the convex quadrilateral $ABCD$ where $\angle BAC = 30°$, $\angle ACD = 40°$, $\angle ADB = 50°$, $\angle CBD = 60°$, and $\angle ABC + \angle ADC = 180°$.

Problem 10.15. (Gordin — 2923): Two circles pass through the vertex of an angle and the point of its bisector. Prove that the segments cut off by them on the sides of the angle are equal.

Problem 10.16. (Gordin — 47): In the inscribed quadrilateral $ABCD$, the angles are given: $\angle DAB = \alpha$, $\angle ABC = \beta$, $\angle BKC = \gamma$, where K is the point of intersection of the diagonals. Find angle ACD in terms of α, β, and γ.

Problem 10.17. (Gordin — 69): Through a point D on the base of an isosceles triangle ABC, a line CD is drawn, intersecting its circumcircle at point E. Find AC if $CE = 3$ and $DE = DC$.

Problem 10.18. (Gordin — 528): In a circle with a radius of 17, a quadrilateral is inscribed, the diagonals of which are mutually perpendicular and are at a distance of 8 and 9 from the center of the circle. Find the length of the sides of the quadrilateral.

Problem 10.19. (AMC — 2019.10A.13): Let $\triangle ABC$ be an isosceles triangle with $BC = AC$ and $\angle ACB = 40°$. Construct the circle with diameter $\overline{BC}$, and let D and E be the other intersection points of the circle with the sides $\overline{AC}$ and $\overline{AB}$, respectively. Let F be the intersection of the diagonals of the quadrilateral $BCDE$. What is the degree measure of $\angle BFC$?
(A) 90 (B) 100 (C) 105 (D) 110 (E) 120

Problem 10.20. (DVI — 2011.5): The medians AL and BM of triangle ABC intersect at point K. Find the length of segment CK if $AB = \sqrt{3}$, and it is known that a circle can be circumscribed around quadrilateral $KLCM$.

Problem 10.21. (DVI — 2012.6): The circle is tangent to the sides AB and BC of triangle ABC at points D and E, respectively, and intersects side AC at points F and G (point F is between points A and G). Find the radius of this circle if it is known that $AF = 5$, $GC = 2$, $AD : DB = 2 : 1$, and $BE = EC$.

Problem 10.22. (MechMat — 2001.07.3): Through the vertices A, B, C of the parallelogram $ABCD$ with sides $AB = 3$ and $BC = 5$, a circle is drawn, intersecting the line BD at point E, where $BE = 9$. Find the length of diagonal BD.

Problem 10.23. (MechMat — 2000.05.3): The circle passing through the vertices B, C, and D of the parallelogram $ABCD$ touches the line AD and intersects the line AB at points B and E. Find the length of segment AE if $AD = 4$ and $CE = 5$.

Problem 10.24. (MechMat — 2001.03.3): In trapezoid $ABCD$ with side $CD = 30$, the diagonals intersect at point E, and angles AED and BCD are equal. A circle with a radius of 17, passing through points C, D, and E, intersects the base AD at point F and is tangent to line BF. Find the height of the trapezoid and its bases.

Problem 10.25. (MechMat — 2001.05.3): Two circles with centers O and Q, intersecting at points A and B, intersect the angle bisector of $\angle OAQ$ at points C and D, respectively. The segments OQ and AD intersect at point E, and the areas of triangles OAE and QAE are equal to 18 and 42, respectively. Find the area of quadrilateral $OAQD$ and the ratio $BC : BD$.

Problem 10.26. (MechMat — 2007.4): Points A, B, and C lie on a circle with a radius of 2 and center O. Point K lies on the line tangent to this circle at point B, and $\angle AKC = 46°$. The lengths of segments AK, BK, and CK form an increasing geometric progression (in that order). Find the angle AKO, the distance between points A and C, and determine which angle is larger: ACK or AOK.

Problem 10.27. (MechMat — 2003.07.4): Through vertices A and B of triangle ABC, a circle is drawn tangent to line BC, and through vertices B and C, another circle is drawn tangent to line AB. The extension of the common chord BD of these circles intersects segment AC at point E, and the extension of chord AD of one circle intersects the other circle at point F. Find the ratio $AE : EC$ if $AB = 5$ and $BC = 9$. Compare the areas of triangles ABC and ABF.

Problem 10.28. (MechMat — 2002.07.4): In the inscribed quadrilateral $ABCD$, point

X lies on side AD such that $BX \| CD$ and $CX \| BA$. Find the value of BC if $AX = \frac{3}{2}$ and $DX = 6$.

Problem 10.29. (AMC — 2016.10A.24): A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of the fourth side?
(A) 200 (B) $200\sqrt{2}$ (C) $200\sqrt{3}$ (D) $300\sqrt{2}$ (E) 500

Problem 10.30. (AMC — 2015.10B.19): In $\triangle ABC$, $\angle C = 90°$ and $AB = 12$. Squares $ABXY$ and $ACWZ$ are constructed outside of the triangle. The points X, Y, Z, and W lie on a circle. What is the perimeter of the triangle?
(A) $12 + 9\sqrt{3}$ (B) $18 + 6\sqrt{3}$ (C) $12 + 12\sqrt{2}$ (D) 30 (E) 32

Problem 10.31. (AMC — 2013.10A.23): In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover, $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?
(A) 11 (B) 28 (C) 33 (D) 61 (E) 72

Problem 10.32. (ARML): A quadrilateral $MATH$ is inscribed in a circle of radius 10. $MA = AT = 12$ and $MH = TH = 16$. Compute the radius of the circle inscribed within $MATH$.

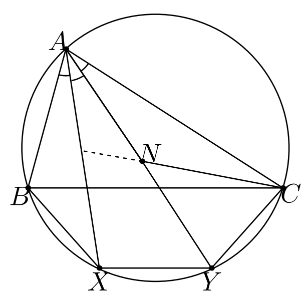
Problem 10.33. (BMO): In the cyclic quadrilateral $ABCD$, the diagonal AC bisects the angle DAB. The side AD is extended beyond D to a point E. Show that $CE = CA$ if and only if $DE = AB$.

Problem 10.34. (European Mathematical Cup): Let ABC be an acute-angled triangle. Let D and E be the midpoints of sides AB and AC, respectively. Let F be the point such that D is the midpoint of EF. Let Γ be the circumcircle of triangle FDB. Let G be a point on the segment CD such that the midpoint of BG lies on Γ. Let H be the second intersection of Γ and FC. Show that the quadrilateral $BHGC$ is cyclic.

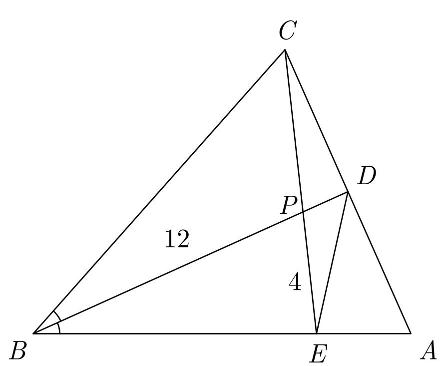
Problem 10.35. (HMMT): Let $ABCD$ be a quadrilateral such that $\angle ABC = \angle CDA = 90^\circ$, and $BC = 7$. Let E and F be on BD such that AE and CF are perpendicular to BD. Suppose that $BE = 3$. Determine the product of the smallest and largest possible lengths of DF.

Problem 10.36. (USAJMO): Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that $\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ$. Prove that lines B_1C_2, C_1A_2, andA_1B_2 are concurrent.

Problem 10.37. (European Mathematical Cup): Let ABC be an acute-angled triangle such that $|AB| < |AC|$. Let X and Y be points on the minor arc BC of the circumcircle of ABC such that $|BX| = |XY| = |YC|$. Suppose that there exists a point N on the segment AY such that $|AB| = |AN| = |NC|$. Prove that the line NC passes through the midpoint of the segment AX.



Problem 10.38. (ARML): In $\triangle ABC$, D is on AC so that BD is the angle bisector of $\angle B$. Point E is on AB and CE intersects BD at P. Quadrilateral $BCDE$ is cyclic, $BP = 12$ and $PE = 4$. Compute the ratio $\frac{AC}{AE}$.



Skill Assessment Problems

Skill Assessment Problem 10.1. a) For an inscribed quadrilateral $ABCD$ with $AC \perp BD$ and P being the point of intersection of the diagonals, find $AP^2 + BP^2 + CP^2 + DP^2$.
b) Find the sum of the squares of the sides of $ABCD$.

Skill Assessment Problem 10.2. In an inscribed quadrilateral $ABCD$, where the diagonals intersect at point O, and the circumcircles of triangles AOB and COD intersect at point M on side AD, prove that O is the center of the inscribed circle of triangle BCM.

Solutions to Skill Assessment Problems

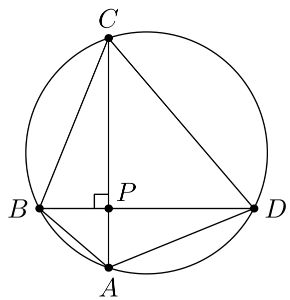


Figure 10.4: Illustration for the problem of diagonal intersection.

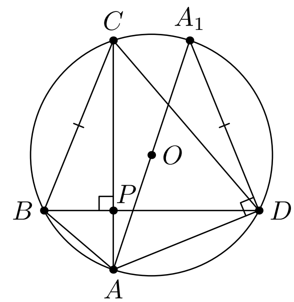


Figure 10.5: Let's make an additional construction.

Solution to Problem 10.1: a) We will use this fact to solve the problem: $90^\circ = \angle APB = \frac{1}{2}(\smile AB + \smile CD)$, so $\smile AB + \smile CD = 180^\circ \Rightarrow \angle AOB + \angle COD = 180^\circ$. Let $\angle AOB = \alpha$, then $\angle COD = 180^\circ - \alpha$.

Extend AO to intersect the circle at point A_1 (Figure 10.5). Recall the following fact: the angle between two chords is equal to half the sum of the arcs between these chords. Therefore, $\frac{\smile AD + \smile BC}{2} = 90^\circ$. Since AA_1 is a diameter, we have

$$\smile AD + \smile DA_1 = 180^\circ \Rightarrow \smile DA_1 = \smile BC \Rightarrow DA_1 = BC.$$

Then,

$$AP^2 + PD^2 + BP^2 + CP^2 = AD^2 + BC^2 =$$
$$AD^2 + DA_1^2 = AA_1^2 = 4R^2.$$

b) Notice that the sum $BC^2 + AD^2$ is also equal to $AP^2 + BP^2 + CP^2 + DP^2$, so

$$AB^2 + BC^2 + CD^2 + DA^2 = 2(AP^2 + BP^2 + CP^2 + DP^2) = 8R^2.$$

□

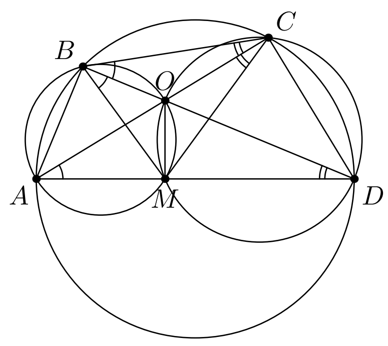


Figure 10.6: Illustration of the problem.

Solution to Problem 10.2: The center of the inscribed circle of a triangle lies at the intersection point of its angle bisectors. Let's prove that AO is the angle bisector of $\angle DAM$. From the inscribed quadrilateral $ABCD$: $\angle DAC = \angle DBC$ because these angles subtend the same arc. From the inscribed quadrilateral $ABMO$: $\angle OBM = \angle OAM$. Hence, $\angle MAO = \angle DAO$, implying that AO bisects $\angle DAM$. Similarly, it can be proved that DO bisects $\angle DAM$: $\angle ADO = \angle ACB = \angle ODM$. Therefore, O is the center of the inscribed circle of triangle ADM, as required. □

Proportional Segments in a Circle

> The advancement and perfection of mathematics are intimately connected with the prosperity of the State.
>
> —Napoleon I

Theory and Practice

Theorem 8. If a diameter and a chord are drawn through a point inside a circle, then the product of the segments of the chord is equal to the product of the segments of the diameter.

Proof. Let's construct the following diagram (Figure 11.1). We need to prove that $AD \cdot AE = AB \cdot AC$.

Figure 11.1: Proportional segments in a circle.

Consider the angle equality: $\angle BAD = \angle CAE$ (vertical angles) and $\angle DBC = \angle CED$ (inscribed angles). Then, $\Delta ABD \sim \Delta ACE$, so

$$\frac{AD}{AB} = \frac{AC}{AE} \Rightarrow AD \cdot AE = AB \cdot AC,$$

which completes the proof. □

Corollary 1. For all chords passing through a given point inside a circle, the product of the segments is constant.

Theorem 9 (Tangent Secant Segment Theorem)**.** If a secant and a tangent are drawn from a point outside the circle, then the product of the secant and its external part is equal to the square of the tangent.

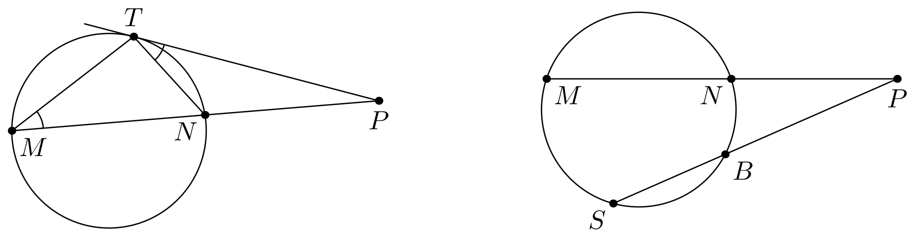


Figure 11.2: Tangent Secant Segment Theorem.

Proof. Let's construct the diagram (Figure 11.2). We need to prove: $PM \cdot PN = PT^2$.

$\Delta MTP \sim \Delta TNP$, because $\angle P$ is common and $\angle TMN = \angle NTP$ = half of arc TN. From the similarity:

$$\frac{PM}{PT} = \frac{PT}{PN} \Rightarrow PM \cdot PN = PT^2.$$

□

Corollary 2. For all secants passing through a given point outside the circle, the product of the secant and its external part is constant.

Theorem 10. If a perpendicular is dropped from a point C on the circumference to the diameter, then the length of the perpendicular is the geometric mean between the lengths of segments of the diameter.

Proof. This theorem is a direct consequence of the theorem about the altitude in a right-angled triangle, discussed in chapter «Let's calculate all the length ratios!» □

Theorem 11. If a perpendicular is dropped from a point C on the circumference to the diameter, then the chord connecting point C to the end of the diameter is the geometric mean between the diameter and the projection of the chord onto the diameter.

Proof. This theorem is proved by a simple consequence of the similarity of the right-angled triangles that arise. □

A very important fact, which follows from the Tangent Secant Segment Theorem, is considered in the following problem. This fact will be extremely useful when you study radical axes in the next book.

Example 11.1. Prove that the line passing through the points of intersection of two circles bisects the common tangent to them.

Solution: Let A and B be the points of intersection of two circles, let MN be the common tangent (M and N are the points of tangency), and let K be the point of intersection of lines AB and MN (A is between K and B) (Figure 11.3).

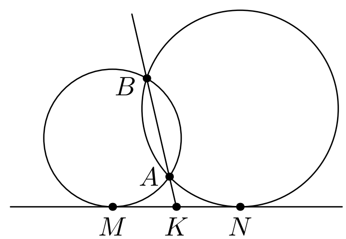


Figure 11.3: Common Secant of Two Circles.

Then, $MK^2 = KB \cdot KA$ and $NK^2 = KB \cdot KA$. Hence, $MK = NK$. □

Problem Set

Problem 11.1. (Gordin — 1331): In a circle with center O, chords AB and CD are drawn, intersecting at point M. Given that $AM = 4$, $MB = 1$, and $CM = 2$, find the angle OMC.

Problem 11.2. (Gordin — 4839): From point A, two rays intersect a given circle: one at points B and C, and the other at points D and E. It is known that $AB = 7$, $BC = 7$, and $AD = 10$. Find the length of DE.

Problem 11.3. (Gordin – 2635): Point M lies inside a circle with radius R and is located at a distance d from the center. Prove that for any chord AB passing through point M, the product $AM \cdot BM$ remains constant. What is this constant?

Problem 11.4. (Gordin – 2873): Through point A, located outside a circle at a distance of 7 from its center, a line is drawn intersecting the circle at points B and C. If $AB = 3$ and $BC = 5$, find the radius of the circle.

Problem 11.5. (Gordin – 4802): Point B is taken on segment AC. Circles are constructed with AB and AC as diameters. Perpendicular to the segment AC is drawn from point B and extended until it intersects the larger circle at point D. From point C, a tangent CK is drawn to the smaller circle. Prove that $CD = CK$.

Problem 11.6. (Gordin – 4621): Through the vertices A, B, and C of a parallelogram $ABCD$ with sides $AB = 3$ and $BC = 5$, a circle is drawn, intersecting the line BD at point E, where $BE = 9$. Find the diagonal BD.

Problem 11.7. (Gordin – 10): In a circle with a radius of $2\sqrt{7}$, a trapezoid $ABCD$ is inscribed, where its base AD is a diameter, and the angle BAD is $60°$. Chord CE intersects the diameter AD at point P, such that $AP : PD = 1 : 3$. Find the area of triangle BPE.

Problem 11.8. (Gordin – 1333): Points A, B, C, and D are located on a straight line in the given order. It is known that $BC = 3$ and $AB = 2CD$. A circle is drawn through points A and C, and another circle is drawn through points B and D. Their common chord intersects segment BC at point K. Find BK.

Problem 11.9. (AMC – 2018.10A.15): Two circles of radius 5 are externally tangent to each other and are internally tangent to a circle of radius 13 at points A and B. The distance AB can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?
(A) 21 (B) 29 (C) 58 (D) 69 (E) 93

Problem 11.10. (MechMat — 1999.05.4): Two circles intersect at points A and B. Through point B, a line is drawn intersecting the circles at points C and D, which lie on opposite sides of line AB. The tangents to these circles at points C and D intersect at point E. Find the length of AE if $AB = 10$, $AC = 16$, and $AD = 15$.

Problem 11.11. (MechMat — 2000.07.4): Two circles touch each other externally at point A. The line passing through point A intersects the first circle at point B and the second circle at point C. The tangent to the first circle passing through point B intersects the second circle at points D and E (D is between B and E). It is known that $AB = 5$ and $AC = 4$. Find the length of segment CE and the distance from point A to the center of the circle, which is tangent to segment AD and to the extensions of segments ED and EA beyond points D and A, respectively.

Problem 11.12. (MechMat — 2005.4): On the base BC of trapezoid $ABCD$, a point E is taken, lying on the same circle as points A, C, and D. Another circle passing through points A, B, and C is tangent to the line CD. Find BC if $AB = 12$ and $BE : EC = 4 : 5$. Determine all possible values of the ratio of the radius of the first circle to the radius of the second circle under these conditions.

Problem 11.13. (MechMat — 2006.5): The segment KB is the bisector of triangle KLM. A circle with a radius of 5 passes through the vertex K and touches side LM at point B, and intersects side KL at point A. Find angle K and the area of triangle KLM if $ML = 9\sqrt{3}$, and $KA : LB = 5 : 6$.

Problem 11.14. (First lemma about sparrows): Point W is the midpoint of the arc ACB of the circumcircle of triangle ABC. Points X and Y simultaneously move from vertices A and B along the lines AC and BC, respectively, and they move in the same direction (either towards point C or away from it). Prove that points W, C, X, Y lie on the same circle if and only if the speeds of the moving points are equal.

Problem 11.15. (Second lemma about sparrows): A circle is inscribed in a triangle ABC and touches sides BC and AC at points A_0 and B_0, respectively. Points X and Y simultaneously move from points A_0 and B_0 along the lines BC and AC, respectively, and they move in opposite directions (one towards point C, the other away from it). Prove that points C, X, Y, and I (the center of the inscribed circle) lie on the same circle if and only if the speeds of the moving points are equal.

Problem 11.16. (OMGO — 2006.8.9): The diagonals of the inscribed quadrilateral $ABCD$ intersect at point K. Prove that the tangent at point K to the circumcircle of triangle ABK is parallel to CD.

Problem 11.17. (AMC — 2017.10B.22): The diameter AB of a circle of radius 2 is extended to point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and line ED is perpendicular to line AD. Segment AE intersects the circle at a point C between A and E. What is the area of $\triangle ABC$?
(A) $\frac{120}{37}$ (B) $\frac{140}{39}$ (C) $\frac{145}{39}$ (D) $\frac{140}{37}$ (E) $\frac{120}{31}$

Problem 11.18. (AMC — 2010.10B.19): A circle with center O has an area of 156π. Triangle ABC is equilateral, $\overline{BC}$ is a chord on the circle, $OA = 4\sqrt{3}$, and point O is outside $\triangle ABC$. What is the side length of $\triangle ABC$?
(A) $2\sqrt{3}$ (B) 6 (C) $4\sqrt{3}$ (D) 12 (E) 18

Problem 11.19. (HMMT): Let ABC be an obtuse triangle with circumcenter O such that $\angle ABC = 15^\circ$ and $\angle BAC > 90^\circ$. Suppose that AO meets BC at D, and that $OD^2 + OC \cdot DC = OC^2$. Find $\angle C$.

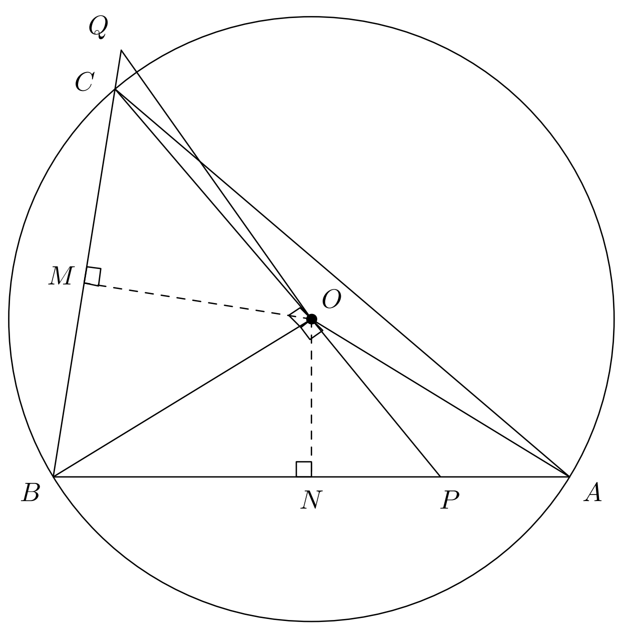
Problem 11.20. (HMMT): Points A, B, C lie on a circle ω such that BC is a diameter. AB is extended past B to point B' and AC is extended past C to point C' such that line $B'C'$ is parallel to BC and tangent to ω at point D. If $B'D = 4$ and $C'D = 6$, compute BC.

Problem 11.21. (AIME — 2014.I.15): n $\triangle ABC$, $AB = 3$, $BC = 4$, and $CA = 5$. Circle ω intersects $\overline{AB}$ at E and B, $\overline{BC}$ at B and D, and $\overline{AC}$ at F and G. Given that $EF = DF$ and $\frac{DG}{EG} = \frac{3}{4}$, length $DE = \frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is a positive integer not divisible by the square of any prime. Find $a + b + c$.

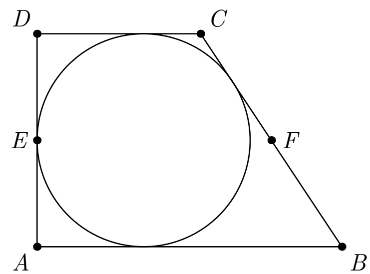
Problem 11.22. (AIME — 2012.II.15): Triangle ABC is inscribed in circle ω with $AB = 5$, $BC = 7$, and $AC = 3$. The bisector of angle A meets side $\overline{BC}$ at D and circle ω at a second point E. Let γ be the circle with diameter $\overline{DE}$. Circles ω and γ meet at E and a second point F. Then $AF^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 11.23. (BMO): Two touching circles S and T share a common tangent, which meets S at A and T at B. Let AP be a diameter of S and let the tangent from P to T touch it at Q. Show that $AP = PQ$.

Problem 11.24. (AIME — 2015.II.11): The circumcircle of acute $\triangle ABC$ has center O. The line passing through point O perpendicular to $\overline{OB}$ intersects lines AB and BC at P and Q, respectively. Also $AB = 5$, $BC = 4$, $BQ = 4.5$, and $BP = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Problem 11.25. (BMO): Quadrilateral $ABCD$ has right angles at A and D. A circle of radius 10 fits neatly inside the quadrilateral and touches all four sides. The length of edge BC is 24. The midpoint of edge AD is called E, and the midpoint of edge BC is called F. What is the length of EF?



Skill Assessment Problems

Skill Assessment Problem 11.1. Two circles intersect at points A and B. Point C lies on AB. Chords DM and LN pass through C on their respective circles. Prove that quadrilateral $DLMN$ is cyclic.

Skill Assessment Problem 11.2. Line DA is tangent to the circle passing through A, and chord BC is parallel to DA. Lines DB and DC intersect the circle again at points K and L. Line KL intersects segment DA at point X. Prove that X is the midpoint of DA.

Skill Assessment Problem 11.3. Given a parallelogram $ABCD$ with $AC > BD$, point M lies on AC such that points B, C, D, and M are concyclic. Prove that BD is a common tangent to the circumcircles of triangles ABM and ADM.

Solutions to Skill Assessment Problems

Solution to Problem 11.1: Since the quadrilaterals $ALBN$ and $AMBD$ are inscribed, we have $CL \cdot CN = CA \cdot CB = CM \cdot CD$. Therefore, the quadrilateral $KLMN$ is inscribed. □

Solution to Problem 11.2: Let DA and BC be oriented in the same direction; M is the intersection point of the lines KL and DA. Then, $\angle LDM = \angle LCB = \angle DKM$, which implies $\triangle KDM \sim \triangle DLM$. Therefore, $DM : KM = LM : DM$, i.e., $DM^2 = KM \cdot LM$. Additionally, $MA^2 = MK \cdot ML$. Thus, $MA = DM$. □

Solution to Problem 11.3: Let AC and BD intersect at O. Then, $MO \cdot OC = BO \cdot OD$. Since $OC = OA$ and $BO = OD$, we have $MO \cdot OA = BO^2$ and $MO \cdot OA = DO^2$. Thus, OB is a tangent to the circle circumscribed around triangle ABM, and OD is a tangent to the circle circumscribed around triangle ADM.

Since points D, O, and B are collinear, BD is a common tangent to the circles circumscribed around the required triangles. □

Right Triangle Altitude Properties

> “When Pythagoras discovered his famous theorem, he offered the gods a sacrifice of a hundred oxen. Since then, the oxen shudder whenever a new truth comes to light.
>
> —Ludwig Börne

Theory and Practice

Certainly, the most well-known «algebraic» theorem in geometry is the Pythagorean theorem.

Theorem 12 (Pythagorean Theorem)**.** The sum of the squares of the legs of a right-angled triangle is equal to the square of its hypotenuse.

At this time, there are about 400 different proofs for this theorem.

Proof 1 *(by similarity).* In the right-angled triangle ABC, draw the altitude CH from the right angle (Figure 12.1).

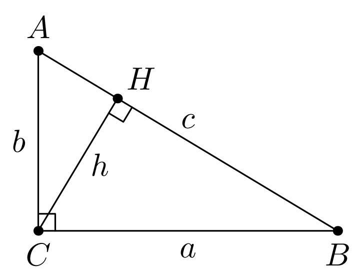


Figure 12.1: Proof by similarity.

From the similarity of triangles ABC, BCH, and ACH, we have:

$$\frac{a}{c} = \frac{|HB|}{a}; \quad \frac{b}{c} = \frac{|AH|}{b}.$$

Hence:

$$a^2 = c \cdot |HB|; b^2 = c \cdot |AH|,$$
$$a^2 + b^2 = c \cdot (|HB| + |AH|) = c^2 \Rightarrow a^2 + b^2 = c^2.$$

□

Proof 2 (by areas). Let's use the «careful observation of the picture» method (Figure 12.2).

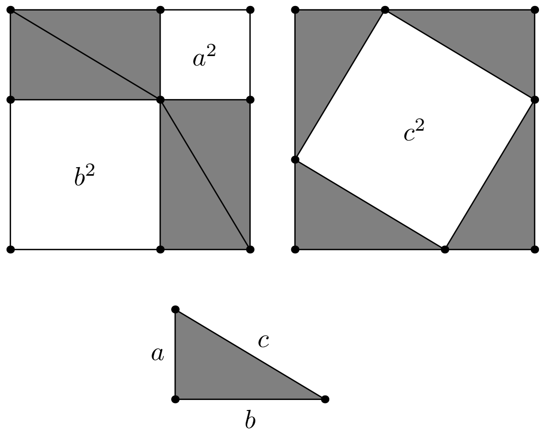


Figure 12.2: Proof by equal areas.

It can be seen that, since the areas of the proposed squares are equal, the areas of their white parts are also equal. □

Proof 3 (Leonardo da Vinci). Let ABC be a right-angled triangle, on the sides of which squares are built. On JH, a triangle symmetrical to ABC is built (with respect to the center of the square), constructing Figure 12.3.

The line CI passes through the center of symmetry, so it divides the square in half. Quadrilaterals $CAJI$ and $ABGD$ coincide by a 90° rotation, so they are congruent. Moreover, from their equality, it follows that the area of half of the square built on the hypotenuse is equal to the sum of the areas of half of the squares built on the legs, which indicates the validity of the theorem. □

The Pythagorean theorem is probably the most well-known theorem in geometry and may be required almost everywhere where there are right angles. Let's consider a few problems related to it.

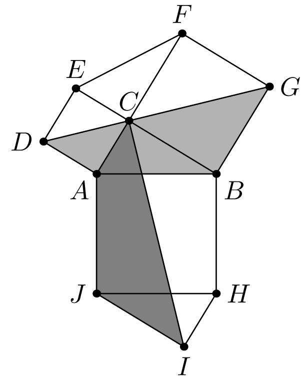


Figure 12.3: Leonardo da Vinci's proof

Of course, in right-angled triangles, we will also use the theorem about the triangle's height.

Example 12.1. On the altitudes BB_1 and CC_1 of triangle ABC, points B_2 and C_2 are taken so that $\triangle AB_2C = \triangle AC_2B = 90°$. Prove that $AB_2 = AC_2$.

Solution: Let's construct the diagram in Figure 12.4.

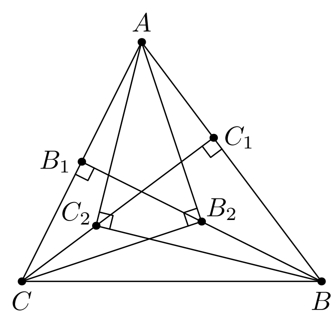


Figure 12.4: Diagram for the problem.

By the theorem of the height of a right-angled triangle:

$$AB_2^2 = AB_1 \cdot AC = AC_1 \cdot AB = AC_2^2.$$

□

Problem Set

Problem 12.1. (Gordin — 2000): The legs of a right-angled triangle are in the ratio $5 : 6$, and the length of the hypotenuse is 122. Find the lengths of segments into which the altitude drawn from the right angle divides the hypotenuse.

Problem 12.2. (Gordin — 3432): Find the height of a right-angled triangle dropped onto the hypotenuse if it is known that the base of this height divides the hypotenuse into segments of length 1 and 4.

Problem 12.3. (Gordin — 420): The chord of a circle is 10. Through one end of the chord, a tangent is drawn to the circle, and through the other end, a secant is drawn parallel to the tangent. Find the radius of the circle if the internal segment of the secant is 12.

Problem 12.4. (Gordin — 2015): In an isosceles triangle ABC, the base AC is 32, and the lateral sides are 20. A perpendicular is drawn from vertex B to the lateral side, dividing the base. Find the lengths of the segments into which it divides the base.

Problem 12.5. (Gordin — 2237): In an isosceles trapezoid $ABCD$ with bases BC and AD, the diagonals intersect at point O. Find the perimeter of the trapezoid if $BO = \frac{7}{8}$, $OD = \frac{25}{8}$, and $\angle ABD = 90^\circ$.

Problem 12.6. (Mos2ARSO — 2016.10.4): From the vertex of the obtuse angle A of triangle ABC, the altitude AD is dropped. A circle with center D and radius DA is drawn, which intersects sides AB and AC at points M and N, respectively. Find AC if $AB = c$, $AM = m$, and $AN = n$.

Problem 12.7. (Gordin — 4401): Point A is chosen on a circle, and on diameter BC, points D and E are taken. Beyond point B, on the extension of the diameter, point F is taken. Find BC if $\angle BAD = \angle ACD$, $\angle BAF = \angle CAE$, $BD = 2$, $BE = 5$, and $BF = 4$.

Problem 12.8. (Gordin — 4403): On the diameter AB of a circle, points C and D are taken, and on its extension beyond point B, point E is taken. On the circle, point F is taken, such that $\angle AFC = \angle BFE$, $\angle DAF = \angle BFD$, $AB = 8$, $CB = 6$, and $DB = 5$. Find BE.

Problem 12.9. (Gordin — 4580): The length of altitude of a right-angled triangle dropped from the vertex of the right angle is equal to h, and the difference between the projections of the legs on the hypotenuse is equal to l. Find the area of this triangle.

Problem 12.10. (Gordin — 4586): In a right-angled triangle, the length of altitude dropped from the vertex of the right angle is equal to h, and the projection of one of the legs on the hypotenuse is equal to l. Find the radius of the circle tangent to the legs if the center of the circle lies on the hypotenuse of a triangle.

Problem 12.11. (Gordin — 1987): The height CD of triangle ABC divides the side AB into segments AD and BD, such that $AD \cdot BD = CD^2$. Is it true that triangle ABC is right-angled?

Problem 12.12. (Gordin — 2058): In the trapezoid $ABCD$, points K and M are the midpoints of the bases $AB = 5$ and $CD = 3$, respectively. Find the area of the trapezoid if triangle AMB is right-angled and DK is the height of the trapezoid.

Problem 12.13. (AMC — 2019.10A.15): Right triangles T_1 and T_2 have areas 1 and 2, respectively. A side of T_1 is congruent to a side of T_2, and a different side of T_1 is congruent to a different side of T_2. What is the square of the product of the other (third) sides of T_1 and T_2?

(A) $\frac{28}{3}$ (B) 10 (C) $\frac{32}{3}$ (D) $\frac{34}{3}$ (E) 12

Problem 12.14. (AMC — 2019.10B.16): In $\triangle ABC$ with a right angle at C, point D lies in the interior of $\overline{AB}$ and point E lies in the interior of $\overline{BC}$ so that $AC = CD$, $DE = EB$, and the ratio $AC : DE = 4 : 3$. What is the ratio $AD : DB$?

(A) $2:3$ (B) $2:\sqrt{5}$ (C) $1:1$ (D) $3:\sqrt{5}$ (E) $3:2$

Problem 12.15. (AMC — 2018.10A.23): Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths of 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air, it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?

(A) $\frac{25}{27}$ (B) $\frac{26}{27}$ (C) $\frac{73}{75}$ (D) $\frac{145}{147}$ (E) $\frac{74}{75}$

Problem 12.16. (AMC — 2017.10A.21): A square with side length x is inscribed in a right triangle with sides of length 3, 4, and 5 so that one vertex of the square coincides with the right-angle vertex of the triangle. A square with side length y is inscribed in another right triangle with sides of length 3, 4, and 5 so that one side of the square lies on the hypotenuse of the triangle. What is $\frac{x}{y}$?

(A) $\frac{12}{13}$ (B) $\frac{35}{37}$ (C) 1 (D) $\frac{37}{35}$ (E) $\frac{13}{12}$

Problem 12.17. (AMC — 2014.10A.9): The two legs of a right triangle, which are altitudes, have lengths of $2\sqrt{3}$ and 6. How long is the third altitude of the triangle?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 12.18. (AMC — 2014.10A.22): In rectangle $ABCD$, $AB = 20$ and $BC = 10$. Let E be a point on $\overline{CD}$ such that $\angle CBE = 15^\circ$. What is AE?

(A) $\frac{20\sqrt{3}}{3}$ (B) $10\sqrt{3}$ (C) 18 (D) $11\sqrt{3}$ (E) 20

Problem 12.19. (AMC — 2013.10A.15): Two sides of a triangle have lengths of 10 and 15. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side?

(A) 6 (B) 8 (C) 9 (D) 12 (E) 18

Problem 12.20. (AMC – 2013.10B.23): In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

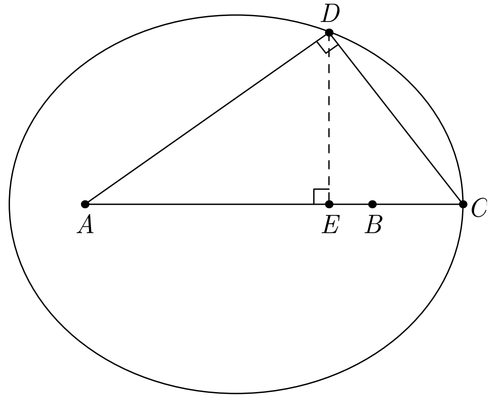
(A) 18 (B) 21 (C) 24 (D) 27 (E) 30

Problem 12.21. (AMC – 2011.10B.14): A rectangular parking lot has a diagonal of 25 meters and an area of 168 square meters. In meters, what is the perimeter of the parking lot?

(A) 52 (B) 58 (C) 62 (D) 68 (E) 70

Problem 12.22. (mathcounts): Triangle PQR is a right triangle with $\angle Q = 90°$, $PQ = 3$ and $QR = 4$. Points S, T and U are on sides PQ, PR and QR, respectively, such that $QSTU$ is a square. Find the length of ST. Express your answer as a common fraction.

Problem 12.23. (Georgia Tech High School Math Day): An ellipse has foci A and B with $AB = 4$. The segment AB is extended to point C on the ellipse, and $BC = 1$. Point D also lies on the ellipse, and $\angle ADC$ is a right angle. Let point E be the base of the altitude of triangle ADC from point D. Compute the length AE. (The diagram below is not necessarily to scale.)



Skill Assessment Problems

Skill Assessment Problem 12.1. (Sharygin — 2016.8.1) In triangle ABC, the height AH divides the median BM in half. Prove that a right-angled triangle can be formed from the medians of triangle ABM.

Skill Assessment Problem 12.2. (Sharygin — 2016.8.2) On a grid paper, mark three nodes so that in the triangle formed by them, the sum of two shorter medians is equal to the semi-perimeter.

Skill Assessment Problem 12.3. The legs of a right-angled triangle are in the ratio $1 : 10$, and the hypotenuse is 2020. Find the segments into which the height from the right angle divides the hypotenuse.

Solutions to Skill Assessment Problems

Solution to Problem 12.1: Let the medians AK, BL, and MN of triangle ABM intersect at point T (see Figure 12.5).

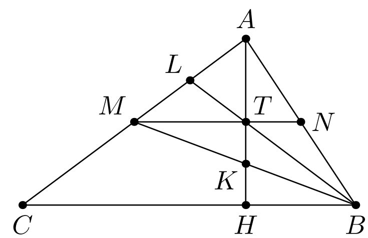


Figure 12.5: Three medians of a triangle

Since $AT = \frac{2}{3}AK$, $BT = \frac{2}{3}BL$, $MT = \frac{2}{3}MN$, it is enough to construct a right-angled triangle from the segments AT, BT, and MT. But this triangle is AMT, as $MN \perp AH$ as the median of triangle ABC, and $AM = 2LT = BT$. □

Solution to Problem 12.2: Mark points A, B, and C, forming a right-angled triangle with legs $AC = 6$ and $BC = 4$. The median from vertex C is half the length of hypotenuse AB, and the median from vertex B by the Pythagorean theorem is $5 = \frac{1}{2}(AC + BC)$, which is the required condition. □

Solution to Problem 12.3: Let the leg $BC = x$, then the leg $AC = 10x$. Draw the altitude CD of triangle ABC (Figure 12.6).

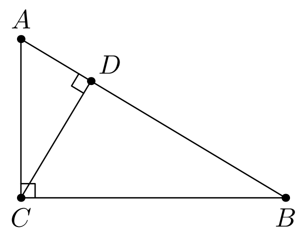


Figure 12.6: Altitude in a right-angled triangle.

Then, $AD = \frac{AC^2}{AB} = \frac{100x^2}{2020}$, $BD = \frac{BC^2}{AB} = \frac{x^2}{2020}$, $AD + BD = 2020$.

This implies $\frac{100x^2}{2020} + \frac{x^2}{2020} = 2020$. Thus, $101x^2 = 2020^2$, $x^2 = 2020 \cdot 20$.

After substitution, we obtain $AD = \frac{100 \cdot 2020 \cdot 20}{2020} = 2000$, $BD = \frac{BC^2}{AB} = \frac{2020 \cdot 20}{2020} = 20$. $\square$

Areas

> The essence of mathematics lies in its freedom.
>
> —Georg Cantor

Theory and Practice

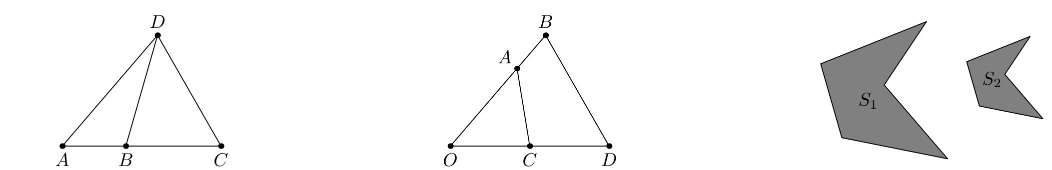
Let's recall various formulas and theorems for areas. The area of triangle ABC with sides $BC = a$, $AC = b$, $AB = c$ can be calculated using one of the following formulas:

1. $A = \frac{a \cdot h_a}{2} = \frac{b \cdot h_b}{2} = \frac{c \cdot h_c}{2}$, where h_a, h_b, h_c are the lengths of altitudes drawn from vertices A, B, C, respectively.

2. $A = p \cdot r$, where p is the semiperimeter and r is the radius of the inscribed circle.

3. $A = \sqrt{p(p-a)(p-b)(p-c)}$ — Heron's formula.

How can we understand the relationships between different areas? The following facts will help us:



a) Common height. b) Common angle. c) Similar figures.

Figure 13.1: Facts about the areas of figures.

1. Let points A, B, C lie on the same line, while point D does not lie on this line. Then $\frac{A_{ABD}}{A_{BCD}} = \frac{AB}{BC}$, since these triangles share a common height (Figure 13.1a).

2. Let points A and B lie on one side of angle O with vertex O, and points C and D on the other side. Then $\frac{A_{OAC}}{A_{OBD}} = \frac{OA \cdot OC}{OB \cdot OD}$, since these triangles share a common angle (Figure 13.1b).

3. The areas of similar figures are in proportion to the square of the similarity

ratio (Figure 13.1c).

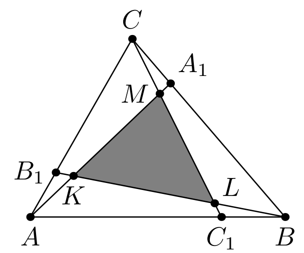


Figure 13.2: Illustration of the problem.

Example 13.1. Consider triangle ABC and points A_1,B_1, C_1 on its sides BC, AC, AB respectively, such that $\frac{AB_1}{B_1C} = \frac{CA_1}{A_1B} = \frac{BC_1}{C_1A} = \frac{1}{3}$. Segments AA_1 and BB_1 intersect at point K, segments BB_1 and CC_1 intersect at point L, and segments CC_1 and AA_1 intersect at point M. Find the ratio of the areas of triangles KLM and ABC (Figure 13.2).

Proof. Let's use the theorem on proportional segments in a triangle: $\frac{AK}{KA_1} = \frac{AB_1}{B_1C} \cdot \left(1 + \frac{CA_1}{A_1B}\right) = \frac{1}{3} \cdot \frac{4}{3} = \frac{4}{9}$. Therefore, $\frac{AK}{AA_1} = \frac{4}{13}$. Denote the area of triangle ABC as A. Then $\frac{A_{AA_1C}}{A_{ABC}} = \frac{A_1C}{BC} = \frac{1}{4} \implies A_{AA_1C} = \frac{A}{4}$. Similarly, the areas of triangles ABB_1 and BCC_1 are also $\frac{A}{4}$. Now, let's find the area of triangle AKB_1, using the fact that triangle AA_1C shares a common angle with it:

$$\frac{A_{AKB_1}}{A_{AA_1C}} = \frac{AK}{AA_1} \cdot \frac{AB_1}{AC} = \frac{4}{13} \cdot \frac{1}{4} = \frac{1}{13} \implies$$

$$A_{AKB_1} = \frac{A_{AA_1C}}{13} = \frac{A}{52}$$

.

Similarly, the areas of triangles C_1LB and A_1MC are also $\frac{A}{52}$. Now we can calculate the area of triangle KLM:

$$A_{KLM} = A_{ABC} - A_{ABB_1} - A_{BCC_1} - A_{CAA_1} + A_{AKB_1} +$$

$$+A_{CMA_1} + A_{BLC_1} =$$
$$= A\left(1 - \frac{1}{4} \cdot 3 + \frac{1}{52} \cdot 3\right) = \frac{4A}{13},$$

which is what we needed to find. □

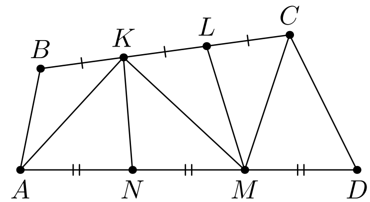


Figure 13.3: Illustration of the problem.

Example 13.2. Consider a convex quadrilateral $ABCD$ with $BK = KL = LC$ on side BC and $AN = NM = MD$ on side AD. Prove that the area of quadrilateral $ABCD$ is three times the area of quadrilateral $KLMN$ (Fig. 13.3).

Proof. The areas of triangles AKN and MKN are equal since they share a common height and equal base. Denote these areas as A_1. Similarly, denote the areas of triangles KLM and CLM as A_2.

For triangles ABC and ABK, their heights coincide, but the bases differ by a factor of 3, so $A_{ABC} = 3A_{ABK}$. Similarly, $A_{ACD} = 3A_{MCD}$. Therefore,

$$A_{ABCD} = A_{ABC} + A_{ACD} = 3(A_{ABK} + A_{MCD}) \Rightarrow$$

$$\Rightarrow A_{AKCM} = 2(A_{ABK} + A_{MCD}) \Rightarrow$$

$$A_{ABK} + A_{CDM} = \frac{A_{AKCM}}{2} = A_1 + A_2.$$

Thus, the area of quadrilateral $ABCD$ is $3(A_1 + A_2)$, while the area of quadrilateral $KLMN$ is $A_1 + A_2$, which is three times smaller. □

Example 13.3. In triangle ABC, $AB = 10$, $BC = 12$, and $AC = 8$. On segment AB, point K is taken such that $AK : KB = 2 : 3$, and on side BC, point M is taken such that $BM : MC = 2 : 1$. Point O is taken on segment KM such that $KO : OM = 4 : 5$. Which of the triangles ABO, BCO, or ACO has the smallest area?

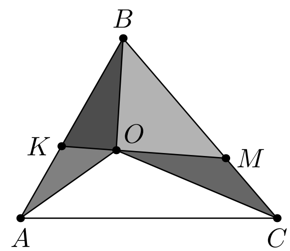


Figure 13.4: Area ratio

Proof. Let $A_{AKO} = s$, then by the property of areas (equal heights, different bases), we have (see Fig. 13.4):

$$\frac{A_{BKO}}{A_{AKO}} = \frac{3}{2}, \text{ then } A_{BKO} = \frac{3}{2} \cdot s,$$

$$\frac{A_{BKO}}{A_{BOM}} = \frac{4}{5}, \text{ then } A_{BOM} = \frac{5}{4} \cdot \frac{3}{2} \cdot s = \frac{15}{8} \cdot s,$$

$$\frac{A_{BOM}}{A_{OMC}} = \frac{1}{2}, \text{ then } A_{OMC} = \frac{1}{2} \cdot \frac{15}{8} \cdot s = \frac{15}{16} \cdot s.$$

Then $A_{AOB} = s + \frac{3}{2} \cdot s = \frac{5}{2} \cdot s = \frac{40}{16} \cdot s$ and $A_{BOC} = \frac{15}{8} \cdot s + \frac{15}{16} \cdot s = \frac{45}{16} \cdot s$. Also, $A_{KBM} = \frac{27}{8} \cdot s$, on the other hand, $A_{KBM} = \frac{3}{5} \cdot \frac{2}{3} \cdot A_{ABC}$, i.e., $A_{ABC} = \frac{5}{2} \cdot \frac{27}{8} \cdot s$. Then $A_{AOC} = A_{ABC} - A_{AOB} - A_{BOC} = \frac{50}{16}s$. Therefore, the area of ABO is the smallest. □

Problem Set

Problem 13.1. (MMG — 2012.11.2): Inside parallelogram $ABCD$, an arbitrary point P is chosen, and segments PA, PB, PC, and PD are drawn. The areas of three of the resulting triangles are $1, 2$, and 3 (in some order). What can the area of the fourth triangle?

Problem 13.2. (MMO — 2005.10.3): On the sides of triangle ABC, squares ABB_1A_2, BCC_1B_2, and CAA_1C_2 are constructed outward. Squares $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are also constructed outward on segments A_1A_2 and B_1B_2 beyond triangles AA_1A_2 and BB_1B_2. Prove that $A_3B_4 \parallel AB$.

Problem 13.3. (Gordin — 2925): Points M and N are located on side BC of triangle ABC, and point K is on side AC. It is given that $BM : MN : NC = 1 : 1 : 2$ and $CK : AK = 1 : 4$. If the area of triangle ABC is equal to 1, find the area of quadrilateral $AMNK$.

Problem 13.4. (MMO — 1967.9.2): Can a square pie be cut into 9 equal parts by choosing two points inside the square and connecting each of them with straight cuts to all four vertices of the square? If possible, which two points should be chosen?

Problem 13.5. (Gordin — 6226): In triangle ABC with $AB = BC \neq AC$, point E is chosen on side AB, and point D is chosen beyond vertex A on the extension of side AC, such that $\angle BDC = \angle ECA$. Prove that the areas of triangles DEC and ABC are equal.

Problem 13.6. (TOT — 1996.9.3): The rectangle $ABCD$ has an area of 1 and is folded along a line so that point C coincides with point A. Prove that the area of the resulting pentagon has an area of less than $\frac{3}{4}$.

Problem 13.7. (AMC — 2019.10B.24): In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?

Problem 13.8. (AMC — 2018.10A.24): Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

(A) 60 (B) 65 (C) 70 (D) 75 (E) 80

Problem 13.9. (MechMat — 2004.03.4): In the convex quadrilateral $KLMN$, diagonals KM and LN are perpendicular to the sides MN and KL respectively, and the length of side KN is $4\sqrt{3}$. Point A is located on the side KN such that $\angle LAK = \angle MAN$. It is known that $\angle MKN - \angle KNL = 15^\circ$. Find the length of the broken line LAM and the area of quadrilateral $KLMN$ if $LA : AM = 1 : \sqrt{3}$.

Problem 13.10. (AMC — 2018.10B.24): Let $ABCDEF$ be a regular hexagon with side length 1. X, Y, and Z are placed at the midpoints of sides $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$, respectively. What is the area of the convex hexagon whose interior is the intersection of the interiors of $\triangle ACE$ and $\triangle XYZ$?

(A)$\frac{3}{8}\sqrt{3}$ (B)$\frac{7}{16}\sqrt{3}$ (C)$\frac{15}{32}\sqrt{3}$ (D)$\frac{1}{2}\sqrt{3}$ (E)$\frac{9}{16}\sqrt{3}$

Problem 13.11. (AMC — 2017.10A.22): Sides $\overline{AB}$ and $\overline{AC}$ of equilateral triangle ABC are tangent to a circle at points B and C, respectively. What fraction of the area of $\triangle ABC$ lies outside the circle?

(A) $\frac{4\sqrt{3}\pi}{27} - \frac{1}{3}$ (B) $\frac{\sqrt{3}}{2} - \frac{\pi}{8}$ (C) $\frac{1}{2}$ (D) $\sqrt{3} - \frac{2\sqrt{3}\pi}{9}$ (E) $\frac{4}{3} - \frac{4\sqrt{3}\pi}{27}$

Problem 13.12. (AMC — 2017.10B.15): Rectangle $ABCD$ has $AB = 3$ and $BC = 4$. Point E is the foot of the perpendicular from B to diagonal $\overline{AC}$. What is the area of $\triangle AED$?

(A) 1 (B) $\frac{42}{25}$ (C) $\frac{28}{15}$ (D) 2 (E) $\frac{54}{25}$

Problem 13.13. (AMC — 2017.10B.19): Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3AB$. Similarly, extend side $\overline{BC}$

beyond C to a point C' so that $CC' = 3BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

(A) $9:1$ (B) $16:1$ (C) $25:1$ (D) $36:1$ (E) $37:1$

Problem 13.14. (AMC — 2017.10B.21): In $\triangle ABC$, $AB = 6$, $AC = 8$, $BC = 10$, and D is the midpoint of $\overline{BC}$. What is the sum of the radii of the circles inscribed in $\triangle ADB$ and $\triangle ADC$?

(A) $\sqrt{5}$ (B) $\frac{11}{4}$ (C) $2\sqrt{2}$ (D) $\frac{17}{6}$ (E) 3

Problem 13.15. (AMC — 2016.10A.21): Circles with centers P, Q and R, having radii $1, 2$ and 3, respectively, lie on the same side of line l and are tangent to l at P', Q' and R', respectively, with Q' between P' and R'. The circle with center Q is externally tangent to each of the other two circles. What is the area of triangle PQR?

(A) 0 (B) $\sqrt{\frac{2}{3}}$ (C) 1 (D) $\sqrt{6} - \sqrt{2}$ (E) $\sqrt{\frac{3}{2}}$

Problem 13.16. (AMC — 2016.10B.10): A thin piece of wood of uniform density in the shape of an equilateral triangle with a side length of 3 inches weighs 12 ounces. A second piece of the same type of wood, with the same thickness and also in the shape of an equilateral triangle, has a side length of 5 inches. Which of the following is closest to the weight, in ounces, of the second piece?

(A) 14.0 (B) 16.0 (C) 20.0 (D) 33.3 (E) 55.6

Problem 13.17. (AMC — 2016.10B.11): Carl decided to fence in his rectangular garden. He bought 20 fence posts, placed one on each of the four corners, and spaced out the rest evenly along the edges of the garden, leaving exactly 4 yards between neighboring posts. The longer side of his garden, including the corners, has twice as many posts as the shorter side, including the corners. What is the area, in square yards, of Carl's garden?

(A) 256 (B) 336 (C) 384 (D) 448 (E) 512

Problem 13.18. (AMC — 2016.10B.23): In regular hexagon $ABCDEF$, points W, X, Y, and Z are chosen on sides $\overline{BC}$, $\overline{CD}$, $\overline{EF}$, and $\overline{FA}$ respectively, so lines AB, ZW, YX, and ED are parallel and equally spaced. What is the ratio of the area of hexagon $WCXYFZ$ to the area of hexagon $ABCDEF$?

(A) $\frac{1}{3}$ (B) $\frac{10}{27}$ (C) $\frac{11}{27}$ (D) $\frac{4}{9}$ (E) $\frac{13}{27}$

Problem 13.19. (AMC — 2015.10A.11): The ratio of the length to the width of a rectangle is $4:3$. If the rectangle has a diagonal of length d, then the area may be expressed as kd^2 for some constant k. What is k?

(A) $\frac{2}{7}$ (B) $\frac{3}{7}$ (C) $\frac{12}{25}$ (D) $\frac{16}{25}$ (E) $\frac{3}{4}$

Problem 13.20. (AMC — 2015.10A.19): The isosceles right triangle ABC has right angle at C and area 12.5. The rays trisecting $\angle ACB$ intersect AB at D and E. What is the area of $\triangle CDE$?

(A) $\frac{5\sqrt{2}}{3}$ (B) $\frac{50\sqrt{3}-75}{4}$ (C) $\frac{15\sqrt{3}}{8}$ (D) $\frac{50-25\sqrt{3}}{2}$ (E) $\frac{25}{6}$

Problem 13.21. (AMC — 2014.10A.13): Equilateral $\triangle ABC$ has a side length of 1, and squares $ABDE$, $BCHI$, $CAFG$ lie outside the triangle. What is the area of hexagon $DEFGHI$?

(A) $\frac{12+3\sqrt{3}}{4}$ (B) $\frac{9}{2}$ (C) $3+\sqrt{3}$ (D) $\frac{6+3\sqrt{3}}{2}$ (E) 6

Problem 13.22. (AMC — 2014.10A.15): In rectangle $ABCD$, $DC = 2CB$ and points E and F lie on $\overline{AB}$ so that $\overline{ED}$ and $\overline{FD}$ trisect $\angle ADC$. What is the ratio of the area of $\triangle DEF$ to the area of rectangle $ABCD$?

(A) $\frac{\sqrt{3}}{6}$ (B) $\frac{\sqrt{6}}{8}$ (C) $\frac{3\sqrt{3}}{16}$ (D) $\frac{1}{3}$ (E) $\frac{\sqrt{2}}{4}$

Problem 13.23. (AMC — 2013.10A.13): Square $ABCD$ has side length 10. Point E is on $\overline{BC}$, and the area of $\triangle ABE$ is 40. What is BE?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Problem 13.24. (AMC — 2013.10A.20): A unit square is rotated 45° about its center. What is the area of the region swept out by the interior of the square?

(A) $1 - \frac{\sqrt{2}}{2} + \frac{\pi}{4}$ (B) $\frac{1}{2} + \frac{\pi}{4}$ (C) $2 - \sqrt{2} + \frac{\pi}{4}$

(D) $\frac{\sqrt{2}}{2} + \frac{\pi}{4}$ (E) $1 + \frac{\sqrt{2}}{4} + \frac{\pi}{8}$

Problem 13.25. (AMC — 2013.10B.7): Six points are equally spaced around a circle of radius 1. Three of these points are the vertices of a triangle that is neither equilateral nor isosceles. What is the area of this triangle?

(A) $\frac{\sqrt{3}}{3}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{2}$ (E) 2

Problem 13.26. (AMC — 2013.10B.15): A wire is cut into two pieces, one of length a and the other of length b. The piece of length a is bent to form an equilateral triangle, and the piece of length b is bent to form a regular hexagon. The triangle and the hexagon have equal area. What is $\frac{a}{b}$?

(A) 1 (B) $\frac{\sqrt{6}}{2}$ (C) $\sqrt{3}$ (D) 2 (E) $\frac{3\sqrt{2}}{2}$

Problem 13.27. (AMC — 2013.10B.16): In triangle ABC, medians AD and CE intersect at P, with $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$?

(A)13 (B)13.5 (C)14 (D)14.5 (E)15

Problem 13.28. (AMC — 2012.10B.14): Two equilateral triangles lie inside of a square whose side length is $2\sqrt{3}$. The bases of these triangles are the opposite sides of the square, and their intersection is a rhombus. What is the area of the rhombus?

(A) $\frac{3}{2}$ (B) $\sqrt{3}$ (C) $2\sqrt{2}-1$ (D) $8\sqrt{3}-12$ (E) $\frac{4\sqrt{3}}{3}$

Problem 13.29. (AMC — 2012.10B.19): In rectangle $ABCD$, $AB = 6$, $AD = 30$, and G is the midpoint of $\overline{AD}$. Segment AB is extended 2 units beyond B to point E, and F is the intersection of $\overline{ED}$ and $\overline{BC}$. What is the area of $BFDG$?

(A) $\frac{133}{2}$ (B) 67 (C) $\frac{135}{2}$ (D) 68 (E) $\frac{137}{2}$

Problem 13.30. (AMC — 2011.10A.11): Square $EFGH$ has one vertex on each side of square $ABCD$. Point E is on $\overline{AB}$ with $AE = 7 \cdot EB$. What is the ratio of the area of $EFGH$ to the area of $ABCD$?

(A) $\frac{49}{64}$ (B) $\frac{25}{32}$ (C) $\frac{7}{8}$ (D) $\frac{5\sqrt{2}}{8}$ (E) $\frac{\sqrt{14}}{4}$

Problem 13.31. (AMC — 2011.10A.18): Circles A, B, and C each have a radius of 1. Circles A and B share one point of tangency. Circle C has a point of tangency with the midpoint of $\overline{AB}$. What is the area inside circle C but outside circle A and circle B ?

(A) $3-\frac{\pi}{2}$ (B) $\frac{\pi}{2}$ (C) 2 (D) $\frac{3\pi}{4}$ (E) $1+\frac{\pi}{2}$

Problem 13.32. (AMC — 2010.10A.19): Equiangular hexagon $ABCDEF$ has side lengths $AB = CD = EF = 1$ and $BC = DE = FA = r$. The area of $\triangle ACE$ is 70% of the area of the hexagon. What is the sum of all possible values of r?

(A) $\frac{4\sqrt{3}}{3}$ (B) $\frac{10}{3}$ (C) 4 (D) $\frac{17}{4}$ (E) 6

Problem 13.33. (AMC — 2010.10B.7): A triangle has side lengths 10, 10, and 12. A rectangle has width 4 and area equal to the area of the triangle. What is the perimeter of this rectangle?

(A) 16 (B) 24 (C) 28 (D) 32 (E) 36

Problem 13.34. (AMC — 2010.10B.16): A square of side length 1 and a circle of radius $\frac{\sqrt{3}}{3}$ share the same center. What is the area inside the circle, but outside the square?

(A) $\frac{\pi}{3} - 1$ (B) $\frac{2\pi}{9} - \frac{\sqrt{3}}{3}$ (C) $\frac{\pi}{18}$ (D) $\frac{1}{4}$ (E) $\frac{2\pi}{9}$

Problem 13.35. (AMC — 2010.10B.20): Two circles lie outside of the regular hexagon $ABCDEF$. The first circle is tangent to $\overline{AB}$, and the second is tangent to $\overline{DE}$. Both are tangent to lines BC and FA. What is the ratio of the area of the second circle to the one of the first circle?

(A) 18 (B) 27 (C) 36 (D) 81 (E) 108

Problem 13.36. (AIME — 2002.II.13): In triangle ABC, point D is on $\overline{BC}$ with $CD = 2$ and $DB = 5$, point E is on $\overline{AC}$ with $CE = 1$ and $EA = 3$, $AB = 8$, and $\overline{AD}$ and $\overline{BE}$ intersect at P. Points Q and R lie on $\overline{AB}$ so that $\overline{PQ}$ is parallel to $\overline{CA}$ and $\overline{PR}$ is parallel to $\overline{CB}$. It is given that the ratio of the area of triangle PQR to the area of triangle ABC is m/n, where m and n are relatively prime positive integers. Find $m + n$.

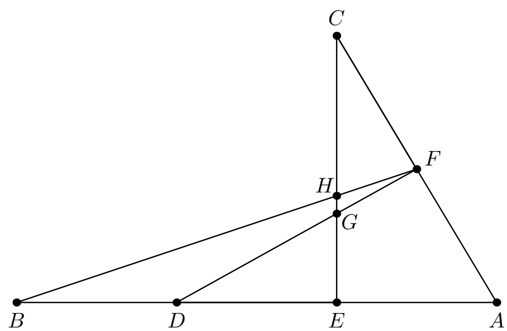
Problem 13.37. (HMMT): Let $ABCD$ be a parallelogram with $AB = 8$, $AD = 11$, and $\angle BAD = 60°$. Let X be on segment CD with $CX/XD = 1/3$ and Y be on segment AD with $AY/YD = 1/2$. Let Z be on segment AB such that AX, BY, and DZ are concurrent. Determine the area of triangle XYZ.

Problem 13.38. (BMO): Let ABC be an equilateral triangle, and P be a point inside this triangle. Let D, E and F be the feet of the perpendiculars from P to the sides BC, CA and AB, respectively. Prove that

a) $AF + BD + CE = AE + BF + CD$ and

b) $[APF] + [BPD] + [CPE] = [APE] + [BPF] + [CPD]$.

Problem 13.39. (ARML): Given noncollinear points A, B, C, segment AB is trisected by points D and E, and F is the midpoint of segment AC. DF and BF intersect CE at G and H, respectively. If $[DEG] = 18$, compute $[FGH]$.



Skill Assessment Problems

Skill Assessment Problem 13.1. Is there a triangle such that the distances from its vertices to the opposite sides are 1, 2, and 3?

Skill Assessment Problem 13.2. $ABCD$ is a convex quadrilateral, and AC intersects BD at point O. If $A_{AOB} = A_{COD}$, prove that the original quadrilateral is either a trapezoid or a parallelogram.

Skill Assessment Problem 13.3. In a convex quadrilateral, points are marked on the sides, dividing them into three equal parts, and the corresponding points on opposite sides are connected (see Figure 13.5). Prove that the area of the middle shaded quadrilateral is 9 times smaller than the area of the original quadrilateral.

Skill Assessment Problem 13.4. In triangle ABC, a line is drawn through vertex A and the midpoint K of median BM, intersecting side BC at point X. Find $\frac{A_{ABK}}{A_{KXMC}}$.

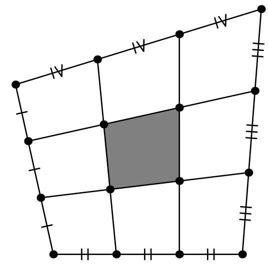


Figure 13.5: Areas when dividing a quadrilateral.

Solutions to Skill Assessment Problems

Solution to Problem 13.1: Assume that such a triangle exists. According to the formula for the area of a triangle, its sides are inversely proportional to the altitudes, i.e., the numbers 1, $\frac{1}{2}$, $\frac{1}{3}$. However, a triangle with these sides does not exist according to the triangle inequality. □

Solution to Problem 13.2: Notice that $A_{ABD} = A_{AOB} + A_{AOD} = A_{COD} + A_{AOD} = A_{ACD}$. The triangles ABD and ACD share the base AD, so they have equal altitudes dropped onto side AD. This implies that points B and C are equidistant from line AB. Since B and C are on the same side of line AB, we have $BC \parallel AD$. □

Solution to Problem 13.3: Prove that $C_2K = KL = LA_1$ and $C_1M = MN = NA_2$.

From the similarity of triangles BC_2D_1 and BAC, it follows that $AC = 3C_2D_1$ and $C_2D_1 \parallel AC$. From the similarity of triangles ACD and B_2A_1D, it follows that $AC = 1.5B_2A_1$ and $AC \parallel B_2A_1$. Hence, $C_2D_1 \parallel B_2A_1$ and $2C_2D_1 = B_2A_1$ (see Figure 13.6). Therefore, $\triangle C_2KD_1 \sim \triangle A_1KB_2$ with a coefficient of 0.5, which implies $KA_1 = 2C_2K \implies C_2K = C_2A_1/3$. Similarly, $LA_1 = C_2A_1/3$; $C_1M = A_2N = C_1A_2/3$.

Using a problem from the covered material: $A_{ABCD} = 3A_{C_1C_2A_1A_2}$ and $A_{C_1C_2A_1A_2} = 3A_{KLMN} \implies A_{ABCD} = 9A_{KLMN}$, as required. □

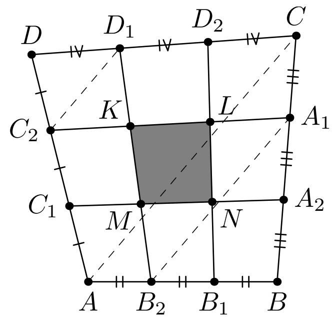


Figure 13.6: Additional construction.

Solution to Problem 13.4: Using the theorem of proportional segments in a triangle, we have $\frac{BK}{KM} = \frac{BX}{XC} \cdot \left(1 + \frac{CM}{MA}\right) \implies \frac{BX}{XC} = \frac{1}{2}$. Let $A_{ABC} = A$. Then $A_{BMC} = A/2$; $\frac{A_{BXK}}{A_{BMC}} = \frac{BK}{BM} \cdot \frac{BX}{BC} = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$, since these triangles have a common angle. Hence, $A_{KXMC} = \frac{5A}{6} \cdot \frac{1}{2} = \frac{5A}{12}$. $\frac{A_{ABK}}{A_{ABM}} = \frac{BK}{BM} = \frac{1}{2}$, as these triangles have the same height, implying $A_{ABK} = \frac{A}{4}$. Now, we can calculate the ratio of the areas:

$$\frac{A_{ABK}}{A_{KXMC}} = \frac{A}{4} \cdot \frac{12}{5A} = \frac{3}{5},$$

as required. □

Locus

> Let no one ignorant of geometry enter.

— Tradition has it that this phrase was engraved at the door of Plato's Academy

Theory and Practice

A locus of points refers to those, and only those, points that possess a specified property.

For example, the geometric set of points equidistant from point O at a distance r defines a circle with center at O and radius r, denoted as (O, r).

In the definition of locus, the words «those and only those» mean essentially the same as the expression «if and only if,» indicating the need to prove two statements: that all described points satisfy the given condition, and that no other points satisfy it. The locus of points equidistant from the sides of an angle is the angle bisector because:

1. Necessity: Any point equidistant from the sides of an angle lies on the bisector.

2. Sufficiency: Any point on the bisector is equidistant from the sides of the angle.

Therefore, the solution to a locus problem should include a proof that:

a) points that possess the required property belong to the figure F, which is the answer to the problem;

b) all points in figure F possess the required property.

If the locus has two properties, there is an intersection or common part of two loci, each possessing the first and second properties, respectively.

For example, the locus of points equidistant from O at a distance not exceeding a but not less than b ($a > b$) defines a ring bounded by the circles (O, a) and (O, b).

Let's recall a few more important loci:

The locus of points equidistant from the two ends of a given segment is the **midpoint perpendicular** to that segment.

The locus of points at a given distance from a given line is two parallel lines on opposite sides of it.

Typically, within the scope of school and Olympiad programs, loci are either lines, rays, segments, circles or arcs, areas bounded by given objects, or a combination of these objects.

Example 14.1. Given points A, B, and N, find the locus of points M such that $AM^2 - BM^2 = AN^2 - BN^2$.

Solution: Let's draw a line l through point N perpendicular to AB. We will prove that this line is the desired locus. Suppose it intersects the segment AB (or its extension) at point X.

a) Sufficiency b) Necessity

Figure 14.1: Problem of the difference of squared distances to the ends of the segment.

Sufficiency (see Figure 14.1a): we will prove that for any point M on this line, the equality $AM^2 - BM^2 = AN^2 - BN^2$ holds. Let's write the Pythagorean theorem for triangles AMX,BMX,ANX,BNX: $AM^2 = AX^2 + XM^2$; $BM^2 = BX^2 + XM^2$; $AN^2 = AX^2 + XN^2$; $BN^2 = BX^2 + XN^2$. Hence, $AM^2 - BM^2 = AX^2 + XM^2 - BX^2 - XM^2 = AX^2 - BX^2$. Also, $AN^2 - BN^2 = AN^2 + XN^2 - BN^2 - XN^2 = AX^2 - BX^2$.

Necessity: we will prove that any point M for which the equality $AM^2 - BM^2 = AN^2 - BN^2$ holds lies on the line l. Suppose this is not the case (see Figure 14.1b). Draw a line through point M perpendicular to AB. Let this line intersect line AB at point Y, which is different from point X. Then, by the Pythagorean theorem for triangles AMY and BMY, $AM^2 - BM^2 = AY^2 - BY^2$. Similarly, $AN^2 - BN^2 = AX^2 - BX^2$. Hence, $AX^2 - BX^2 = AY^2 - BY^2$. Let's introduce a numerical axis on line AB. Assume point A has a coordinate of 0, B has a coordinate of b, X has a coordinate of x, and Y has a coordinate of y. Then $AX^2 - BX^2 = x^2 - (b-x)^2 = 2bx - b^2$; $AY^2 - BY^2 = y^2 - (b-y)^2 = 2by - b^2$. From the equality $2bx - b^2 = 2by - b^2$, we conclude that $x = y$, which means points X and Y coincide. This contradiction completes the proof. □

Example 14.2. A kitten is sleeping in the middle of a ladder that is leaning against a wall. What trajectory will the kitten follow when the ladder starts to slide on the floor?

Solution: Finding the trajectory of the ladder means finding the locus of midpoints of segments of a given length, the ends of which lie on two given rays (walls). Let the length of the ladder be $2l$. We will prove that the desired locus will be a quarter circle with center O and radius l.

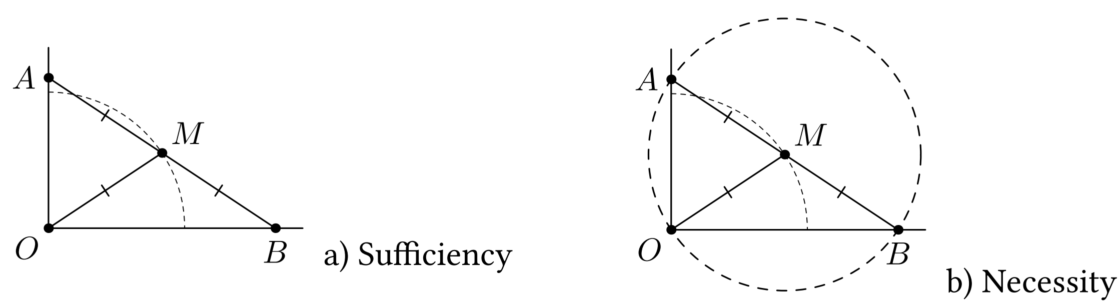


Figure 14.2: The problem of the kitten on the ladder.

Sufficiency (see Figure 14.2a): we will prove that the midpoint M of any position of the ladder AB will be on the circle. Indeed, in the right triangle AOB, the median OM is equal to half the hypotenuse, i.e., $OM = l$.

Necessity (see Figure 14.2b): any point M lying on a quarter circle will be the midpoint of some position of the ladder; let's draw a circle with center M and radius l. Suppose it intersects the rays (walls) at points A and B. Then $AB = 2l$, so M is indeed the midpoint of some position of the ladder. □

Problem Set

Problem 14.1. (MF — 1990.7.2): Draw a set of midpoints of all segments whose endpoints lie on a) a given semicircle and b) the diagonals of a given square.

Problem 14.2. (MMO — 1965.8.3): Given a line a and two non-parallel segments AB and CD on the same side of the line. Find a point M on line a such that triangles ABM and CDM have equal areas.

Problem 14.3. (MMG — 2012.8.4): Consider all triangles ABC with fixed positions for vertices B and C, while vertex A moves within the plane of the triangle so that median CM remains a constant length. Determine the trajectory of point A.

Problem 14.4. (Gordin — 2411): Using a compass and straightedge, draw a tangent to a given circle such that the given line would cut off on this tangent line a given segment.

Problem 14.5. (Gordin — 1769): Given points A and B. Circles with centers at B and radii not exceeding AB are drawn, and tangents are drawn through point A to these circles. Find the locus of points of tangency.

Problem 14.6. (Gordin — 2399): Given a circle, find the locus of points from which tangents are drawn to the circle, and the lengths of these tangents are equal to a given segment.

Problem 14.7. (Gordin — 2446): Given non-parallel sides AB and CD of a convex quadrilateral $ABCD$ with area A. Find the locus of points X lying inside the quadrilateral for which $A_{ABX} + A_{CDX} = \frac{A}{2}$.

Problem 14.8. (Gordin — 2491): Through the intersection point of two circles, draw a line that intersects the circles again at points A and B. Find the locus of midpoints of segment AB.

Problem 14.9. (AMC — 2019.10A.6): For how many of the following types of quadrilaterals does there exist a point in the plane of the quadrilateral that is equidistant from all four vertices of the quadrilateral?

- a square
- a rectangle that is not a square
- a rhombus that is not a square
- a parallelogram that is not a rectangle or a rhombus
- an isosceles trapezoid that is not a parallelogram

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 14.10. (AMC — 2018.10B.12): Line segment $\overline{AB}$ is the diameter of a circle with $AB = 24$. Point C, not congruent with A or B, lies on the circle. As point C moves around the circle, the centroid (center of mass) of $\triangle ABC$ traces out a closed curve missing two points. To the nearest positive integer, what is the area of the region bounded by this curve?

(A) 25 (B) 38 (C) 50 (D) 63 (E) 75

Problem 14.11. (AMC — 2011.10B.18): Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on the side AB so that $\angle AMD = \angle CMD$. What is the degree measure of $\angle AMD$?

(A) 15 (B) 30 (C) 45 (D) 60 (E) 75

Problem 14.12. (AMC − 2011.10B.20): Rhombus $ABCD$ has side length 2 and $\angle B = 120°$. Region R consists of all points inside the rhombus that are closer to vertex B than any of the other three vertices. What is the area of R?

(A) $\frac{\sqrt{3}}{3}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{2\sqrt{3}}{3}$ (D) $1+\frac{\sqrt{3}}{3}$ (E) 2

Problem 14.13. (UCT Mathematics Competition): Find the locus of all points P for which the sum of the distances from P to two given intersecting lines is a given constant.

Skill Assessment Problems

Skill Assessment Problem 14.1. Find the locus of centers of circles that are tangent to a given circle at a given point.

Skill Assessment Problem 14.2. In a circle inscribed in triangle ABC, point P moves along the arc ACB. Determine the locus of centers of the inscribed circles of all possible triangles ABP.

Solutions to Skill Assessment Problems

Solution to Problem 14.1: Let's denote the given circle as ω, and let its center be point O. Denote the given point as A (see Figure 14.3). Draw a line l through points A and O. We will prove that the locus of the centers of circles tangent to ω at point A will be the line l excluding point O.

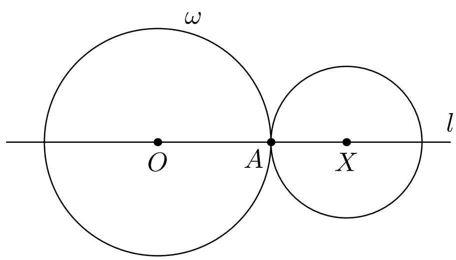


Figure 14.3: Illustration for the problem about the locus of centers of circles.

1) Prove that the center of any circle tangent to ω at point A will lie on the line l. This is indeed true because the centers of tangent circles and their points of tangency always lie on the same line.

2) Prove that any point X lying on line l, excluding point O, will be the center of some circle tangent to ω at point A. Indeed, the circle with radius AX will be the desired circle. If A is to the right of point O, then the circles touch externally, and if to the left, then internally. □

Solution to Problem 14.2: Let us remember one interesting geometric fact, the so-called *trillium theorem.*

Let I be the center of the inscribed circle of triangle KLM, inscribed in circle ω. Let T be the midpoint of the arc KL. Then $TI = TK = TL$ (see Figure 14.4).

The reverse statement is also true: let I lie on the bisector of angle KML of triangle KLM inscribed in circle ω. Then, if $TI = TK = TL$, then I is the center of the

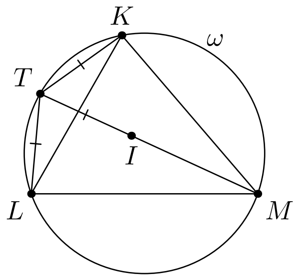


Figure 14.4: Trillium theorem.

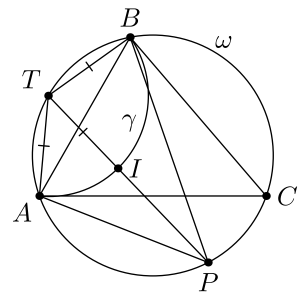


Figure 14.5: Locus is an arc γ.

inscribed circle of triangle KLM.

Now, let's get back to the problem. We will prove that the locus of points is the arc γ with the center at the midpoint T of arc AB, starting at point A and ending at point B (see Figure 14.5).

1) Consider an arbitrary point P on the arc ACB and prove that the center of the inscribed circle of triangle ABP will lie on γ. Indeed, using the trillium theorem, we get the desired result.

2) Consider an arbitrary point I on arc γ and prove that it is the center of the inscribed circle of some triangle ABP. Extend TI until it intersects arc ACB at point P. Since arcs AT and BT are equal, point I lies on the bisector of angle APB, and from the reverse trillium theorem, we get the desired result. □

Geometry of Masses

> When talking to a mathematician, you may not have any knowledge of mathematics. But at the same time, a sense of humor and an awareness of your insignificance are absolutely necessary…
>
> —K. Dzevanovskiy

Theory and Practice

One of the less popular but quite interesting topics in Olympiad mathematics is the so-called «geometry of masses». This method helps solve certain geometric problems by transforming them into a «physical» form through the introduction of «weights» for certain vertices or edges. The most common type of such problem involves finding certain ratios, which are determined by finding the center of mass of a particular system. Let's recall the concept of the center of mass. Consider two small balls with masses m_1 and m_2, connected by a rigid «weightless» rod. There is a point on this rod such that if the entire system is suspended at this point, it will be in equilibrium – none of the balls will «pull down» the other. This point is the center of mass of the two considered material points m_1, m_2. Similarly, the concept of the center of mass, or barycenter, can be introduced for a system of N points.

Let's recall some fundamental facts from mechanics:

1. Any system consisting of a finite number of material points has a center of mass, and it is unique.

2. The center of mass of two material points is located on the segment connecting these points; its position is determined by the law of the lever (Figure 15.1): the product of the mass of a material point by the distance from it to the center of mass is the same for both points, i.e., $m_1 \cdot d_1 = m_2 \cdot d_2$.

3. If, in a system consisting of a finite number of material points, several points are marked, and the masses of all marked points are moved to their center of mass, then this does not change the position of the center of mass of the entire system.

How can this concept help us solve geometry problems? Let's solve, for example, the following problem:

Example 15.1. Through the midpoint of median AA_1 and the vertex B of triangle ABC, a line is drawn. In what ratio does it divide side AC (Figure 15.2)?

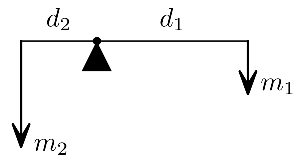


Figure 15.1: Law of the lever.

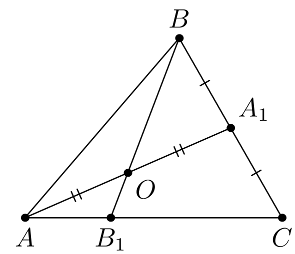


Figure 15.2: Illustration for the problem of segment ratios.

Solution: We could solve this problem, for example, using the theorem on proportional segments in a triangle (a consequence of Menelaus's theorem). However, not everyone remembers Menelaus' theorem, and the calculation in this problem becomes quite tedious. We will take a different approach. If we have midpoints of segments, we need to place equal masses at the ends of the specified segments. Let's place 1 gram at points B and C. Then, at point A_1, a mass of 2 grams is concentrated. We consider the midpoint of segment AA_1, so we place a mass of 2 grams at point A. Then, the barycenter of the triangle is point O. On the other hand, this barycenter is located at the intersection of segments BB_1 and AC, i.e., B_1 is the center of mass of segment AC. At point A, a mass of 2 grams is concentrated, and at the center C, a mass of 1 gram is concentrated. Therefore, the required ratio is $1 : 2$. □

Example 15.2. In triangle ABC, point F divides the base BC in the ratio $3 : 1$, counting from vertex B. Points M and P cut off one-sixth from the lateral sides AB and AC, respectively, counting from vertex A and vertex C. In what ratio are each of the segments MP and AF divided by their point of intersection (Figure 15.3)?

Solution: Place masses in points B and C such that their mass center is point F. By the lever law, we can place mass 1 at B and mass 3 at C. Next, find mass x at A such that point M becomes the center of mass for points B and A. According to the lever law, $1 \cdot BM = x \cdot MA$, so $x = 5$. Finally, find mass y at A such that point P becomes

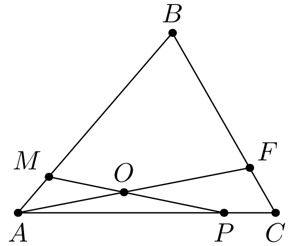


Figure 15.3: Intersecting secants in a triangle.

the center of mass for points C and A. According to the lever law, $3 \cdot CP = y \cdot PA$, so $y = 0.6$.

Now we have a new situation: at point A, we have two different masses, 5 and 0.6. Consider a system of all four masses. Denote its center of mass as Z. Move the masses of points B and A to their center of mass M, and the masses of points C and A to their center of mass P. Then, Z is the center of mass for points M and P. Alternatively, group the same four masses differently: move the masses of points B and C to their center of mass F, and consider the two masses at 5 and 0.6 from point A as one mass 5.6. Then Z will be the center of mass for points F and A. Therefore, Z lies on segment MP. As a result, Z is the point of intersection of segments MP and AF, and it coincides with the point O. Since Z is the center of mass for the system of points A and F, by the lever law, $5.6 \cdot AZ = 4 \cdot FZ$, so $AZ : ZF = 5 : 7$. Similarly, $6 \cdot MZ = 3.6 \cdot PZ$, so $MZ : ZP = 3 : 5$. □

Using the mass method, you can prove known facts quite easily.

Example 15.3. Prove that the medians of triangle ABC intersect at one point and divide it in the ratio $2 : 1$, counting from the vertex.

Proof. Place unit masses at points A, B, and C. Let O be the center of mass of this system of points. Point O is also the center of mass of point A with mass 1 and point A_1 with mass 2, where A_1 is the center of mass of points B and C with unit masses, i.e., A_1 is the midpoint of segment BC. Therefore, point O lies on median AA_1 and divides it in the ratio $AO : OA_1 = 2 : 1$. Similarly, it can be proved that the other medians pass through point O and divide in the ratio $2 : 1$.

□

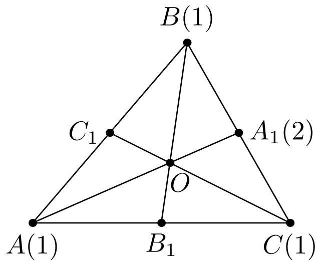


Figure 15.4: Property of medians in a triangle.

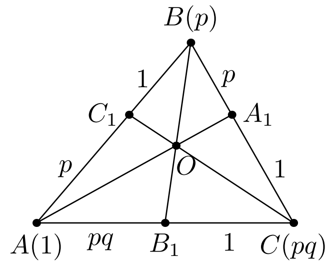


Figure 15.5: Ceva's theorem.

Example 15.4. Prove Ceva's theorem using mass grouping.

Proof. Let lines AA_1 and CC_1 intersect at point O; $AC_1 : C_1B = p$ and $BA_1 : A_1C = q$. We need to prove that line BB_1 passes through the point O if and only if $CB_1 : B_1A = 1 : pq$.

Place masses 1, p, and pq at points A, B, and C, respectively. Then point C_1 is the center of mass of points A and B, and point A_1 is the center of mass of points B and C. Therefore, the center of mass of points A, B, and C with these masses is point O at the intersection of lines CC_1 and AA_1. On the other hand, point O lies on the segment connecting point B with the center of mass of points A and C. If B_1 is the center of mass of points A and C with masses 1 and pq, then $AB_1 : B_1C = pq : 1$. It remains to be noticed that there is a unique point in segment AC that divides it in the given ratio $AB_1 : B_1C$.

□

Problem Set

Problem 15.1. (Prasolov – 14.005): Let $ABCD$ be a convex quadrilateral, and K, L, M, and N be the midpoints of sides AB, BC, CD, and DA respectively. Prove that the intersection point of segments KM and LN is the midpoint of both segments, and the midpoint of the segment joins the midpoints of the diagonals.

Problem 15.2. (Prasolov – 14.006): Let $A_1, B_1, \ldots, F_1$ be the midpoints of sides $AB, BC, \ldots, FA$ of an arbitrary hexagon. Prove that the centroids of triangles $A_1C_1E_1$ and $B_1D_1F_1$ coincide.

Problem 15.3. (MMO – 1956.9.1): In a convex quadrilateral $ABCD$, consider the quadrilateral $KLMN$ formed by the centroids of triangles ABC, BCD, CDA, and DAB. Prove that the lines connecting the midpoints of opposite sides of quadrilateral $ABCD$ intersect at the same point as the lines connecting the midpoints of opposite sides of quadrilateral $KLMN$.

Problem 15.4. (AIME – 2003.I.15): In $\triangle ABC, AB = 360, BC = 507$, and $CA = 780$. Let M be the midpoint of $\overline{CA}$, and let D be the point on $\overline{CA}$ such that $\overline{BD}$ bisects angle ABC. Let F be the point on $\overline{BC}$ such that $\overline{DF} \perp \overline{BD}$. Suppose that $\overline{DF}$ meets $\overline{BM}$ at E. The ratio $DE : EF$ can be written in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

Problem 15.5. (AIME – 2002.II.13): In triangle ABC, point D is on $\overline{BC}$ with $CD = 2$ and $DB = 5$, point E is on $\overline{AC}$ with $CE = 1$ and $EA = 3$, $AB = 8$, and $\overline{AD}$ and $\overline{BE}$ intersect at P. Points Q and R lie on $\overline{AB}$ so that $\overline{PQ}$ is parallel to $\overline{CA}$ and $\overline{PR}$ is parallel to $\overline{CB}$. It is given that the ratio of the area of triangle PQR to the area of triangle ABC is m/n, where m and n are relatively prime positive integers. Find $m + n$.

Problem 15.6. (AIME – 2013.II.13): In $\triangle ABC$, $AC = BC$, and point D is on $\overline{BC}$ so that $CD = 3 \cdot BD$. Let E be the midpoint of $\overline{AD}$. Given that $CE = \sqrt{7}$ and $BE = 3$, the area of $\triangle ABC$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Skill Assessment Problems

Skill Assessment Problem 15.1. Given six points, no three of which are collinear. Consider all possible partitions of these points into groups of three. Connect the intersection points of the medians of a triangle formed by the points in the first group with the intersection points of the medians of a triangle formed by the points in the other three groups. Prove that the resulting ten segments intersect at a single point.

Skill Assessment Problem 15.2. Prove that the midlines of a quadrilateral intersect at their midpoints.

Solutions to Skill Assessment Problems

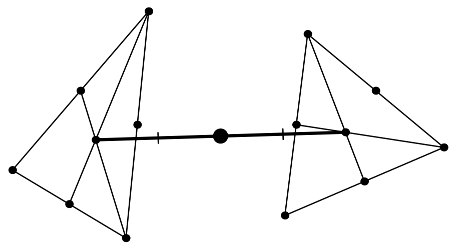


Figure 15.6: Illustration for the problem.

Solution to Problem 15.1: Place a mass of 1 at each of the given points. Then, the center of mass of each of the triangles will be at the intersection of its medians, and a mass of 3 will be concentrated there. For each partition into two triangles with the ends of the considered segments, the barycenters of the triangles will be the points of consideration. Therefore, the barycenter of the entire system will be the midpoint of the considered segment (see Figure 15.6). However, the barycenter for the system is unique. This implies that all considered segments intersect at one point, and even a stronger condition is satisfied than required: the intersection point divides the given segments in half. □

Solution to Problem 15.2: Place a mass of 1 at each vertex of the quadrilateral. Then, the barycenter of systems from points A and B, as well as points C and D, will be the midpoints of segments AB and CD – that is, points M and N – and masses of 2 will be concentrated there. Therefore, the barycenter of the system from points A, B, C, D lies on the segment MN, and, due to the equality of the concentrated masses at points M and N, it is the midpoint of this segment. Similarly, we prove that the barycenter of the system from points A, B, C, D lies at the midpoint of the segment PQ. The uniqueness of the center of mass for the system implies that the barycenter lies at the intersection of segments MN and PQ, dividing them in half, as required. □

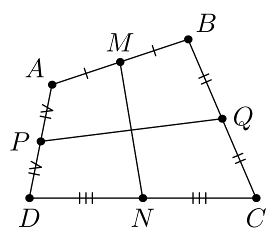


Figure 15.7: Illustration for the problem.

Coordinate Method

> There is no royal road to geometry.
>
> —Attributed to Euclid by Proclus

Theory and Practice

In conclusion, in our book dedicated to geometry, we will present a purely algebraic approach to geometry.

Any (literally any) geometry problem can be reduced to algebra by introducing a coordinate system. However, solving a problem using the coordinate method can be much longer than a geometric solution.

Let's recall some key facts that will help you in this «algebraization.»

The equation of a **line with a given slope** $y = kx + b$, where $k = \tan\alpha$ (α is the angle between the x-axis and the line, measured counterclockwise).

If the lines have $k_1 = k_2$ and $b_1 \neq b_2$, then the lines are parallel.

If the lines have $k_1 = k_2$ and $b_1 = b_2$, then the lines coincide.

If the lines have $k_1 \neq k_2$, then the lines intersect.

If the lines have $k_1 \cdot k_2 = -1$, then the lines are perpendicular.

The equation of a **line passing through** 2 **points** (x_1, y_1) and (x_2, y_2) is given by: $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}$.

The equation of a **circle** with center at $(x_0; y_0)$ and radius R is: $(x-x_0)^2+(y-y_0)^2 = R^2$.

Let M be the **midpoint** of the segment AB. Its coordinates are found using the formulas:

$$x_M = \frac{x_A + x_B}{2}; \ \ y_M = \frac{y_A + y_B}{2}.$$

The coordinates of point $M(x,y)$, which divides the segment M_1M_2 between points $M_1(x_1,y_1)$ and $M_2(x_2,y_2)$ in the ratio $M_1M : MM_2 = \lambda$, are determined by the formulas:

$$x = \frac{x_1 + \lambda x_2}{1+\lambda}; \quad y = \frac{y_1 + \lambda y_2}{1+\lambda}.$$

The **distance** between points $A(x_1,y_1)$ and $B(x_2,y_2)$ is determined by the formula:

$$\rho(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Let's consider an example of a problem that can be solved using coordinates (although the geometric solution provided by the authors is much more elegant).

Example 16.1. (Sharygin — 2019.8.3): Inside the circle, there is a rectangle $ABCD$. Rays BA and DA intersect the circle at points A_1 and A_2. Point A_0 is the midpoint of chord A_1A_2. Similarly, points B_0, C_0, D_0 are defined. Prove that the segments A_0C_0 and B_0D_0 are equal.

Author's solution: Let X and Y be the projections of the center of the circle onto the lines AB and CD, respectively (see Figure 16.1). Then $BB_1 - AA_1 = (XB_1 - XB) - (XA_1 - XA) = AX - BX = DY - CY = CC_1 - DD_1$. Hence, the projection of the segment A_0C_0 onto the line AB, equal to $\frac{A_1B_1 + C_1D_1 - AA_1 - CC_1}{2}$, is equal to the projection of the segment B_0D_0 onto the same line. Similarly, the projections of these segments onto the line AD are equal, and therefore the segments themselves are equal.

□

Now let's turn off our brains. Let's draw Figure 16.2 for this problem.

Assume that the given circle has the equation $x^2 + y^2 = 1$.

Let point A have coordinates $(a;b)$, and point C have coordinates $(c;d)$, with coordinate axes parallel to the sides of the rectangle. Then point B has coordinates (a,d),

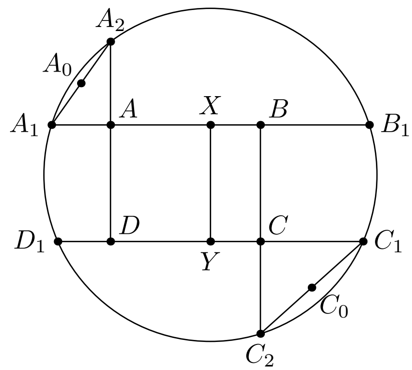


Figure 16.1: Projections in the problem.

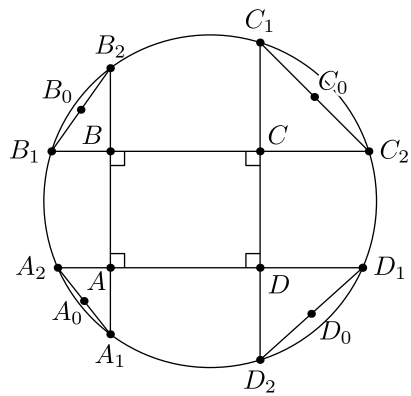


Figure 16.2: Coordinate method.

and point D has coordinates (c,b).

Then, the coordinate x of the point A_1 is equal to a. From this, the y coordinate of this point is $-\sqrt{1-a^2}$. Similarly, the coordinates of the other points on the circle are determined (see Figure 16.3).

Thus, the point A_0, as the midpoint of the segment A_1A_2, has coordinates $(a-\sqrt{1-b^2}; b-\sqrt{1-a^2})$, the point B_0 has coordinates $(a-\sqrt{1-d^2}; d+\sqrt{1-a^2})$, the point C_0 has coordinates $(c+\sqrt{1-d^2}; d+\sqrt{1-c^2})$, and the point D_0 has coordinates $(c+\sqrt{1-b^2}; b-\sqrt{1-c^2})$.

The square of the length of the segment A_0C_0 will be $(a-\sqrt{1-b^2}-(c+\sqrt{1-d^2}))^2+(b-\sqrt{1-a^2}-(d+\sqrt{1-c^2}))^2$, and the square of the length of the segment B_0D_0

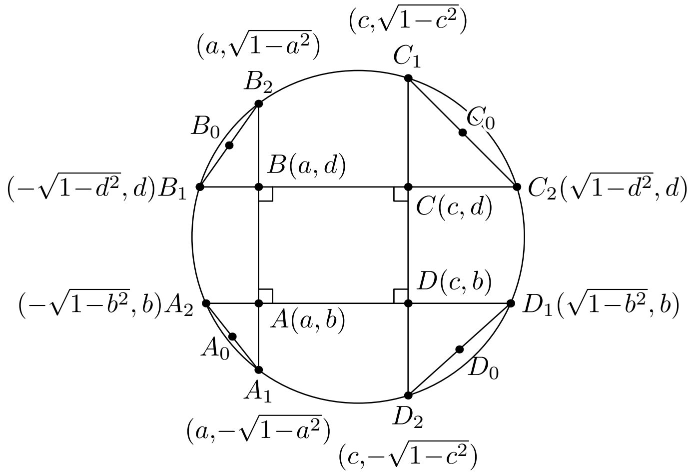


Figure 16.3: Coordinate method: assigned coordinates.

will be $((a-\sqrt{1-d^2}-(c+\sqrt{1-b^2}))^2+(d+\sqrt{1-a^2}-(b-\sqrt{1-c^2}))^2$.

To avoid writing horrifying calculations, let's make temporary substitutions: $\sqrt{1-a^2}=x$, $\sqrt{1-b^2}=y$, $\sqrt{1-c^2}=z$, $\sqrt{1-d^2}=t$.

Let's compare the obtained values.

$(a-y-(c+t))^2+(b-x-(d+z))^2 \bigvee ((a-t-(c+y))^2+(d+x-(b-z))^2$.

Expand the brackets and make additional substitutions: $a-y=\alpha$, $c+t=\beta$, $b-d=\gamma$, $x+z=\delta$:

$$(\alpha-\beta)^2+(\gamma-\delta)^2 \bigvee (\alpha-\beta)^2+(\delta-\gamma)^2,$$

where, obviously, equality is achieved. Thus, the required segments are equal, which was to be proved. □

Of course, geometrical solutions are often much more pleasant than coordinate ones. However, if you have already been breaking your brain for a sufficient amount of time, sometimes it's worth simply turning off your brain and starting to calculate. However, you should be very careful and attentive.

We don't offer many problems in the problem set (as we mentioned, any problem can be calculated in coordinates). So, you can now practice any of the previously suggested problems or any problem that appeals to you.

Problem Set

Problem 16.1. (TOT — 1987.4): The length of the altitude AB of a rectangular trapezoid $ABCD$ is equal to the sum of the lengths of the bases AD and BC. In what ratio does the angle bisector at B divide the side CD?

Problem 16.2. (Gordin — 8098): Prove that the sums of the squares of the distances from an arbitrary point in space to the opposite vertices of a rectangle are equal.

Problem 16.3. (AMC — 2020.10A.16): A point is chosen at random within the square in the coordinate plane whose vertices are $(0,0)$, $(2020,0)$, $(2020,2020)$, and $(0,2020)$. The probability that the point is within d units of a lattice point is $\frac{1}{2}$. (A point (x,y) is a lattice point if x and y are both integers.) What is d to the nearest tenth?

(A) 0.3 (B) 0.4 (C) 0.5 (D) 0.6 (E) 0.7

Problem 16.4. (AMC — 2019.10A.7): Two lines with slopes $\frac{1}{2}$ and 2 intersect at (2,2). What is the area of the triangle enclosed by these two lines and the line $x+y=10$?

(A) 4 (B) $4\sqrt{2}$ (C) 6 (D) 8 (E) $6\sqrt{2}$

Problem 16.5. (AMC — 2019.10B.5): Triangle ABC lies in the first quadrant. Points A, B, and C are reflected across the line $y=x$ to points A', B', and C', respectively. Assume that none of the vertices of the triangle lie on the line $y=x$. Which of the following statements is not always true?

(A) Triangle $A'B'C'$ lies in the first quadrant.

(B) Triangles ABC and $A'B'C'$ have the same area.

(C) The slope of line AA' is -1.

(D) The slopes of lines AA' and CC' are the same.

(E) Lines AB and $A'B'$ are perpendicular to each other

Problem 16.6. (AMC — 2019.10B.23): Points $A(6{,}13)$ and $B(12{,}11)$ lie on circle ω in the plane. Suppose that the tangent lines to ω at A and B intersect at a point on the x-axis. What is the area of ω?

(A) $\frac{83\pi}{8}$ (B) $\frac{21\pi}{2}$ (C) $\frac{85\pi}{8}$ (D) $\frac{43\pi}{4}$ (E) $\frac{87\pi}{8}$

Problem 16.7. (AMC — 2018.10B.17): In rectangle $PQRS$, $PQ = 8$ and $QR = 6$. Points A and B lie on $\overline{PQ}$, points C and D lie on $\overline{QR}$, points E and F lie on $\overline{RS}$, and points G and H lie on $\overline{SP}$ so that $AP = BQ < 4$ and the convex octagon $ABCDEFGH$ is equilateral. The length of a side of this octagon can be expressed in the form $k + m\sqrt{n}$, where k, m, and n are integers and n is not divisible by the square of any prime. What is $k + m + n$?

(A) 1 (B) 7 (C) 21 (D) 92 (E) 106

Problem 16.8. (AMC — 2017.10B.8): Points $A(11, 9)$ and $B(2, -3)$ are vertices of $\triangle ABC$ with $AB = AC$. The altitude from A meets the opposite side at $D(-1, 3)$. What are the coordinates of point C?

(A) $(-8, 9)$ (B) $(-4, 8)$ (C)$(-4, 9)$ (D) $(-2, 3)$ (E) $(-1, 0)$

Problem 16.9. (AMC — 2017.10B.24): The vertices of an equilateral triangle lie on the hyperbola $xy = 1$, and a vertex of this hyperbola is the centroid of the triangle. What is the square of the area of the triangle?

(A) 48 (B) 60 (C) 108 (D) 120 (E) 169

Problem 16.10. (AMC — 2016.10A.16): A triangle with vertices $A(0,2)$, $B(-3,2)$, and $C(-3,0)$ is reflected about the x-axis, then the image $\triangle A'B'C'$ is rotated counterclockwise about the origin by 90° to produce $\triangle A''B''C''$. Which of the following transformations will return $\triangle A''B''C''$ to $\triangle ABC$?

- (A) counterclockwise rotation about the origin by 90°.
- (B) clockwise rotation about the origin by 90°.
- (C) reflection about the x-axis
- (D) reflection about the line $y = x$
- (E) reflection about the y-axis.

Problem 16.11. (AMC — 2016.10B.9): All three vertices of $\triangle ABC$ are lying on the parabola defined by $y = x^2$, with A at the origin and $\overline{BC}$ parallel to the x-axis. The area of the triangle is 64. What is the length of BC?

(A) 4 (B) 6 (C) 8 (D) 10 (E) 16

Problem 16.12. (AMC — 2016.10B.20) A dilation of the plane — that is, a size transformation with a positive scale factor — sends the circle of radius 2 centered at A(2,2) to the circle of radius 3 centered at A(5,6). What distance does the origin O(0,0), move under this transformation?

(A) 0 (B) 3 (C) $\sqrt{13}$ (D) 4 (E) 5

Problem 16.13. (AMC — 2016.10B.21): What is the area of the region enclosed by the graph of the equation $x^2 + y^2 = |x| + |y|$?

(A) $\pi + \sqrt{2}$ (B) $\pi + 2$ (C) $\pi + 2\sqrt{2}$ (D) $2\pi + \sqrt{2}$ (E) $2\pi + 2\sqrt{2}$

Problem 16.14. (AMC – 2015.10A.17): A line that passes through the origin intersects both the line $x = 1$ and the line $y = 1 + \frac{\sqrt{3}}{3}x$. The three lines create an equilateral triangle. What is the perimeter of the triangle?

(A) $2\sqrt{6}$ (B) $2 + 2\sqrt{3}$ (C) 6 (D) $3 + 2\sqrt{3}$ (E) $6 + \frac{\sqrt{3}}{3}$

Problem 16.15. (AMC – 2015.10B.13): The line $12x + 5y = 60$ forms a triangle with the coordinate axes. What is the sum of the lengths of the altitudes of this triangle?

(A) 20 (B) $\frac{360}{17}$ (C) $\frac{107}{5}$ (D) $\frac{43}{2}$ (E) $\frac{281}{13}$

Problem 16.16. (AMC – 2014.10A.14): The y-intercepts, P and Q, of two perpendicular lines intersecting at the point $A(6,8)$ have a sum of zero. What is the area of $\triangle APQ$?

(A) 45 (B) 48 (C) 54 (D) 60 (E) 72

Problem 16.17. (AMC – 2014.10A.18): A square in the coordinate plane has vertices whose y-coordinates are 0, 1, 4, and 5. What is the area of the square?

(A) 16 (B) 17 (C) 25 (D) 26 (E) 27

Problem 16.18. (AMC – 2013.10A.16): A triangle with vertices $(6, 5)$, $(8, -3)$, and $(9, 1)$ is reflected about the line $x = 8$ to create a second triangle. What is the area of the union of the two triangles?

(A) 9 (B) $\frac{28}{3}$ (C) 10 (D) $\frac{31}{3}$ (E) $\frac{32}{3}$

Problem 16.19. (AMC – 2013.10A.18): Let points $A = (0, 0)$, $B = (1, 2)$, $C = (3, 3)$, and $D = (4, 0)$. Quadrilateral $ABCD$ is cut into equal area pieces by a line passing through A. This line intersects $\overline{CD}$ at point $\left(\frac{p}{q}, \frac{r}{s}\right)$, where these fractions are in lowest terms. What is $p + q + r + s$?

(A) 54 (B) 58 (C) 62 (D) 70 (E) 75

Problem 16.20. (AMC — 2012.10B.3): The point in the xy-plane with coordinates $(1000, 2012)$ is reflected across the line $y = 2000$. What are the coordinates of the reflected point?

(A) (998,2012) (B) (1000,1988) (C) (1000,2024)
(D) (1000,4012) (E) (1012,2012)

Problem 16.21. (AMC — 2011.10A.9): A rectangular region is bounded by the graphs of the equations $y = a, y = -b, x = -c$, and $x = d$, where a,b,c, and d are all positive numbers. Which of the following represents the area of this region?

(A) $ac+ad+bc+bd$ (B) $ac-ad+bc-bd$ (C) $ac+ad-bc-bd$ (D) $-ac-ad+bc+bd$ (E) $ac-ad-bc+bd$

Problem 16.22. (HMMT): Trapezoid $ABCD$ is inscribed in the parabola $y = x^2$ such that $A = (a, a^2)$, $B = (b, b^2)$, $C = (-b, b^2)$, and $D = (-a, a^2)$ for some positive reals a, b with $a > b$. If $AD + BC = AB + CD$, and $AB = 0.75$, what is a?

Skill Assessment Problems

Skill Assessment Problem 16.1. On the line given by the equation $3x-4y+34 = 0$, lies point M. Point N lies on the circle defined by the equation $x^2+y^2-8x+2y-8 = 0$. Find the minimum distance between points M and N.

Skill Assessment Problem 16.2. M is the centroid of triangle ABC. Prove that the coordinates of point M are the arithmetic means of the corresponding coordinates of the vertices of the triangle.

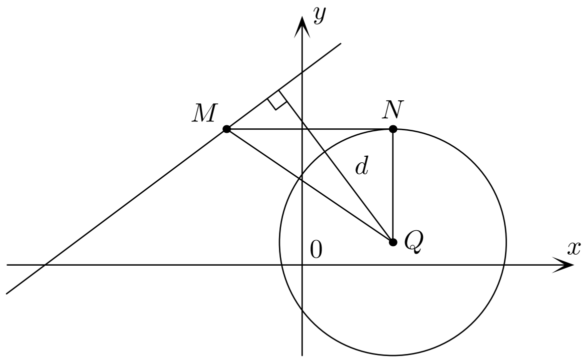


Figure 16.4: Coordinate sketch

Solutions to Skill Assessment Problems

Solution to Problem 16.1: Notice that

$$x^2 + y^2 - 8x + 2y - 8 = 0 \Leftrightarrow x^2 - 8x + 16 + y^2 + 2y + 1 = 25 \Leftrightarrow$$

$$(x-4)^2 + (y+1)^2 = 5^2.$$

This implies that the center of the circle is the point $Q(4;-1)$, and the radius is 5 (see Figure 16.4).

Let d be the distance from the point Q to the line $3x - 4y + 34 = 0$. Then,

$$d = \frac{|3 \cdot 4 - 4 \cdot (-1) + 34|}{\sqrt{3^2 + 4^2}} = \frac{50}{5} = 10 > 5.$$

This means that all points of the given line are outside the given circle. Therefore, for each point M on the line and each point N on the circle,

$$MN > MQ - QN = MQ - 5 > d - 5 = 10 - 5 = 5,$$

and this distance is equal to 5 if M is the projection of point Q onto the given line.

□

Solution to Problem 16.2: Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ be the vertices of the triangle, and $M(x_0, y_0)$ be the intersection point of its medians. It is known that the medians of a triangle intersect at a point, dividing them in a ratio of $2 : 1$, counting from the vertex. Therefore, if

$$D(x_4; y_4) = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$$

is the midpoint of segment BC, then $AM : MD = 2 : 1$. Since the point $M(x_0, y_0)$ divides the segment with endpoints at $A(x_1, y_1)$ and $D(x_4, y_4)$ in the ratio of $2 : 1$, counting from point A, by the theorem on proportional segments, the projection of point M onto the OX axis divides the projection of segment AD onto this axis in the same ratio, that is

$$\frac{x_0 - x_1}{x_4 - x_0} = 2.$$

From this, we get

$$x_0 = \frac{x_1 + 2x_4}{3} = \frac{x_1 + x_2 + x_3}{3}.$$

Similarly,

$$y_0 = \frac{y_1 + y_2 + y_3}{3}.$$

□

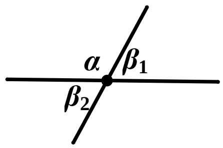
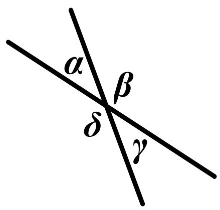
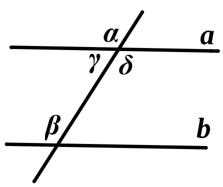
Important Theory

17.1 Notations

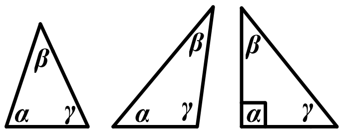
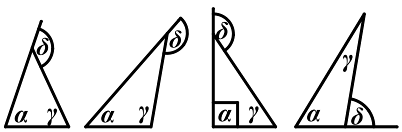
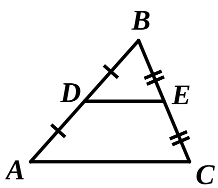
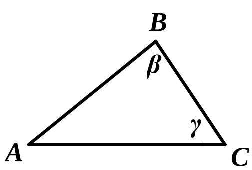
For the triangle ABC, we will use the following notations:

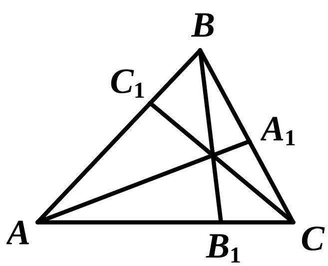
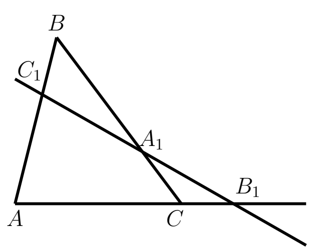
- $a = BC$, $b = AC$, $c = AB$ — lengths of the sides or just the sides;
- $\alpha = \angle CAB$, $\beta = \angle ABC$, $\gamma = \angle BCA$ — angle measures;
- h_a,h_b,h_c — altitudes dropped from vertices A, B, C to sides a, b, c (or their extensions);
- m_a,m_b,m_c — medians drawn from vertices A,B,C to sides a,b,c;
- l_a,l_b,l_c — angle bisectors drawn from vertices A,B,C to sides a,b,c;
- r — radius of the inscribed circle;
- I — point of intersection of bisectors;
- R and O — radius and center, respectively, of the circumscribed circle;
- M — point of intersection of medians (centroid, center of mass);
- H — point of intersection of altitudes or their extensions (orthocenter or external orthocenter);
- p — semiperimeter of the triangle $\left(p = \frac{a+b+c}{2}\right)$;
- $p_a = p - a,\ p_b = p - b,\ p_c = p - c$;
- $A(A_{\triangle ABC})$ — area of the triangle.

17.2 Properties of Angles and Parallel Lines

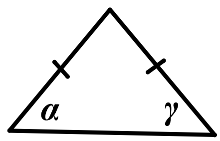
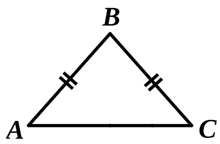
	$\alpha + \beta_1 = 180°$ $\alpha + \beta_2 = 180°$
	$\alpha = \gamma$ $\beta = \delta$
	if $a \parallel b$, then $\alpha = \beta$ $\beta = \delta$ $\beta + \gamma = 180°$ inverse: from any of these equalities $\Rightarrow a \parallel b$

17.3 Properties of an Arbitrary Triangle

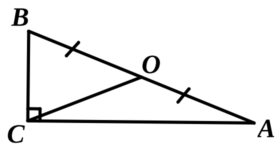
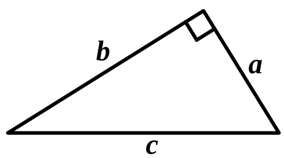
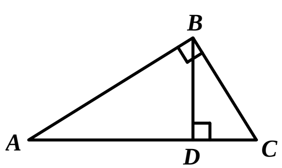
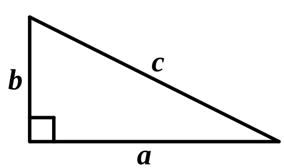
	$\alpha + \beta + \gamma = 180^\circ$
	external angle of a triangle $\delta = \alpha + \gamma$
	middle line $DE \parallel AC$ $DE = \frac{1}{2}AC$
	if $AC > AB,$ then $\beta > \gamma$

sin rule	$\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma} = 2R$
cos rule	$a^2 = b^2 + c^2 - 2bc \cdot \cos\alpha$ $b^2 = c^2 + a^2 - 2ca \cdot \cos\beta$ $c^2 = a^2 + b^2 - 2ab \cdot \cos\gamma$
	**Ceva's theorem** segments AA_1, BB_1, CC_1 intersect in one point if and only if $\dfrac{AC_1}{C_1B} \cdot \dfrac{BA_1}{A_1C} \cdot \dfrac{CB_1}{B_1A} = 1$
	**Menelaus's theorem** $\dfrac{AC_1}{C_1B} \cdot \dfrac{BA_1}{A_1C} \cdot \dfrac{CB_1}{B_1A} = 1$

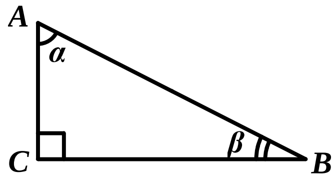
17.4 Properties of an Isosceles Triangle

	$\angle\alpha = \angle\gamma$
	in isosceles triangle $l_b = m_b = h_b$ **inverse:** each of there equalities mean, that $\triangle ABC$ isosceles $(AB = BC)$

17.5 Properties of a Right Triangle

	$m_c = R = \frac{1}{2}c = OC,$ and vice versa: **Criterion for a right-angled triangle:** If the median of a triangle is equal to the half of adverse side, then the triangle is right-angled.
	**Pythagorean theorem** $c^2 = a^2 + b^2$
	$\triangle ABD \sim \triangle CBD,$ $\triangle ABC \sim \triangle ABD,$ $\triangle ABC \sim \triangle CBD$
	$r = \dfrac{a+b-c}{2}$

17.6 Basic Trigonometric Relations

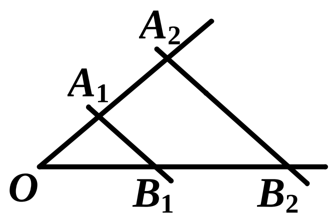
	$\sin\alpha = \dfrac{BC}{AB}$ $\cos\alpha = \dfrac{AC}{AB}$ $\tan\alpha = \dfrac{BC}{AC}$ $\cot\alpha = \dfrac{AC}{BC}$

If $\alpha + \beta = 90°$, then $\sin\alpha = \cos\beta$ $\tan\alpha = \cot\beta$	$\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$ $\tan\alpha \cdot \cot\alpha = 1$ $1 + \tan^2\alpha = \dfrac{1}{\cos^2\alpha}$	$\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$ $\sin^2\alpha + \cos^2\alpha = 1$ $1 + \cot^2\alpha = \dfrac{1}{\sin^2\alpha}$

17.7 Congruency and Similarity of Triangles

- 3 criteria for the congruency of triangles:
  - by three sides;
  - by two sides and the angle between them;
  - by a side and 2 adjacent angles.
- 3 criteria for the similarity of triangles:
  - by two angles;
  - by two proportional sides and the angle between them;
  - by three proportional sides.

17.8 Proportional Segments

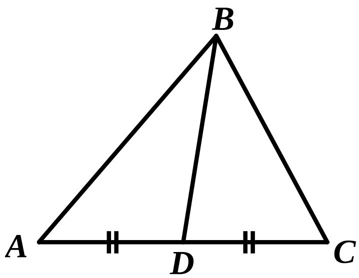
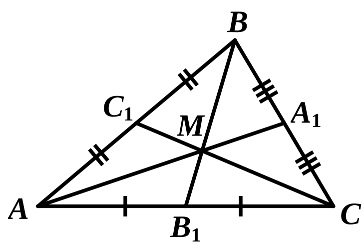
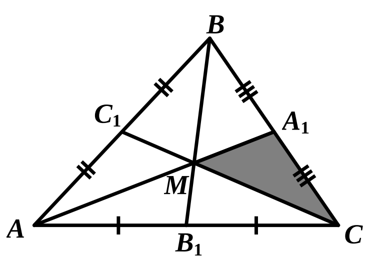
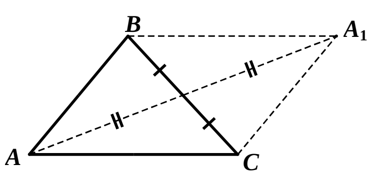


if $A_1B_1 \parallel A_2B_2$, then

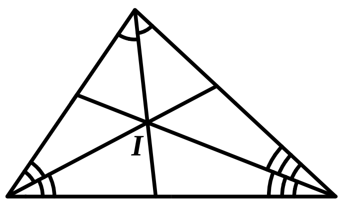
$$\frac{OA_1}{OA_2} = \frac{OB_1}{OB_2} = \frac{A_1B_1}{A_2B_2};$$

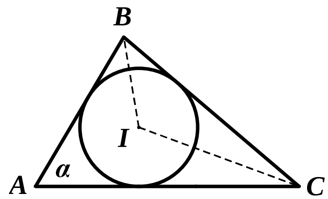
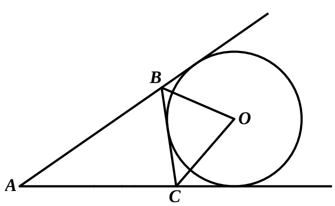
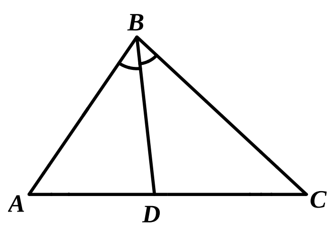
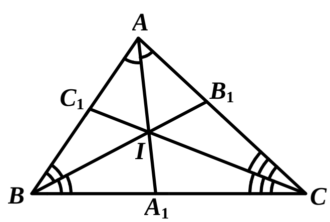
$$\frac{OA_1}{A_1A_2} = \frac{OB_1}{B_1B_2}$$

17.9 Medians

<table>
<tr><td>
</td><td>$A_{\triangle ABD} = A_{\triangle CBD}$</td></tr>
<tr><td>
</td><td>The medians of a triangle intersect at one point
$\frac{AM}{MA_1} = \frac{BM}{MB_1} = \frac{CM}{MC_1} = \frac{2}{1}$</td></tr>
<tr><td>
</td><td>The medians of a triangle split the triangle into 6 triangles with equal area</td></tr>
<tr><td>
</td><td>quadrilateral $ABA_1C -$ parallelogram</td></tr>
</table>

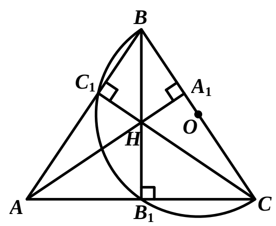
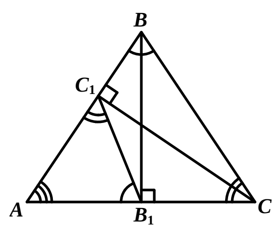
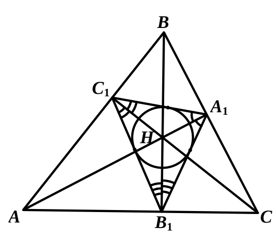
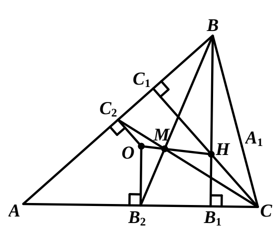
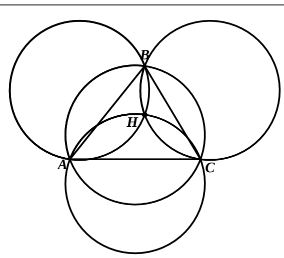
17.10 Bisectors

	Bisectors of a triangle intersect at one point $I -$ the center of the inscribed circle in a triangle

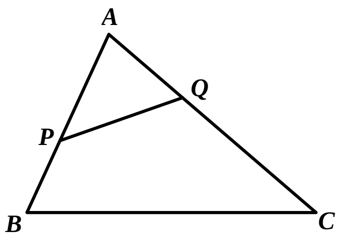
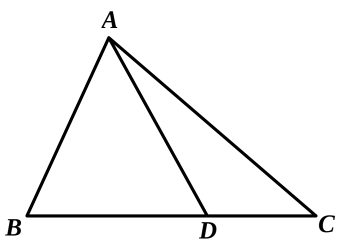
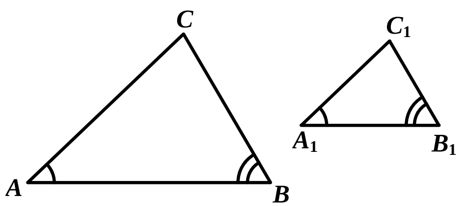
	$\angle CIB = 90^\circ + \dfrac{\alpha}{2}$
	If $O -$ center of the excircle, is touching side BC and extentions of sides AB and AC of ABC, then the equality holds: $\angle COB = 90^\circ - \dfrac{\angle A}{2}$
	$\dfrac{AD}{DC} = \dfrac{AB}{BC}$ $\dfrac{A_{\triangle ABD}}{A_{\triangle CBD}} = \dfrac{AB}{BC}$
	$\dfrac{AI}{IA_1} = \dfrac{b+c}{a}$ $\dfrac{BI}{IB_1} = \dfrac{a+c}{b}$ $\dfrac{CI}{IC_1} = \dfrac{a+b}{c}$ $AA_1^2 = BA \cdot AC - BA_1 \cdot A_1C$

17.11 Altitudes

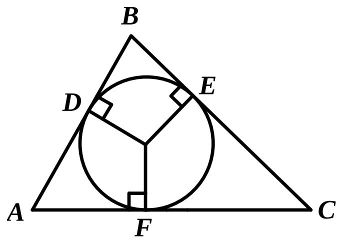
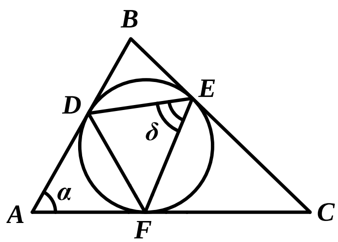
Some properties of altitudes and the point of their intersection — the orthocenter of a triangle (AA_1, BB_1, CC_1 are altitudes of a non-right triangle ABC, H is the orthocenter of the triangle):

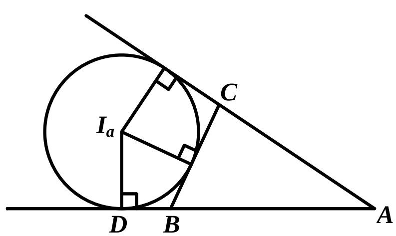
<table>
<tr><td>
</td><td>Points $B, C, B_1, C_1 -$ lie on the same circle, and $BC -$ its diameter.</td></tr>
<tr><td>
</td><td>Triangle ABB_1 is similar to ACC_1.
Triangle AB_1C_1 is similar to ABC with coefficient $|\cos\angle A|$</td></tr>
<tr><td>
</td><td>$\angle C_1A_1A = \angle B_1A_1A$,
$\angle A_1B_1B = \angle C_1B_1B$,
$\angle B_1C_1C = \angle A_1C_1C$,
$H -$ center of a incircle of $\triangle A_1B_1C_1$</td></tr>
<tr><td>
</td><td>Points O, H and M lie on the same straight line (Euler line), and M lies on OH and $OM : MH = 1 : 2$</td></tr>
<tr><td>
</td><td>If $H -$ orthocenter of the triangle, then the radii of circumscribed circles of triangles ABC, ABH, BCH, ACH are equal.</td></tr>
</table>

17.12 Relationships between Areas of Triangles

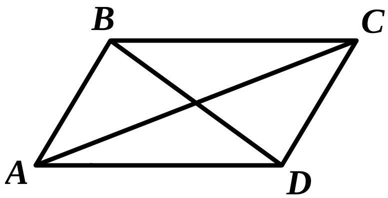
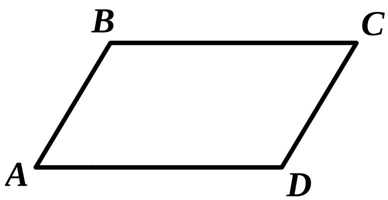
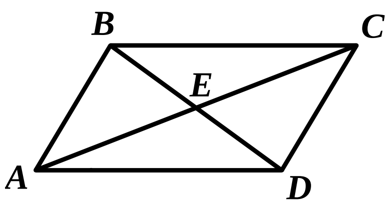
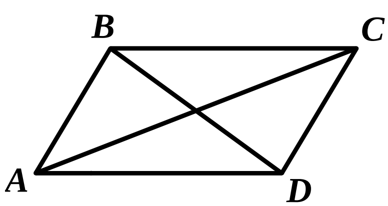
<table>
<tr><td>
</td><td>$$\frac{A_{\triangle APQ}}{A_{\triangle ABC}} = \frac{AP}{AB} \cdot \frac{AQ}{AC}$$</td></tr>
<tr><td>
</td><td>$$\frac{A_{\triangle ABD}}{A_{\triangle ADC}} = \frac{BD}{DC}$$</td></tr>
<tr><td>
</td><td>$$\frac{A_{\triangle ABC}}{A_{\triangle A_1B_1C_1}} = \frac{AB^2}{A_1B_1^2}$$</td></tr>
</table>

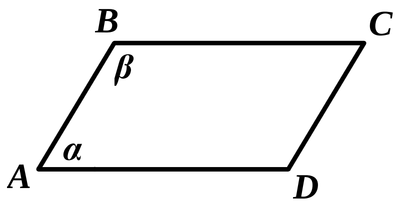
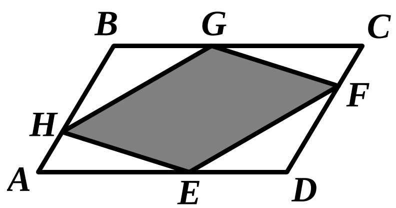
17.13 Incircle and Excircle

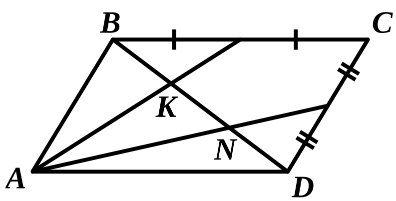
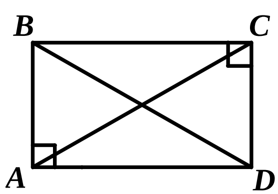
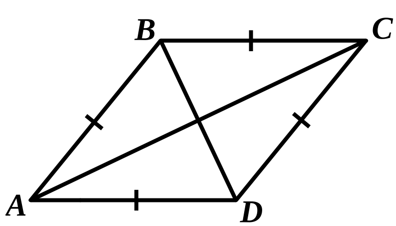
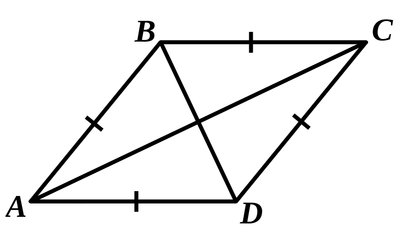
	$AF = AD = p - a$ $BD = BE = p - b$ $CE = CF = p - c$
	If incircle of ABC, is touching AB, BC and AC in D, E, and F, then $\angle DEF = 90^\circ - \frac{\alpha}{2}$

	$AD = p$, where $D -$ touching point of excircle, which touches extention of AB.
Euler's theorem	$IO^2 = R^2 - 2rR$

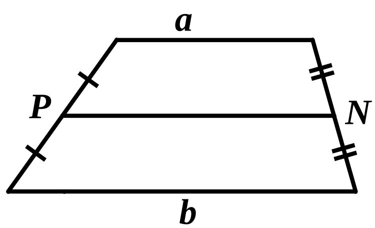
17.14 Parallelogram

	$\triangle ABD = \triangle CBD$ $\triangle ABC = \triangle ADC$
	$AB = CD$ $BC = AD$ $\angle DAB = \angle BCD$ $\angle ABC = \angle ADC$
	$AE = EC$ $BE = ED$
	$BD^2 + AC^2 = 2(AB^2 + BC^2)$

<table>
<tr><td>
</td><td>$\sin\alpha = \sin\beta$
$\cos\alpha = -\cos\beta$
$\tan\alpha = -\tan\beta$</td></tr>
<tr><td>
</td><td>if $\dfrac{AE}{ED} = \dfrac{DF}{FC} =$
$= \dfrac{CG}{GB} = \dfrac{BH}{HA}$, then

$HEFG -$ parallelogram</td></tr>
<tr><td>
</td><td>$BK = KN = ND$</td></tr>
<tr><td>
</td><td>$AC = BD \Leftrightarrow$
$ABCD -$ rectangle</td></tr>
<tr><td>
</td><td>$AC \perp BD \Leftrightarrow$
$ABCD -$ rhombus</td></tr>
<tr><td>
</td><td>in rhombus $ABCD$
AC and $BD -$ angle bisectors</td></tr>
</table>

17.15 Trapezoid

<table>
<tr>
<td>


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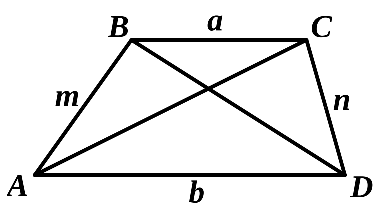
<td>

$PN \parallel a$

$PN = \dfrac{a+b}{2}$

</td>
</tr>
<tr>
<td>


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<td>

$AC^2 + BD^2 = m^2 + n^2 + 2ab$

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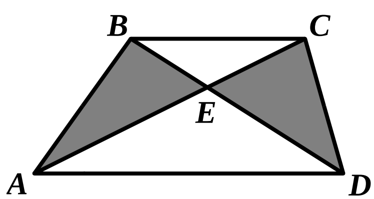
</tr>
<tr>
<td>


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$S_{\triangle ABE} = S_{\triangle DCE}$

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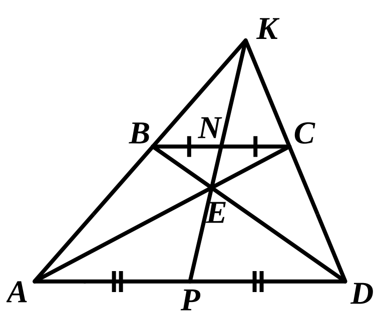
<tr>
<td>


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<td>

Points P,E,N,K are collinear

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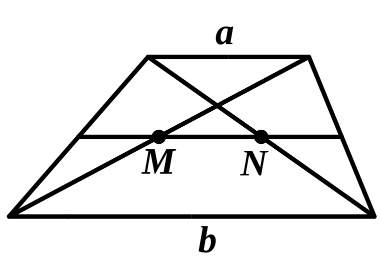
<tr>
<td>


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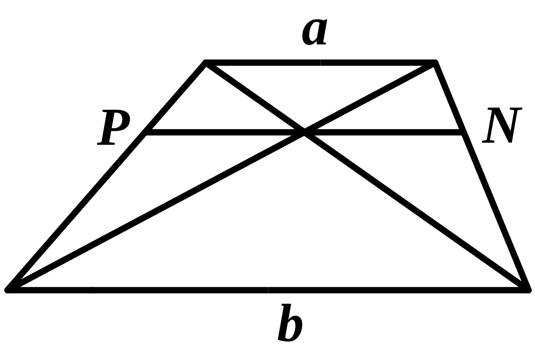
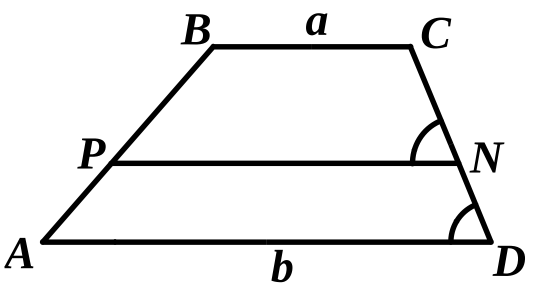
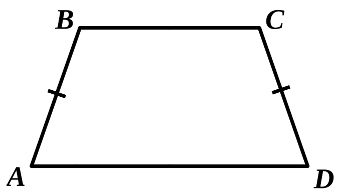
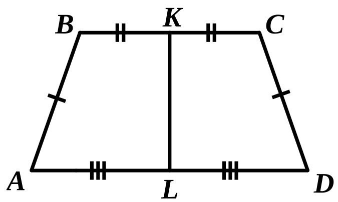
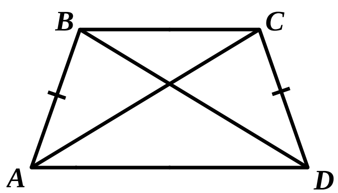
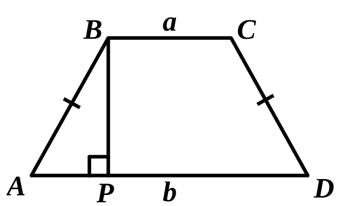
<td>

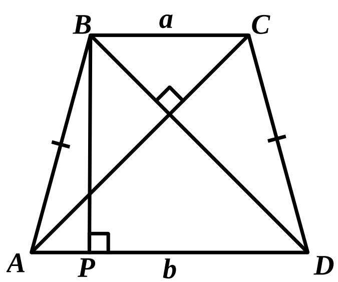
Let N and Mbe the midpoints of the diagonals
Then $MN = \dfrac{1}{2}|a-b|$

</td>
</tr>
</table>

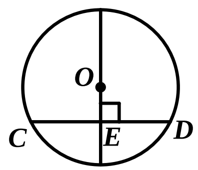
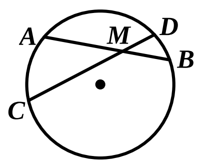
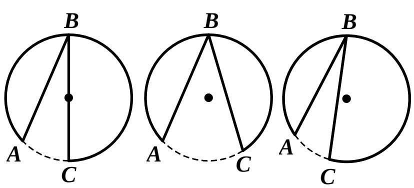
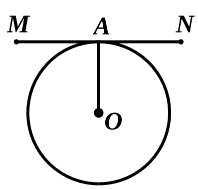
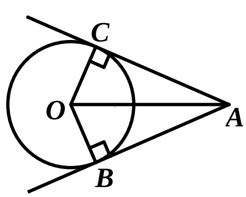
	If $PN \parallel a$, then $PN = \dfrac{2ab}{a+b}$
	If $S_{PBCN} = S_{APND}$, then $PN = \dfrac{\sqrt{a^2+b^2}}{2}$

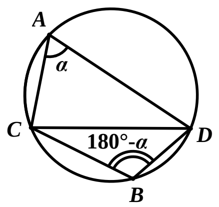
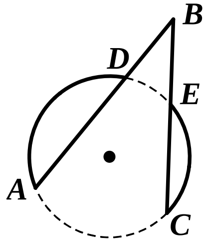
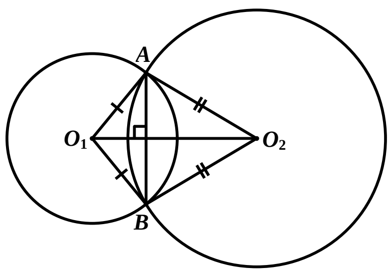
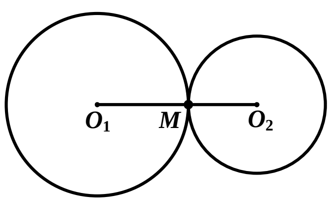
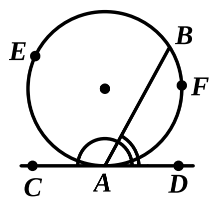
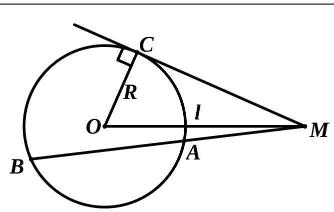
17.16 Isosceles Trapezoid

	$\angle BAD = \angle CDA$
	$KL \perp AD$
	$AC = BD$
	$AP = \dfrac{b-a}{2}, \quad PD = \dfrac{a+b}{2}$

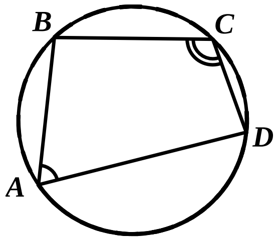
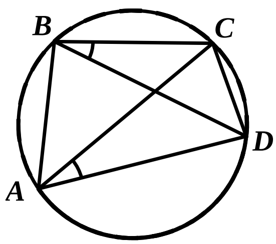
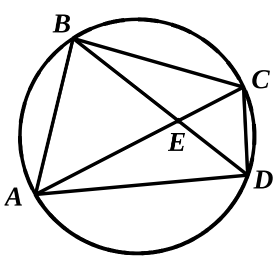
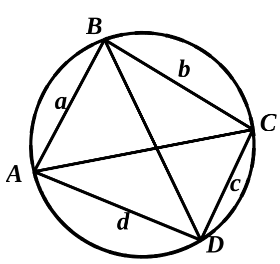
	$BD \perp AC \Leftrightarrow$ $BP = \dfrac{a+b}{2}$

17.17 Circle. Properties of Chords and Angles

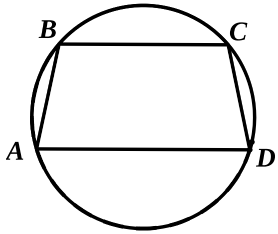
	$CE = ED$ ($O -$ centre of circle)
	$AM \cdot MB = CM \cdot MD$
	$\angle ABC = \frac{1}{2} \smile AC$
	MN touches the cercle in A $\Rightarrow MN \perp OA$; if MN passes through point A of cercle and $MN \perp OA$ $\Rightarrow MN -$ tangent line
	$AC = AB$, $\angle CAO = \angle BAO$

	$\angle CAD + \angle CBD = 180°$
	$\angle ABC = \frac{1}{2}(\smile AC - \smile DE)$
	The line joining the centers of two intersecting circles is perpendicular to their common chord and bisects it.
	The line joining the centers of two tangent circles passes through their point of tangency.
	$\angle BAC = \frac{1}{2} \smile AEB$ $\angle BAD = \frac{1}{2} \smile AFB$
	$MA \cdot MB = MC^2$ $MA \cdot MB = l^2 - R^2$

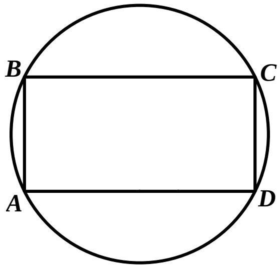
17.18 Cyclic Quadrilateral

<table>
<tr><td>
</td><td>$\angle BAD + \angle BCD = 180^\circ$</td></tr>
<tr><td>
</td><td>$\angle DAC = \angle DBC$</td></tr>
<tr><td>
</td><td>$AE \cdot EC = BE \cdot ED$</td></tr>
<tr><td>
</td><td>Ptolemy's theorem
$AC \cdot BD = ac + bd$</td></tr>
</table>

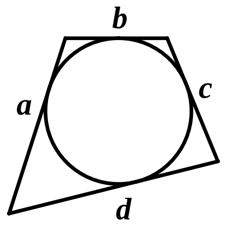
17.19 Inscribed Trapezoid

	the inscribed trapezoid is isosceles: $AB = DC$

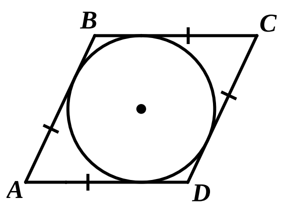
17.20 Inscribed Parallelogram

	inscribed parallelogram is a rectangle: $\angle DAB = \angle ABC = \angle BCD = \angle CDA = 90^\circ$

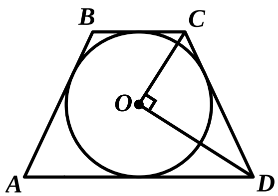
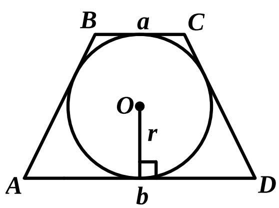
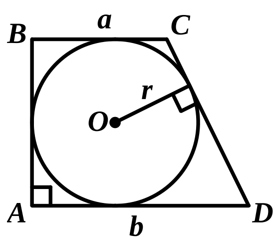
17.21 Tangential Quadrilateral

	$a + c = b + d$

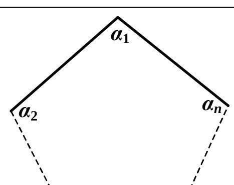
17.22 Tangential Parallelogram

	tangential parallelogram is a rhombus: $AB = BC = CD = DA$

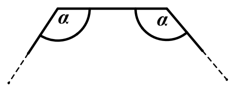
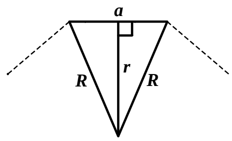
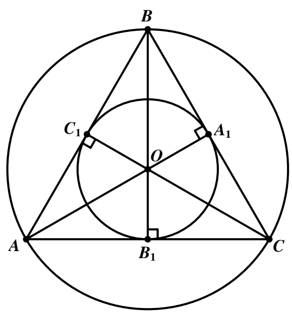
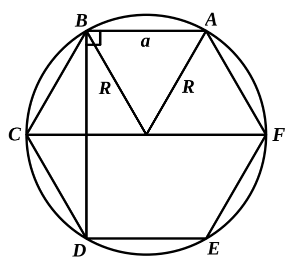
17.23 Tangential Trapezoid

	$\angle DOC = 90°$ (O – centre of the circle)
	if $AB = CD$, then $r = \frac{\sqrt{ab}}{2}$
	$r = \frac{ab}{a+b}$

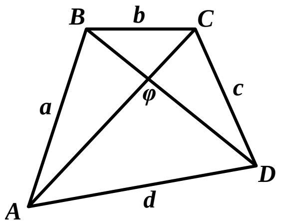
17.24 Sum of Angles of an N-gon

	$\alpha_1 + \alpha_2 + \ldots \alpha_n = 180°(n-2)$

17.25 Regular N-gon

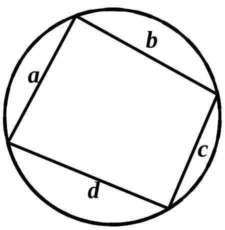
	$\alpha = \dfrac{180^\circ(n-2)}{n}$
	$a = 2R \cdot \sin\dfrac{180^\circ}{n}$ $a = 2r \cdot \tan\dfrac{180^\circ}{n}$
	For a regular triangle: $a = AB = BC = CA$ $h = \dfrac{\sqrt{3}}{2}a$ $R = \dfrac{2}{3}h$ $r = \dfrac{1}{3}h$
	For a regular hexagon: $a = R$ $AB \parallel CF$ $CF = 2a$ $BD \perp BA$ $BD = \sqrt{3}a$

17.26 Area of a Quadrilateral

<table>
<tr>
<td>
</td>
<td>

$$S = \frac{1}{2} AC \cdot BD \cdot \sin\varphi$$

$$S = \frac{1}{4}\left|d^2 - c^2 + b^2 - a^2\right| \cdot \tan\varphi, \;\; \varphi < 90^\circ$$

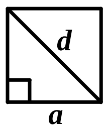
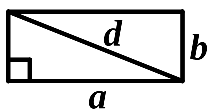
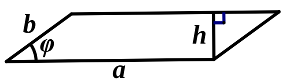
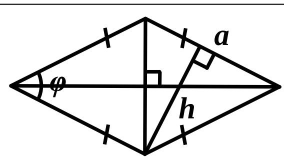
</td>
</tr>
<tr>
<td>
</td>
<td>

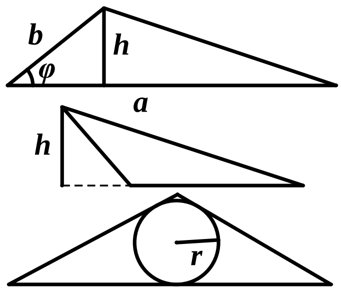
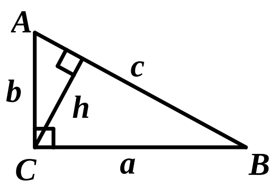
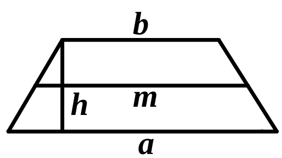
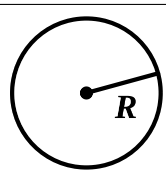
Brahmagupta's formula:

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d)}$$

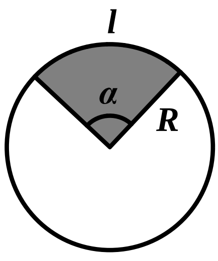
</td>
</tr>
</table>

17.27 Areas of Random Figures

 square	$S = a^2$ $P = 4a$ $d = \sqrt{2}a$
 rectangle	$S = ab$ $P = 2(a+b)$ $d = \sqrt{a^2+b^2}$
 parallelogram	$S = ah = ab\sin\varphi$ $P = 2(a+b)$ $d_1 = \sqrt{a^2+b^2-2ab\cos\varphi}$ $d_2 = \sqrt{a^2+b^2+2ab\cos\varphi}$
 rhombus	$S = ah = a^2\sin\varphi = \frac{d_1 d_2}{2}$, where d_1,d_2 − diagonals of a rhombus

 triangle	$S = \frac{ah}{2} = \frac{1}{2}ab\sin\varphi = pr = \frac{abc}{4R} =$ $= \sqrt{p(p-a)(p-b)(p-c)}$
 right triangle	$S = \frac{ab}{2} = \frac{ch}{2}$ $h = \frac{ab}{c}$
 trapezoid	$S = \frac{a+b}{2}h = mh$ where m – midline of trapezoid
 circle	$S = \pi R^2$ $L = 2\pi R = \pi D$

17.28 Arc Length, Segment and Sector Area

	Arc length: $l = R\alpha$, where α is measured in radians sector area: $S = \frac{1}{2}\alpha R^2 = \frac{1}{2} l \cdot R$ segment area: $S = \frac{1}{2} R^2(\alpha - \sin\alpha)$

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